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PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

A SERIES INVERSION FORMULA.

By E. C. TITCHMARSH.

[Received 8 November, 1925.—Read 12 November, 1925.]

1. *Introduction.* Suppose that the functions $f(x)$ and $e^{i\lambda(x-\pi)}f(x)$, where λ is not an integer, are expansible in Fourier series in the form

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}, \quad e^{i\lambda(x-\pi)}f(x) = \sum_{n=-\infty}^{\infty} b_n e^{-inx},$$

so that

$$\sum_{n=-\infty}^{\infty} a_n e^{inx} = \sum_{n=-\infty}^{\infty} b_n e^{-i(n+\lambda)x+i\lambda\pi}.$$

If we multiply by $e^{i(m+\lambda)x}$ and integrate term by term from 0 to 2π, we obtain

$$\frac{\sin\pi\lambda}{\pi} \sum_{n=-\infty}^{\infty} \frac{a_n}{m+n+\lambda} = b_m. \tag{1.1}$$

Similarly, if we multiply by e^{-imx} and integrate, we obtain the reciprocal formula

$$a_m = \frac{\sin\pi\lambda}{\pi} \sum_{n=-\infty}^{\infty} \frac{b_n}{m+n+\lambda}. \tag{1.2}$$

This can be verified directly by substituting (1.1) in (1.2), changing the order of summation, and using the formula

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+r+\lambda)(n+s+\lambda)} = \frac{\pi^2}{\sin^2\pi\lambda} \quad (r=s), \quad 0 \quad (r\neq s).$$

In either case the work is so far purely formal, and the Fourier method could not be justified without introducing unnecessary restrictions. We have, for example,

$$a_n = e^{iny}, \quad b_n = e^{-i(n+\lambda)y + i\lambda\pi} \quad (0 < y < 2\pi),$$

and, in particular, if $a_n = (-1)^n$, then $b_n = (-1)^n$. Such a case would not be included in the Fourier theory.

It will, therefore, be assumed simply that the numbers a_n are given, and that the numbers b_n are defined by (1 . 1), and the object is to discover in what circumstances and in what sense the formula (1 . 2) holds. We shall consider the case $\lambda = \frac{1}{2}$, other cases being practically the same. In a recent paper* it was shown that, if $\Sigma |a_n|^p$, where $p > 1$, is convergent, then the formulae hold, and $\Sigma |b_n|^p$ is also convergent. But this condition is by no means necessary.

The series considered, of the form

$$\sum_{n=-\infty}^{\infty} u_n, \tag{1 . 3}$$

are convergent in the ordinary sense if the series

$$\sum_{n=1}^{\infty} u_n, \quad \sum_{n=1}^{\infty} u_{-n}$$

are both convergent, and they are summable $(C, 1)$ in the ordinary sense if these two series are both summable $(C, 1)$. The series (1 . 3) will be said to be convergent in the restricted sense if the series

$$\sum_{n=1}^{\infty} (u_n + u_{-n})$$

is convergent, and to be summable $(C, 1)$ in the restricted sense if this series is summable $(C, 1)$.

It is assumed throughout that the series (1 . 1) *are convergent in the ordinary sense.* To show the inadequacy of restricted convergence, let $a_n = 1$ for all values of n $(\lambda = \frac{1}{2})$. Then the series (1 . 1) converge in the restricted sense to the sum zero, and (1 . 2) is not satisfied.

E. C. Titchmarsh, *Math. Zeitschrift*, 25 (1926), 321–347. See also M. Riesz, *Math. Annalen* (to appear shortly).

The results are as follows :—

THEOREM I. *The series* (1 . 2) *are summable* $(C, 1)$ *in the restricted sense to the sums stated.*

The special interest of the theorem is that it shows that, if the series (1 . 2) converge at all, then they have the sums stated. The remaining theorems may be expressed in terms of the numbers

$$R_n = \sum_{m=n}^{\infty} \frac{a_m + a_{-m}}{m}, \quad R'_n = \sum_{m=n}^{\infty} \frac{a_m - a_{-m}}{m},$$

which may be any two sequences tending to zero.

THEOREM II. *The necessary and sufficient condition for the convergence of* (1 . 2) *in the restricted sense is*

$$\lim_{q \to \infty} \sum_{(1-\theta)q}^{(1+\theta)q} \frac{R_n}{n-q-\frac{1}{2}} = 0 \qquad (0 < \theta < 1). \tag{1 . 4}$$

THEOREM III. *The necessary and sufficient condition for the summability* $(C, 1)$ *in the ordinary sense of* (1 . 2) *is the convergence of the series*

$$\sum_{n=1}^{\infty} \frac{R'_n}{n+\frac{1}{2}}. \tag{1 . 5}$$

THEOREM IV. *The necessary and sufficient conditions for convergence in the ordinary sense are the conditions of Theorems II and III, together with*

$$\lim_{q \to \infty} \sum_{(1-\theta)q}^{(1+\theta)q} \frac{R'_n}{n-q-\frac{1}{2}} = 0. \tag{1 . 6}$$

2. LEMMA. *The series*

$$\frac{1}{\pi} \sum_{m=-\infty}^{\infty} \left(\frac{b_m}{m+n+\frac{1}{2}} - \frac{b_m}{m+\frac{1}{2}} \right)$$

converges in the ordinary sense to the sum $a_n - a_0$.

The series is

$$\frac{1}{\pi^2} \sum_{m=-\infty}^{\infty} \left(\frac{1}{m+n+\frac{1}{2}} - \frac{1}{m+\frac{1}{2}} \right) \sum_{r=-\infty}^{\infty} \frac{a_r}{m+r+\frac{1}{2}},$$

and we have merely to prove that the order of the summation may be inverted. It will be sufficient to prove that

$$\sum_{m=-\infty}^{\infty} \sum_{r=s}^{\infty} = \sum_{r=s}^{\infty} \sum_{m=-\infty}^{\infty},$$

where $s > n$, $s > 0$, since the remaining part can be dealt with similarly; and for this it is sufficient to prove that

$$\lim_{p\to\infty} \sum_{r=s}^{\infty} \sum_{m=-\infty}^{-p} = 0, \quad \lim_{q\to\infty} \sum_{r=s}^{\infty} \sum_{m=q}^{\infty} = 0.$$

Now

$$\left(\frac{1}{m+n+\frac{1}{2}} - \frac{1}{m+\frac{1}{2}}\right)\frac{1}{m+r+\frac{1}{2}} = \frac{n}{r}\,\frac{1}{m+n+\frac{1}{2}}\left(\frac{1}{m+r+\frac{1}{2}} - \frac{1}{m+\frac{1}{2}}\right),$$

and obviously

$$\sum_{r=s}^{\infty} \frac{a_r}{r} \sum_{m=-\infty}^{-q} \frac{1}{(m+n+\frac{1}{2})(m+\frac{1}{2})}, \quad \sum_{r=s}^{\infty} \frac{a_r}{r} \sum_{m=q}^{\infty} \frac{1}{(m+n+\frac{1}{2})(m+\frac{1}{2})}$$

tend to zero. Also

$$\sum_{m=q}^{\infty} \frac{1}{(m+n+\frac{1}{2})(m+r+\frac{1}{2})}$$

is a steadily decreasing function of r, its maximum tending to zero as $q \to \infty$. Hence, by partial summation,

$$\sum_{r=s}^{\infty} \frac{a_r}{r} \sum_{m=q}^{\infty} \frac{1}{(m+n+\frac{1}{2})(m+r+\frac{1}{2})} \to 0.$$

A similar partial summation gives

$$\sum_{r=s}^{\infty} \frac{a_r}{r} \sum_{m=-\infty}^{-p} \frac{1}{(m+n+\frac{1}{2})(m+r+\frac{1}{2})}$$

$$= \sum_{r=s}^{\infty} \frac{R_r''}{(r-n)(r-n+1)} \sum_{m=n-p+1}^{r-p} \frac{1}{m+\frac{1}{2}} - \sum_{r=s}^{\infty} \frac{R_r''}{(r-n+1)(r-p+\frac{3}{2})}$$

$$- \frac{R_{s-1}''}{s-n} \sum_{m=n-p+1}^{s-p} \frac{1}{m+\frac{1}{2}},$$

where $R_n'' = \frac{1}{2}(R_n + R_n')$. The last term is $O(1/p)$, the middle term is

$$O\left\{\int_s^{p-1} \frac{du}{u(p-u)} + \int_{p+1}^{\infty} \frac{du}{u(u-p)}\right\} = O\left(\frac{\log p}{p}\right),$$

and since

$$\left| \sum_{m=n-p+1}^{r-p} \frac{1}{m+\frac{1}{2}} \right| < \frac{K}{\sqrt{p}} \quad (r < \sqrt{p}), \qquad < K \log rp \quad (r \geqslant \sqrt{p}),$$

the first term is

$$\sum_{r<\sqrt{p}} o\left(\frac{1}{\sqrt{p}}\right) + \sum_{r\geqslant\sqrt{p}} O\left(\frac{\log r + \log p}{r^2}\right) = o(1).$$

This proves the lemma.

3. *Convergence in the restricted sense.* In view of the lemma it is sufficient to consider the series

$$\sum_{m=-\infty}^{\infty} \frac{b_m}{m+\frac{1}{2}}. \tag{3.1}$$

We have

$$\sum_{m=-q-1}^{q} \frac{b_m}{m+\frac{1}{2}} = \frac{1}{\pi} \sum_{m=-q-1}^{q} \frac{1}{m+\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{a_n}{m+n+\frac{1}{2}}$$

$$= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} a_n \sum_{m=-q-1}^{q} \frac{1}{(m+\frac{1}{2})(m+n+\frac{1}{2})}.$$

Now, if $n \neq 0$,

$$\sum_{m=-q-1}^{q} \frac{1}{(m+\frac{1}{2})(m+n+\frac{1}{2})} = \frac{1}{n} \sum_{m=-q-1}^{q} \left(\frac{1}{m+\frac{1}{2}} - \frac{1}{m+n+\frac{1}{2}}\right)$$

$$= -\frac{1}{n} \sum_{m=n-q-1}^{n+q} \frac{1}{m+\frac{1}{2}}$$

$$= -u_{n,q}/n,$$

say, while, if $n = 0$, we obtain

$$\sum_{m=-q-1}^{q} \frac{1}{(m+\frac{1}{2})^2} = \pi^2 + O\left(\frac{1}{q}\right).$$

Hence

$$\sum_{m=-q-1}^{q} \frac{b_m}{m+\frac{1}{2}} = \pi a_0 - \frac{1}{\pi} \sum_{n=-\infty}^{\infty}{}' \frac{a_n}{n} u_{n,q} + O\left(\frac{1}{q}\right)$$

$$= \pi a_0 - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{a_n + a_{-n}}{n} u_{n,q} + O\left(\frac{1}{q}\right),$$

since $u_{n,q} = -u_{-n,q}$. Now

$$\sum_{n=1}^{\infty} \frac{a_n + a_{-n}}{n} u_{n,q} = \sum_{n=1}^{\infty} R_n (u_{n,q} - u_{n-1,q})$$

$$= \sum_{n=1}^{k} + \sum_{n=k+1}^{(1-\theta)q} + \sum_{(1-\theta)q}^{(1+\theta)q} + \sum_{(1+\theta)q}^{\infty},$$

where $k = [\sqrt{q}]$, and $0 < \theta < 1$.* Now $|R_n| \leqslant R$ (all n), $|R_n| < \epsilon$ $(n > \sqrt{q_0})$; also $u_{n,q}$ increases steadily for $n < q$, and then decreases steadily to zero, while

$$u_{n,q} < \frac{2n}{q-n+\frac{3}{2}} \quad (n < q), \quad u_{n,q} < \frac{2q+2}{n-q-\frac{1}{2}} \quad (n > q).$$

Hence, for $q > q_1$,

$$\left| \sum_{n=1}^{k} \right| < R \frac{2\sqrt{q}}{q-\sqrt{q}+\frac{3}{2}}, \quad \left| \sum_{k+1}^{(1-\theta)q} \right| < \epsilon \frac{2(1-\theta)q}{\theta q + \frac{3}{2}}, \quad \left| \sum_{(1+\theta)q}^{\infty} \right| < \epsilon \frac{2q+2}{\theta q - \frac{1}{2}}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{a_n + a_{-n}}{n} u_{n,q} = \sum_{(1-\theta)q}^{(1+\theta)q} R_n (u_{n,q} - u_{n+1,q}) + o(1)$$

$$= \sum_{(1-\theta)q}^{(1+\theta)q} R_n \left(\frac{1}{n+q+\frac{1}{2}} - \frac{1}{n-q-\frac{3}{2}} \right) + o(1)$$

$$= - \sum_{(1-\theta)q}^{(1+\theta)q} \frac{R_n}{n-q-\frac{1}{2}} + o(1),$$

and this proves Theorem II.

---

* I write $\sum_{n=a}^{b} = \sum_{a \leqslant n \leqslant b}$, whether a and b are integers or not. To avoid trivial difficulties with the limits, let θ be irrational.

4. *Summability in the restricted sense.* In view of what has been proved already, Theorem I is true if

$$\sum_{q=1}^{Q} \sum_{n=(1-\theta)q}^{(1+\theta)q} \frac{R_n}{n-q-\frac{1}{2}} = o(Q). \tag{4.1}$$

The left-hand side is equal to

$$\begin{aligned}
&-\sum_{n=1}^{(1-\theta)Q} R_n \sum_{q=n/(1+\theta)}^{n/(1-\theta)} \frac{1}{q-n+\frac{1}{2}} - \sum_{n=(1-\theta)Q}^{(1+\theta)Q} R_n \sum_{q=n/(1+\theta)}^{Q} \frac{1}{q-n+\frac{1}{2}} \\
&= -\sum_{n=1}^{(1-\theta)Q} R_n \sum_{p=-\theta n/(1+\theta)}^{\theta n/(1-\theta)} \frac{1}{p+\frac{1}{2}} - \sum_{n=(1-\theta)Q}^{(1+\theta)Q} R_n \sum_{p=-\theta n/(1+\theta)}^{Q-n} \frac{1}{p+\frac{1}{2}} \\
&= -\sum_{n=1}^{(1-\theta)Q} R_n \sum_{\theta n/(1+\theta)-1}^{\theta n/(1-\theta)} \frac{1}{p+\frac{1}{2}} - \sum_{n=(1-\theta)Q}^{(1+\theta)(Q+1)/(1+2\theta)} R_n \sum_{\theta n/(1+\theta)-1}^{Q-n} \frac{1}{p+\frac{1}{2}} \\
&\quad + \sum_{(1+\theta)(Q+2)/(1+2\theta)}^{Q} R_n \sum_{Q-n+1}^{\theta n/(1+\theta)-1} \frac{1}{p+\frac{1}{2}} + \sum_{n=Q+1}^{(1+\theta)Q} R_n \sum_{n-Q-1}^{\theta n/(1+\theta)-1} \frac{1}{p+\frac{1}{2}}.
\end{aligned}$$

For $Q \geq Q_0(\epsilon)$, the first two terms give

$$\sum_{n \leq \sqrt{Q}} O(1) + \epsilon \sum_{n > \sqrt{Q}}^{(1-\theta)Q} O(1) + \epsilon \sum_{(1-\theta)Q}^{(1+\theta)(Q+1)/(1+2\theta)} O\left(\frac{Q}{n}\right) = O(\sqrt{Q}) + \epsilon O(Q).$$

Also replacing R_n by ϵ in the last two terms is equivalent to replacing R_n by ϵ in (4.1), with error $O(\sqrt{Q}) + \epsilon O(Q)$, as has just been proved; and

$$\epsilon \sum_{q=1}^{Q} \sum_{(1-\theta)q}^{(1+\theta)q} \frac{1}{n-q-\frac{1}{2}} = \epsilon \sum_{q=1}^{Q} O(1) = \epsilon O(Q).$$

This proves Theorem I.

5. *Summability in the ordinary sense.* Suppose that the series (3.1) is summable $(C, 1)$ in the ordinary sense. Then

$$\sum_{m=1}^{q} \left(1 - \frac{m-1}{q}\right) \frac{b_m}{m+\frac{1}{2}} = A + o(1). \tag{5.1}$$

Let $$b'_m = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{m+n+\frac{1}{2}}, \quad b''_m = \frac{1}{\pi} \sum_{n=-\infty}^{-1} \frac{a_n}{m+n+\frac{1}{2}}.$$

Then $$\sum_{m=1}^{q} \left(1 - \frac{m-1}{q}\right) \frac{b'_m + b''_m}{m+\frac{1}{2}} = A' + o(1).$$

But, by Theorem I,

$$\sum_{m=1}^{q}\left(1-\frac{m-1}{q}\right)\frac{b''_m-b''_{-m-1}}{m+\frac{1}{2}}=o(1).$$

Hence
$$\sum_{m=1}^{q}\left(1-\frac{m-1}{q}\right)\frac{b'_m+b''_{-m-1}}{m+\frac{1}{2}}=A'+o(1).$$

Now
$$b'_m+b''_{-m-1}=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{a_n-a_{-n}}{m+n+\frac{1}{2}}=o(1).$$

Hence
$$\sum_{m=1}^{q}\frac{m-1}{q}\frac{b'_m+b''_{-m-1}}{m+\frac{1}{2}}=\frac{1}{q}\sum_{m=1}^{q}o(1)=o(1).$$

Hence

$$\begin{aligned}\pi A'+o(1)&=\pi\sum_{m=1}^{q}\frac{b'_m+b''_{-m-1}}{m+\frac{1}{2}}\\&=\sum_{m+1}^{q}\frac{1}{m+\frac{1}{2}}\sum_{n-1}^{\infty}\frac{a_n-a_{-n}}{m+n+\frac{1}{2}}\\&=\sum_{m=1}^{q}\sum_{n=1}^{\infty}\left(\frac{1}{m+\frac{1}{2}}-\frac{1}{m+n+\frac{1}{2}}\right)(R'_n-R'_{n+1})\\&=\sum_{n=1}^{\infty}\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n+q+\frac{1}{2}}\right)R'_n\\&=\sum_{n=1}^{q}\frac{R'_n}{n+\frac{1}{2}}-\sum_{n=1}^{q}\frac{R'_n}{n+q+\frac{1}{2}}+q\sum_{n=q+1}^{\infty}\frac{R'_n}{(n+\frac{1}{2})(n+q+\frac{1}{2})}\\&=\sum_{n=1}^{q}\frac{R'_n}{n+\frac{1}{2}}+\sum_{n=1}^{q}o\left(\frac{1}{q}\right)+q\sum_{n=q+1}^{\infty}o\left(\frac{1}{n^2}\right)\\&=\sum_{n=1}^{q}\frac{R'_n}{n+\frac{1}{2}}+o(1).\end{aligned}$$

This proves that the condition stated is necessary. Conversely, if the condition is satisfied, we find, on reversing the above steps, that (5 . 1) holds. But, by Theorem I,

$$\sum_{m=1}^{q}\left(1-\frac{m-1}{q}\right)\frac{b_m-b_{-m-1}}{m+\frac{1}{2}}$$

tends to a limit. Hence

$$\sum_{m=1}^{q}\left(1-\frac{m-1}{q}\right)\frac{b_{-m-1}}{m+1}$$

also tends to a limit. Hence the series is summable $(C, 1)$ in the ordinary sense.

6. *Convergence in the ordinary sense.* If the series (3 . 1) is convergent in the ordinary sense, then

$$\sum_{m=0}^{q} \frac{b'_m}{m+\frac{1}{2}} + \sum_{m=0}^{q} \frac{b''_m}{m+\frac{1}{2}} \tag{6 . 1}$$

and

$$\sum_{m=-q-1}^{-1} \frac{b'_m}{m+\frac{1}{2}} + \sum_{m=-q-1}^{-1} \frac{b''_m}{m+\frac{1}{2}} \tag{6 . 2}$$

both tend to limits when $q \to \infty$. Also it follows from the working of § 5 that the existence of the limit of

$$\sum_{m=0}^{q} \frac{b'_m}{m+\frac{1}{2}} - \sum_{m=-q-1}^{-1} \frac{b''_m}{m+\frac{1}{2}} \tag{6 . 3}$$

is a necessary condition. Adding (6 . 2) and (6 . 3), we see that the restricted convergence of

$$\sum_{m=-\infty}^{\infty} \frac{b'_m}{m+\frac{1}{2}}$$

is a necessary condition, and similarly for b''_m. Applying Theorem II to each of these series, we obtain the conditions (1 . 5) and (1 . 6); and, as we have seen, the existence of the limit of (6 . 3) is equivalent to the convergence of (1 . 5). Thus the conditions are necessary, and it is easily seen that they are also sufficient.

It may be noted that a simple sufficient condition for convergence in the ordinary sense is the convergence in the ordinary sense of

$$\sum_{-\infty}^{\infty} \frac{a_m}{m+\frac{1}{2}} \log|m+\tfrac{1}{2}|.$$

But the condition is not necessary; *e.g.* let

$$a_n = (-1)^n n (\log n)^{-\frac{1}{2}} \quad (n > 1), \quad a_n = 0 \quad (n < 1).$$

7. *Example.* Let

$$a_m = \frac{1}{(m+\frac{1}{2})^{2k}},$$

where k is a positive integer. Then

$$\pi b_n = \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k}(m+n+\frac{1}{2})}$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k-1}} \frac{1}{n} \left\{ \frac{1}{(m+\frac{1}{2})} - \frac{1}{(m+n+\frac{1}{2})} \right\}$$

$$= \frac{1}{n} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k}} - \frac{1}{n} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k-1}(m+n+\frac{1}{2})}.$$

This process may be repeated until we arrive at the series

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})(m+n+\frac{1}{2})},$$

which is zero if $n \neq 0$. Hence, if $n \neq 0$,

$$\pi b_n = \frac{1}{n} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k}} - \frac{1}{n^2} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k-1}} + \ldots + \frac{1}{n^{2k-1}} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^2}.$$

Now, if r is even,

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^r} = 2(2^r-1)\,\zeta(r),$$

while the sum is zero if r is odd. Hence, if $n \neq 0$,

$$b_n = \frac{2}{\pi} \sum_{p=1}^{k} \frac{(2^{2p}-1)\,\zeta(2p)}{n^{2k-2p+1}}.$$

Also, clearly $b_0 = 0$. Hence the relation

$$\sum_{n=-\infty}^{\infty} a_n^2 = \sum_{n=-\infty}^{\infty} b_n^2$$

gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^{4k}} = \frac{4}{\pi^2} \sum_{n=-\infty}^{\infty}{}' \left\{ \sum_{p=1}^{k} \frac{(2^{2p}-1)\,\zeta(2p)}{n^{2k-2p+1}} \right\}^2$$

$$= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \sum_{p=1}^{k} \sum_{q=1}^{k} \frac{(2^{2p}-1)(2^{2q}-1)\,\zeta(2p)\,\zeta(2q)}{n^{4k-2p-2q+2}},$$

that is to say

$$(8.51) \quad (2^{4k}-1)\,\zeta(4k)$$

$$= \frac{4}{\pi^2} \sum_{p=1}^{k} \sum_{q=1}^{k} (2^{2p}-1)(2^{2q}-1)\,\zeta(2p)\,\zeta(2q)\,\zeta(4k-2p-2q+2).$$

Similarly if we take

$$a_m = \frac{1}{(m+\frac{1}{2})^{2k+1}}$$

we obtain

$$b_0 = \frac{1}{\pi}(2^{2k+2}-1)\,\zeta(2k+2), \quad b_n = -\frac{2}{\pi}\sum_{p=1}^{k}\frac{(2^{2p}-1)\,\zeta(2p)}{n^{2k-2p+2}} \quad (n \neq 0).$$

This leads to the relation

$$(8.52) \quad (2^{4k+2}-1)\,\zeta(4k+2) = \frac{2}{\pi^2}(2^{2k+2}-1)^2\,\zeta^2(2k+2) + \frac{4}{\pi^2}\sum_{p=1}^{k}\sum_{q=1}^{k}(2^{2p}-1)(2^{2q}-1)\,\zeta(2p)\,\zeta(2q)\,\zeta(4k-2p-2q+4).$$

We have thus proved quite independently of the circular functions that the numbers $\zeta(2n)$ are all rational multiples of integral powers of $\zeta(2)$ [since the constant π^2 enters the above formulae as the value of $6\,\zeta(2)$].

ON FOURIER'S INTEGRAL

By S. POLLARD.

[Received 4 June, 1925.—Received in revised form 9 March, 1926.
Read 11 March, 1926.]

1. The account which follows contains investigations of various aspects of Fourier's integral, the dominant idea being that the theory of Fourier's integral should, so far as possible, be independent of the theory of Fourier's series. It is certainly true that many of the properties of the integral may be deduced from those of the series by a passage to the limit; but this passage to the limit is usually far from obvious and not particularly easy, while the independent treatment is on occasion quite straightforward and, once the idea is grasped, quite simple to apply.

2. The basis of the theory of the series, as is well known, is the fact that they are associated with a certain set of orthogonal functions. The same thing holds good, although in not quite such a simple way, for Fourier's integrals. We have, in fact, if r and s are integers,

$$\int_{-\infty}^{\infty} \cos rx \cos sx \, dx = 0, \tag{2.1}$$ (C 1)

$$\int_{-\infty}^{\infty} \sin rx \sin sx \, dx = 0, \tag{2.2}$$ (C 1)

whenever $r \neq \pm s$, and

$$\int_{-\infty}^{\infty} \cos rx \sin sx \, dx = 0 \tag{2.3}$$ (C 1)

always, the integrals being evaluated by Cesàro's first mean.

Thus, if $r \neq \pm s$,

$$\frac{1}{\lambda}\int_0^\lambda d\xi \int_{-\xi}^{\xi} \cos rx \cos sx\, dx = \frac{1}{\lambda}\int_0^\lambda \left\{\frac{\sin(r+s)\xi}{r+s} + \frac{\sin(r-s)\xi}{r-s}\right\} d\xi$$

$$= \frac{1}{\lambda}\left\{\frac{1-\cos(r+s)\lambda}{(r+s)^2} + \frac{1-\cos(r-s)\lambda}{(r-s)^2}\right\}$$

$$\to 0$$

as $\lambda \to \infty$, which is what is required by (2 . 1).

The other equation (2 . 2) follows in the same way, while (2 . 3) is even easier, since, the integrand being an odd function,

$$\int_{-\xi}^{\xi} \cos rx \sin sx\, dx = 0.$$

It must be noticed, however, that the system $\cos rx$, $\sin rx$, which we have shown to be orthogonal relative to a certain functional operation in $(-\infty, \infty)$, presents the difficulty that it cannot be normalized, since

$$\int_{-\infty}^{\infty} \cos^2 rx\, dx = \infty, \quad \int_{-\infty}^{\infty} \sin^2 rx\, dx = \infty.$$

This effectively prevents us from applying directly most of the arguments which occur in the theory of the series, and what happens is that, as will be seen, while many of the properties of the integral can be proved along lines similar to those followed in the case of the series, the actual proof requires, as a rule, an extra passage to the limit which is not required in the case of the series.

Indeed, this is already apparent, since the orthogonal properties just obtained require the passage to the limit $\lambda \to \infty$, and this does not occur with the orthogonal system associated with the series.

3. In order to overcome the difficulty introduced by the fact that the range of integration is infinite we may divide functions into two classes. In class **F** we place all functions which vanish for all sufficiently large values of their argument, so that $f(x)$ belongs to **F** if there are finite numbers a, b such that $f(x)$ differs from zero only in the interval (a, b). In class **G** we place all the other functions.

The convenience of this classification is due to the fact that any function of the kind considered in our theory can be exhibited in the form

$f(x)+g(x)$, where $f(x)$ belongs to **F**, $g(x)$ to **G**, and, in addition, $g(x)$ vanishes in the neighbourhood of any particular point ξ given in advance at which we may be investigating the behaviour of Fourier's integral.

We may say, roughly, that the artifices which apply to Fourier's series apply equally well to Fourier's integrals of functions of class **F**.

Functions of class **G** we consider as limits of functions of class **F**.

4. Suppose that $f(x)$, $\phi(x)$ are any two members of **F**, and (a, b) is any finite interval outside which they vanish. Suppose further that

$$F^p = \int_a^b |f(x)|^p \, dx < \infty, \tag{4.11}$$

$$\Phi^q = \int_a^b |\phi(x)|^q \, dx < \infty, \tag{4.12}$$

where p and q are positive numbers such that

$$\frac{1}{p}+\frac{1}{q} = 1.$$

By the extended Schwarz inequality,

$$\int_a^b |f(x)|\,dx \leqslant \left\{\int_a^b |f(x)|^p \, dx\right\}^{1/p} \left\{\int_a^b 1^q \, dx\right\}^{1/q} = F(b-a)^{1/q} < \infty.$$

$$\int_a^b |\phi(x)|\,dx \leqslant \left\{\int_a^b |\phi(x)|^q \, dx\right\}^{1/q} \left\{\int_a^b 1^p \, dx\right\}^{1/p} = \Phi(b-a)^{1/p} < \infty.$$

Thus, regard being paid to the special property of $f(x)$ and $\phi(x)$,

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty, \quad \int_{-\infty}^{\infty} |\phi(x)|\,dx < \infty,$$

and so plainly the integrals

$$a_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos sx \, dx, \quad b_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin sx \, dx,$$

and the corresponding integrals α_s, β_s associated with $\phi(x)$ converge absolutely and have finite values.

Write now $$F(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\,\phi(x)\,dx.$$

By a theorem of Young*, $F(t)$ is a continuous function of t, since the range of integration is really the finite interval (a, b), although for convenience we write it as $(-\infty, \infty)$,

It is also plain that $f(x+t)\,\phi(x) = 0$

whenever $|t| > b-a$, so that $F(t)$ is also a function of class **F**.

Further, as is easily seen, the integral being really that of the continuous function $F(x)$ in the finite range $(a-b,\ b-a)$,

$$\int_{-\infty}^{\infty} |F(t)|\,dt < \infty,$$

so that the integrals

$$A_s = \frac{1}{\pi}\int_{-\infty}^{\infty} F(t)\cos st\,dt, \quad B_s = \frac{1}{\pi}\int_{-\infty}^{\infty} F(t)\sin st\,dt$$

converge absolutely and have finite values.

These integrals we can evaluate by Young's method.

For the value of the repeated integral

$$\int_{-\infty}^{\infty} |\cos st|\,dt \int_{-\infty}^{\infty} |f(x+t)\,\phi(x)|\,dx$$

is finite, being not greater than $2(b-a)F\Phi$, since

$$\int_{-\infty}^{\infty} |f(x+t)\,\phi(x)|\,dx \leqslant \left\{\int_{-\infty}^{\infty} |f(x+t)|^p\,dx\right\}^{1/p}\left\{\int_{-\infty}^{\infty} |\phi(x)|^q\,dx\right\}^{1/q} = F\Phi,$$

and we may therefore change the order of integration in the integral for A_s, obtaining

$$A_s = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\cos st\,dt\int_{-\infty}^{\infty} f(x+t)\,\phi(x)\,dx$$

$$= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\phi(x)\,dx\int_{-\infty}^{\infty} f(x+t)\cos st\,dt.$$

Similarly

$$B_s = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\phi(x)\,dx\int_{-\infty}^{\infty} f(x+t)\sin st\,dt.$$

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* W. H. Young, "On a class of parametric integrals and their applications in the theory of Fourier series", *Proc. Royal Soc.* (A), 85 (1911), 401–414 (407).

Now

$$\int_{-\infty}^{\infty} f(x+t)\cos st\,dt = \int_{-\infty}^{\infty} f(t)\cos s(t-x)\,dt$$

$$= \cos sx \int_{-\infty}^{\infty} f(t)\cos st\,dt + \sin sx \int_{-\infty}^{\infty} f(t)\sin st\,dt$$

$$= \pi(a_s \cos sx + b_s \sin sx).$$

Similarly

$$\int_{-\infty}^{\infty} f(x+t)\sin st\,dt = \pi(b_s \cos sx - a_s \sin sx).$$

These values, substituted above, yield

$$A_s = a_s\alpha_s + b_s\beta_s, \quad B_s = b_s\alpha_s - a_s\beta_s,$$

so that

$$F(t) \leqslant \int_0^{\infty} \{(a_s\alpha_s + b_s\beta_s)\cos st + (b_s\alpha_s - a_s\beta_s)\sin st\}\,ds,$$

the "correspondence sign" $\leqslant$ being employed to denote that the integral on the right is the Fourier's integral of the function on the left.

Since $F(t)$ is continuous, the integral in question is summable to $F(t)$ by first order Cesàro means, *i.e.*

$$F(t) = \lim_{\lambda\to\infty} \frac{1}{\lambda}\int_0^{\lambda} d\sigma \int_0^{\sigma} \{(a_s\alpha_s + b_s\beta_s)\cos st + (b_s\alpha_s - a_s\beta_s)\sin st\}\,ds,$$

or, what is equivalent,

$$F(t) = \lim_{\lambda\to\infty} \int_0^{\lambda} \left(1 - \frac{s}{\lambda}\right)\{(a_s\alpha_s + b_s\beta_s)\cos st + (b_s\alpha_s - a_s\beta_s)\sin st\}\,ds.$$

5. Consider now the case $p = q = 2$, $\phi(x) = f(x)$. The above yields

$$(5.1) \qquad \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)f(x)\,dx = \lim_{\lambda\to\infty} \int_0^{\lambda} \left(1 - \frac{s}{\lambda}\right)(a_s^2 + b_s^2)\cos st\,ds.$$

With $t = 0$ we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f^2(x)\,dx = \lim_{\lambda\to\infty} \int_0^{\lambda} \left(1 - \frac{s}{\lambda}\right)(a_s^2 + b_s^2)\,ds = \lim_{\lambda\to\infty} \int^{\infty} \psi(\lambda, s)\,ds,$$

where
$$\psi(\lambda, s) = \left(1-\frac{s}{\lambda}\right)(a_s^2+b_s^2) \quad (0 \leqslant s < \lambda),$$
$$= 0 \qquad\qquad (s \geqslant \lambda).$$

It is plain that $\psi(\lambda, s)$ increases with λ. Thus we may make the passage to the limit under the sign of integration, obtaining

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f^2(x)\,dx = \int_0^{\infty}(a_s^2+b_s^2)\,ds. \tag{5.2}$$

6. Let us abandon the restriction that $f(x)$ belongs to **F**, assuming only

$$\int_{-\infty}^{\infty} f^2(x)\,dx < \infty, \tag{6.1}$$

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty. \tag{6.2}$$

The coefficients a_s, b_s exist as before. Further, if we take

$$f_n(x) = f(x) \quad (-n < x < n)$$
$$= 0 \qquad (x \leqslant -n,\ x \geqslant n),$$

so that $f_n(x)$ belongs to **F**, then, as $n \to \infty$,

$$f_n(x) \to f(x),$$

and, in addition, $\qquad a_{n,s} \to a_s, \quad b_{n,s} \to b_s,$

where

$$a_{n,s} = \frac{1}{\pi}\int_{-\infty}^{\infty} f_n(x)\cos sx\,dx, \quad b_{n,s} = \frac{1}{\pi}\int_{-\infty}^{\infty} f_n(x)\sin sx\,dx,$$

since

$$|a_{n,s}-a_s|, \quad |b_{n,s}-b_s| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} |f_n(x)-f(x)|\,dx$$
$$= \frac{1}{\pi}\int_{-\infty}^{-n} |f(x)|\,dx + \frac{1}{\pi}\int_n^{\infty} |f(x)|\,dx \to 0.$$

By (5 . 2) $$\frac{1}{\pi}\int_{-\infty}^{\infty} f_n^2(x)\,dx = \int_0^{\infty} (a_{n,s}^2+b_{n,s}^2)\,ds,$$

i.e.

$$\frac{1}{\pi}\int_{-n}^{n} f^2(x)\,dx = \int_0^{\infty} (a_{n,s}^2+b_{n,s}^2)\,ds. \tag{6 . 3}$$

Making $n\to\infty$, we have by a known theorem (since the integrand on the right is never negative)

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f^2(x)\,dx \geqslant \int_0^{\infty} (a_s^2+b_s^2)\,ds.$$

In this inequality replace $f(x)$ by $f(x)-f_n(x)$. We get

$$\frac{1}{\pi}\int_{-\infty}^{-n} f^2(x)\,dx+\frac{1}{\pi}\int_{n}^{\infty} f^2(x)\,dx \geqslant \int_0^{\infty} \{a_s-a_{n,s})^2+(b_s-b_{n,s})^2\}\,ds,$$

and so, as $n\to\infty$,

$$\int_0^{\infty} \{(a_s-a_{n,s})^2+(b_s-b_{n,s})^2\}\,ds \to 0.$$

By Schwarz's inequality,

$$\left\{\int_0^{\infty}(a_s^2-a_{n,s}^2)\,ds\right\}^2 \leqslant \int_0^{\infty}(a_s-a_{n,s})^2\,ds \times \int_0^{\infty}(a_s+a_{n,s})^2\,ds$$

$$\leqslant \int_0^{\infty}(a_s-a_{n,s})^2\,ds \times \frac{4}{\pi}\int_{-\infty}^{\infty} f^2(x)\,dx,$$

since $$(a_s+a_{n,s})^2 \leqslant 2a_s^2+2a_{n,s}^2.$$

Thus $$\int_0^{\infty}(a_s^2-a_{n,s}^2)\,ds \to 0,$$

i.e. $$\int_0^{\infty} a_{n,s}^2\,ds \to \int_0^{\infty} a_s^2\,ds.$$

Similarly $$\int_0^{\infty} b_{n,s}^2\,ds \to \int_0^{\infty} b_s^2\,ds.$$

Thus (6 . 3) yields, as $n\to\infty$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f^2(x)\,dx = \int_0^{\infty} (a_s^2+b_s^2)\,ds. \tag{6 . 4}$$

7. If we do not assume (6 . 2), but only (6 . 1), then a_s and b_s may not exist in the ordinary way.

This difficulty we may overcome in a variety of ways, out of which we select, for the sake of illustration, the following.

Write $$|x| = X,$$

and take $$\omega > 0.$$

If we take $$f_\omega(x) = f(x)\, e^{-\omega X},$$

we have a function which satisfies (6 . 2) as well as (6 . 1). For

$$\left\{\int_{-\infty}^{\infty} e^{-\omega X} |f(x)|\, dx\right\}^2 \leqslant \int_{-\infty}^{\infty} e^{-2\omega X}\, dx \times \int_{-\infty}^{\infty} f^2(x)\, dx = \frac{1}{\omega}\int_{-\infty}^{\infty} f^2(x)\, dx < \infty.$$

Write

$$a_{\omega, s} = \frac{1}{\pi}\int_{-\infty}^{\infty} f_\omega(x) \cos sx\, dx, \quad b_{\omega, s} = \frac{1}{\pi}\int_{-\infty}^{\infty} f_\omega(x) \sin sx\, dx.$$

By (6 . 4)

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f_\omega^2(x)\, dx = \int_0^\infty (a_{\omega, s}^2 + b_{\omega, s}^2)\, ds. \tag{7 . 1}$$

In the same way, if ω', ω'' are any two values of ω,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \{f_{\omega'}(x) - f_{\omega''}(x)\}^2\, dx = \int_0^\infty \{(a_{\omega', s} - a_{\omega'', s})^2 + (b_{\omega', s} - b_{\omega'', s})^2\}\, ds. \tag{7 . 2}$$

Now $$\int_{-\infty}^{\infty} \{f_{\omega'}(x) - f_{\omega''}(x)\}^2\, dx = \int_{-\infty}^{\infty} f^2(x) \{e^{-\omega' X} - e^{-\omega'' X}\}^2\, dx.$$

Also $$\{e^{-\omega' X} - e^{-\omega'' X}\}^2 \leqslant 1$$

and $$\lim_{\omega', \omega'' \to 0} \{e^{-\omega' X} - e^{-\omega'' X}\} = 0.$$

Thus, in virtue of (6 . 1),

$$\lim_{\omega', \omega'' \to 0} \int_{-\infty}^{\infty} \{f_{\omega'}(x) - f_{\omega''}(x)\}^2\, dx = \int_{-\infty}^{\infty} \lim_{\omega', \omega'' \to 0} \{f_{\omega'}(x) - f_{\omega''}(x)\}^2\, dx = 0.$$

From (7 . 2) we now obtain

$$\lim_{\omega', \omega'' \to 0} \int_0^\infty (a_{\omega', s} - a_{\omega'', s})^2\, ds = 0, \quad \lim_{\omega', \omega'' \to 0} \int_0^\infty (b_{\omega', s} - b_{\omega'', s})^2\, ds = 0,$$

which show that $a_{\omega,s}$, $b_{\omega,s}$ respectively converge in the mean in $(0, \infty)$ to limits which we will call a_s, b_s. These limits are such that, as $\omega \to 0$,

$$\int_0^\infty (a_s - a_{\omega,s})^2 \, ds \to 0, \quad \int_0^\infty (b_s - b_{\omega,s})^2 \, ds \to 0.$$

Further, there exist sequences $\{\omega_r\}$ such that, as $r \to \infty$,

$$a_{\omega_r,s} \to a_s, \quad b_{\omega_r,s} \to b_s,$$

for almost all values of s in $(0, \infty)$.

Now, since $$0 < e^{-\omega_r X} \leqslant 1,$$

we have, in virtue of (6 . 1), as $r \to \infty$,

$$\int_{-\infty}^\infty f_{\omega_r}^2(x) \, dx \to \int_{-\infty}^\infty f^2(x) \, dx.$$

Also

$$\frac{1}{\pi} \int_{-\infty}^\infty f^2(x) \, dx \geqslant \int_0^\infty (a_{\omega_r,s}^2 + b_{\omega_r,s}^2) \, ds,$$

and, therefore, since the integrand is non-negative,

$$\frac{1}{\pi} \int_{-\infty}^\infty f^2(x) \, dx \geqslant \int_0^\infty \lim (a_{\omega_r,s}^2 + b_{\omega_r,s}^2) \, ds = \int_0^\infty (a_s^2 + b_s^2) \, ds.$$

The argument of § 6 now shows that

$$\int_0^\infty a_{\omega,s}^2 \, ds \to \int_0^\infty a_s^2 \, ds, \quad \int_0^\infty b_{\omega,s}^2 \, ds \to \int_0^\infty b_s^2 \, ds,$$

and, when $\omega \to 0$, (7 . 1) yields

$$\frac{1}{\pi} \int_{-\infty}^\infty f^2(x) \, dx = \int_0^\infty (a_s^2 + b_s^2) \, ds.$$

8. It is not difficult to see that the results of § 7 remain true when the convergence factor $e^{-\omega X}$ is replaced by any other factor $\phi(\omega, x)$ such that

(8 . 1) $$| \phi(\omega, x) | \leqslant 1;$$

(8 . 2) $$\phi(\omega, x) \to 1 \quad \text{as} \quad \omega \to 0;$$

(8 . 3) $$\int_{-\infty}^\infty \phi^2(\omega, x) \, dx < \infty.$$

9. The functions of class **F** have the property that the convergence and summability of their Fourier integrals are, roughly speaking, subject to exactly the same conditions as the convergence and summability of Fourier series.

Thus, let $f(x)$ be any function of **F**, vanishing outside the finite interval (a, b). Further, let $f(x)$ be integrable in the general Denjoy sense within this interval.

Then the coefficients

$$a_s = \frac{1}{\pi}\int_a^b f(t)\cos st\,dt, \quad b_s = \frac{1}{\pi}\int_a^b f(t)\sin st\,dt$$

are definite and have finite values, since $\cos st$ and $\sin st$ are functions of bounded variation.

Moreover, since, for each value of s, $\cos st$ and $\sin st$ have bounded derivatives and therefore are absolutely continuous, we may integrate by parts, obtaining

$$a_s = \frac{1}{\pi}[F(t)\cos st]_a^b + \frac{s}{\pi}\int_a^b F(t)\sin st\,dt$$

$$= \frac{1}{\pi}F(b)\cos sb + \frac{s}{\pi}\int_a^b F(t)\sin st\,dt, \tag{9.1}$$

and, similarly,

$$b_s = \frac{1}{\pi}F(b)\sin sb - \frac{s}{\pi}\int_a^b F(t)\cos st\,dt, \tag{9.2}$$

where

$$F(t) = \int_a^t f(u)\,du.$$

Write

$$a_s' = \frac{1}{\pi}F(b)\cos sb, \quad a_s'' = \frac{s}{\pi}\int_a^b F(t)\sin st\,dt,$$

$$\beta_s' = \frac{1}{\pi}F(b)\sin sb, \quad \beta_s'' = -\frac{s}{\pi}\int_a^b F(t)\cos st\,dt.$$

Then

$$a_s = a_s' + a_s'', \quad b_s = \beta_s' + \beta_s'',$$

and

$$f(x) \backsim \int_0^\infty (a\cos sx + \beta_s'\sin sx)\,ds + \int_0^\infty (a_s''\cos sx + \beta_s''\sin sx)\,ds = I' + I'',$$

say.

By the argument of § 2 we find that, if $a < x < b$,

$$I' = 0, \tag{C 1}$$

i.e. as $\lambda \to \infty$,

$$\frac{1}{\lambda}\int_0^\lambda d\sigma \int_0^\sigma (a_s' \cos sx + \beta_s' \sin sx)\, ds \to 0.$$

The integral I'' we attempt to sum by Hölder means. We have

$$\frac{1}{\lambda}\int_0^\lambda d\sigma \int_0^\sigma (a_s'' \cos sx + \beta_s'' \sin sx)\, ds$$

$$= \frac{1}{\pi\lambda}\int_0^\lambda d\sigma \int_0^\sigma ds \int_a^b F(t)(s \sin st \cos sx - s \cos st \sin sx)\, dt$$

$$= \frac{1}{\pi\lambda}\int_0^\lambda d\sigma \int_0^\sigma ds \int_a^b F(t)\, s \sin s(t-x)\, dt$$

$$= \frac{1}{\pi\lambda}\int_a^b F(t)\, dt \int_0^\lambda d\sigma \int_0^\sigma s \sin s(t-x)\, ds,$$

since, $F(x)$ being continuous and therefore bounded, the integral remains finite when the integrand is replaced by its absolute value,

$$= \frac{1}{\pi}\int_a^b F(t) \left\{ -\frac{\sin \lambda(t-x)}{(t-x)^2} + \frac{2}{\lambda}\,\frac{1-\cos\lambda(t-x)}{(t-x)^3} \right\} dt.$$

It is convenient to consider this as a sum of two terms. The second is

$$\frac{4}{\pi\lambda}\int_a^b F(t)\,\frac{\sin^2 \tfrac{1}{2}\lambda(t-x)}{(t-x)^3}\, dt = \frac{4}{\pi\lambda}\int_{a-x}^{b-x} F(x+t)\,\frac{\sin^2 \tfrac{1}{2}\lambda t}{t^3}\, dt.$$

We suppose that x is interior to the interval (a, b). Then, for sufficiently small values of ϵ, so are $x-\epsilon$ and $x+\epsilon$.

Plainly

$$\left| \frac{1}{\lambda}\int_\epsilon^{b-x} F(x+t)\,\frac{\sin^2 \tfrac{1}{2}\lambda t}{t^3}\, dt \right| \leqslant \frac{1}{\lambda\epsilon^3}\int_\epsilon^{b-x} |F(x+t)|\, dt \to 0,$$

$$\left| \frac{1}{\lambda}\int_{a-x}^{-\epsilon} F(x+t)\,\frac{\sin^2 \tfrac{1}{2}\lambda t}{t^3}\, dt \right| \leqslant \frac{1}{\lambda\epsilon^3}\int_{a-x}^{\epsilon} |F(x+t)|\, dt \to 0,$$

as $\lambda \to \infty$, while

$$\frac{4}{\pi\lambda}\int_{-\epsilon}^{\epsilon} F(x+t)\,\frac{\sin^2 \tfrac{1}{2}\lambda t}{t^3}\, dt = \frac{1}{\pi\lambda}\int^{\epsilon} \frac{F(x+t)-F(x-t)}{t}\,\frac{\sin^2 \tfrac{1}{2}\lambda t}{(\tfrac{1}{2}t)^2}\, dt.$$

This last integral is well known. As $\lambda \to \infty$, it tends to $2l$, where

$$l = \lim_{t \to 0} \frac{F(x+t)-F(x-t)}{2t},$$

whenever this is finite.

The other term which we have to consider may be written as

$$-\frac{1}{\pi} \int_{a-x}^{b-x} F(t+x) \frac{\sin \lambda t}{t^2}\, dt.$$

By the theorem of Riemann-Lebesgue,

$$\int_{\epsilon}^{b-x} \frac{F(t+x)}{t^2} \sin \lambda t\, dt \to 0, \quad \int_{a-x}^{-\epsilon} \frac{F(t+x)}{t^2} \sin \lambda t\, dt \to 0,$$

as $\lambda \to \infty$, while

$$-\frac{1}{\pi} \int_{-\epsilon}^{\epsilon} F(t+x) \frac{\sin \lambda t}{t^2}\, dt = -\frac{1}{\pi} \int_0^{\epsilon} \frac{F(x+t)-F(x-t)}{t} \frac{\sin \lambda t}{t}\, dt.$$

This, again, is a well known expression. It converges by Cesàro means of all positive orders to $-l$ whenever l exists and is finite.

Gathering together the various results obtained above and taking into account the consistency theorems relating to summation by Cesàro means, we find that

$$\int_0^{\infty} (a_s \cos sx + b_s \sin sx)\, ds = 2l - l = l \quad (H, 1+\epsilon)$$

whenever l exists finitely. In other words :

The Fourier's integral of a function which is integrable in the general Denjoy sense in some finite interval (a, b) and vanishes outside this interval is summable $(H, 1+\epsilon)$ for every positive value of ϵ to the symmetrical first derivative of the indefinite integral of the function at every point interior to (a, b) at which this symmetrical first derivative exists and is finite.

For summability to $f(x)$ itself at these points we must have

$$\frac{F(x+t)-F(x-t)}{2t} \to f(x),$$

i.e.

$$\frac{1}{2t} \int_{x-t}^{x+t} f(u)\, du \to f(x)$$

or

$$\frac{1}{2t} \int_{-t}^{t} f(x+u)\, du \to f(x),$$

which is equivalent to saying that the function is, in Hardy's phrase, continuous in mean at x.

It is plain that this is the case wherever $F'(x) = f(x)$. When $f(x)$ is integrable in the *special* Denjoy sense this is known to hold almost everywhere in (a, b). Thus, when $f(x)$ is integrable in the special sense,

$$\int_0^\infty (a_s \cos sx + b_s \sin sx)\, ds = f(x) \quad (H, 1+\epsilon)$$

almost everywhere in (a, b), which we may express by writing

$$\int_0^\infty (a_s \cos sx + b_s \sin sx)\, ds \sim f(x) \quad (H, 1+\epsilon).$$

In virtue of a theorem of Hardy's* we may replace

$$\frac{F(x+t)-F(x-t)}{2t} \to l$$

by

$$\frac{1}{\theta}\int_0^\theta \left| \frac{F(x+t)-F(x-t)}{2t} - l \right| dt \to 0$$

as $\theta \to 0$, which may be rewritten as

$$\frac{1}{\theta}\int_0^\theta \frac{1}{2t} \left| \int_{-t}^{t} \{f(x+u)-l\}\, du \right| \to 0,$$

and is, theoretically at any rate, more general than the above.

* G H. Hardy, "On the summability of Fourier series", *Proc. London Math. Soc.* (2), 12 (1913), 365–372 (365).

ON PARSEVAL'S THEOREM FOR DIRICHLET SERIES

By A. S. BESICOVITCH.

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PARSEVAL'S theorem for Dirichlet series was proved by Carlson* in the following form :

Let $f(s)$ be an analytic function of $s = \sigma + it$ represented by the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \quad (0 < \lambda_1 < \lambda_2 < \ldots ;\ \lambda_n \to +\infty)$$

in its region of convergence. Let us write

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |f(\sigma+it)|^2 \, dt = k(\sigma)$$

and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |f(\sigma+it)|^2 dt = M\{|f(\sigma+it)|^2\} \tag{1}$$

when this limit exists. We shall call the number $M\{|f(\sigma_1+it)|^2\}$ the mean value of the function $f(s)$ for $\sigma = \sigma_1$. Then the theorem may be stated as follows :

Let us suppose that for all $\sigma > \sigma_0$

(i) $k(\sigma)$ *has a finite value,*

(ii) $$|f(\sigma+it)| < C(|t|+1)^m \quad (-\infty < t < +\infty)$$

* F. Carlson, "Contributions à la théorie des séries de Dirichlet", *Arkiv för Math.*, 16 (1922), No. 18.

where C, m are some positive numbers. Then

$$M\{|f(\sigma+it)|^2\} = \Sigma |a_n|^2 e^{-2\lambda_n \sigma} \tag{2}$$

for any $\sigma > \sigma_0$.

The question may now be raised: if (i), (ii) are true not only for $\sigma > \sigma_0$, but even for $\sigma \geqslant \sigma_0$, is it necessarily true that (2) also holds for $\sigma \geqslant \sigma_0$?

In this paper I shall show that neither (i), (ii) (for $\sigma \geqslant \sigma_0$), nor even some much stronger conditions, are sufficient to ensure the truth of (2). I shall, in fact, construct a Dirichlet series

$$f(s) = \Sigma a_n e^{-\lambda_n s},$$

bounded for $\sigma \geqslant \sigma_0$ and even possessing a mean value $M\{|f(\sigma_0+it)|^2\}$, which, however, does not satisfy the equation (2).

Let $\chi(t)$ be a positive continuous function of bounded variation, periodic and with period 1, and let

$$\chi(0) = \chi(1) = 0, \qquad \chi(t) > 0 \quad (0 < t < 1),$$

$$\int_0^1 [\chi(t)]^2 dt = 1. \tag{3}$$

In accordance with (1) we can write

$$M\{|F(t)|^2\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |F(t)|^2 dt$$

for any function $F(t)$, and call the number $M\{|F(t)|^2\}$ the mean value of the function $F(t)$. A periodic function evidently possesses a mean value and it is easy to see from (3) that

$$M\{|\chi(t)|^2\} = 1. \tag{4}$$

For any positive integer l we next define the periodic function $\phi(t, l)$, with period l, in the following way:

$$\phi(t, l) = \chi(t) \quad (0 \leqslant t \leqslant 1),$$

$$= 0 \qquad (1 < t < l).$$

Now let $$\Phi(t) = \phi(t, 2^{k_1}) + \phi(t-a_2, 2^{k_2}) + \phi(t-a_3, 2^{k_3}) + \ldots, \tag{5}$$

where $2 \leqslant k_1 < k_2 < k_3 \ldots$ are positive integers, and a_2 is the nearest point to the origin (or one of the two nearest points) for which $\phi(t, 2^{k_1})$ is zero throughout the interval (a_2, a_2+1); a_3 is the nearest point to the origin (or one of the two nearest points) for which the functions $\phi(t, 2^{k_1})$ and $\phi(t-a_2, 2^{k_2})$ are both zero throughout the interval (a_3, a_3+1), and so on. Writing for shortness

$$\phi(t, 2^{k_1}) = \phi_1(t), \quad \phi(t-a_2, 2^{k_2}) = \phi_2(t), \quad \ldots$$

we have
$$\Phi(t) = \phi_1(t)+\phi_2(t)+\ldots. \tag{6}$$

We shall now show that

$$\Phi(t) = \chi(t). \tag{7}$$

Let us first observe that the function ϕ_j differs from zero only in the intervals

$$(a_j+n2^{k_j},\ a_j+n2^{k_j}+1) \quad (n \text{ is any integer}). \tag{8}$$

The functions $\phi_1, \phi_2, \ldots, \phi_{j-1}$ are all identically zero in the interval (a_j, a_j+1), and so, since they have period 2^{k_j}, they are identically zero in all the intervals (8). Thus, in any interval $(a, a+1)$ (a integer) where ϕ_j differs from zero, all the preceding functions $\phi_1, \phi_2, \ldots, \phi_{j-1}$ are identically zero, and consequently in any interval $(a, a+1)$ (a integer) only one term of the series (6) can differ from zero. For, if one of $\phi_{j+1}, \phi_{j+2}, \ldots$ were not identically zero in $(a, a+1)$, then, by what we have just proved, ϕ_j would be identically zero in $(a, a+1)$. Now it is easy to see that there always is such a term. For, for every j, the interval (a_j, a_j+1) is the nearest to the origin (or one of the two nearest) throughout which all the functions $\phi_1, \phi_2, \ldots, \phi_{j-1}$ are zero. Therefore if all the terms of the series (6) were zero throughout the interval $(a, a+1)$, all the intervals (a_j, a_j+1) would be at least as near to the origin as $(a, a+1)$, which is evidently impossible, since all the intervals (a_j, a_j+1) are different. Now from the definition of the function $\phi(t, l)$ it is easy to see that in every interval $(a, a+1)$ (a integer) the function is equal throughout the interval to zero or to $\chi(t)$. The same is true for all the functions ϕ_j. Since in every interval $(a, a+1)$ just one term of the series (6) differs from zero, the equation (7) follows immediately. Hence

$$M\{|\Phi(t)|^2\} = 1. \tag{9}$$

Let us consider the Fourier series of the function $\phi(t, l)$,

$$\phi(t, l) = \sum_{\nu=-\infty}^{+\infty} b_\nu e^{(2\pi/l)\nu ti} \tag{10}$$

where $$b_\nu = \frac{1}{l}\int_0^l \phi(u, l)\, e^{-(2\pi/l)\nu ui}\, du.$$

Let t be some point outside the set of intervals of values t' of t for which $\phi(t', l) = \chi(t')$ (the isolated roots of this equation are not included in the set), and d the distance from this set to the point t. Also let n_0 and n be any positive integers. We shall have

$$\sum_{\nu=n_0}^{n_0+n-1} b_\nu e^{(2\pi/l)\nu ti} = \frac{1}{l}\int_0^l \phi(u, l)\, e^{-(\pi/l)(2n_0+n-1)(u-t)i}\, \frac{\sin(\pi/l)\, n(t-u)}{\sin(\pi/l)(t-u)}\, du,$$

$$\left|\sum_{\nu=n_0}^{n_0+n-1} b_\nu e^{(2\pi/l)\nu ti}\right| < \frac{1}{l}\int_0^l \phi(u, l)\, \frac{1}{|\sin(\pi/l)(t-u)|}\, du$$

$$= \frac{1}{l}\int_0^1 \chi(u)\, \frac{1}{|\sin(\pi/l)(t-u)|}\, du$$

$$= \frac{1}{l|\sin(\pi/l)(t-u_0)|}\int_0^1 \chi(u)\, du \quad (0 < u_0 < 1).$$

But from Schwarz's inequality we have

$$\int_0^1 \chi(u)\, du \leqslant \sqrt{\left\{\int_0^1 [\chi(u)]^2\, du\right\}} \leqslant 1,$$

and from our assumption about t

$$\left|\sin\frac{\pi}{l}(t-u_0)\right| > \frac{2d}{l};$$

consequently $$\left|\sum_{\nu=n_0}^{n_0+n-1} b_\nu e^{(2\pi/l)\nu ti}\right| < \frac{1}{2d}. \tag{11}$$

Thus (11) gives an upper limit for the modulus of the sum of any number of consecutive terms of the series (10). We shall use (11) later on.

Now suppose

$$\phi_j(t) = \sum_{\nu=-\infty}^{+\infty} b_{\nu,j}\, e^{\nu(2\pi/2^{kj})ti} = s_j(ti) + r_j(ti), \tag{12}$$

where $$s_j(ti) = \sum_{\nu=-n}^{+n_j} b_{\nu,j}\, e^{\nu(2\pi/2^{kj})ti}, \tag{13}$$

and n_j is a positive integer chosen large enough to make

$$|r_j(ti)| < \frac{\epsilon}{2^j}, \tag{14}$$

ϵ being some small positive number: this is evidently possible, since all the series (12) are uniformly convergent. Thus we shall have

$$|s_j(ti)| < \frac{\epsilon}{2^j} \tag{15}$$

for all values of t which satisfy the equation $\phi_j(t) = 0$.

Let us suppose the terms in each of the sums s_j arranged in order of decreasing exponents (more precisely, of decreasing coefficients of ti in the exponents). We can choose μ_1, a positive multiple of $2\pi/2^{k_1}$, so that all the exponents of the product $s_1(ti)\,e^{-\mu_1 ti}$ are negative. We can then choose μ_2, a positive multiple of $2\pi/2^{k_2}$, so that all the exponents of $s_2(ti)\,e^{-\mu_2 ti}$ are negative and smaller than any of the exponents of the first product, and so on. We may suppose further that $\mu_n \to \infty$ as $n \to \infty$. The products

$$s_1(ti)\,e^{-\mu_1 ti}, \quad s_2(ti)\,e^{-\mu_2 ti}, \quad \ldots$$

have the same periods as the functions ϕ_1, ϕ_2, ... respectively.

Let us now consider the series

$$\Psi(t) = b_1 s_1(ti)\,e^{-\mu_1 ti} + b_2 s_2(ti)\,e^{-\mu_2 ti} + \ldots, \tag{16}$$

where b_1, b_2, ... are positive constants whose values will now be chosen. Since we have

$$\begin{aligned}\int_0^1 |s_1(ti)\,e^{-\mu_1 ti}|^2\,dt &= \int_0^1 |s_1(ti)|^2\,dt = \int_0^1 |\phi_1(t) - r_1(ti)|^2\,dt \\ &= \int_0^1 |\phi_1(t)|^2\,dt + \theta_1' \int_0^1 \{|2r_1(ti)\,\phi_1(t)| + |r_1(ti)|^2\}\,dt \\ &= 1 + \tfrac{3}{2}\theta_1 \epsilon \quad (|\theta_1'| < 1,\ |\theta_1| < 1),\end{aligned}$$

we can choose the positive number b_1 so that

$$\int_0^1 |b_1 s_1(ti)\,e^{-\mu_1 ti}|^2\,dt = 1; \tag{17}$$

b_1 is evidently near to 1 and *a fortiori* less than 2. In the same way we have

$$\int_{a_2}^{a_2+1} |\, s_2(ti)\, e^{-\mu_2 ti} \,|^2 dt = 1+\tfrac{3}{4}\theta_2\epsilon,$$

$$\int_{a_3}^{a_3+1} |\, s_3(ti)\, e^{-\mu_3 ti} \,|^2 dt = 1+\tfrac{3}{8}\theta_3\epsilon,$$

... ...

and we can thus choose the positive numbers b_2, b_3, ... so that

$$\int_{a_2}^{a_2+1} |\, b_1 s_1(ti)\, e^{-\mu_1 ti}+b_2 s_2(ti)\, e^{-\mu_2 ti} \,|^2 dt = 1, \tag{18}$$

$$\int_{a_3}^{a_3+1} |\, b_1 s_1(ti)\, e^{-\mu_1 ti}+b_2 s_2(ti)\, e^{-\mu_2 ti}+b_3 s_3(ti)\, e^{-\mu_3 ti} \,|^2 dt = 1, \tag{19}$$

... ...

Observing that, in the integral (18),

$$|\, s_1(ti)\,| < \frac{\epsilon}{2},$$

in the integral (19),

$$|\, s_1(ti)\,| < \frac{\epsilon}{2}, \quad |\, s_2(ti)\,| < \frac{\epsilon}{4},$$

and so on, we conclude that when ϵ is small all the numbers b_2, b_3, ... are near to 1 and *a fortiori* less than 2.

The series (16) converges at every point of the whole interval $-\infty < t < +\infty$. For at every point t only one of the functions ϕ_1, ϕ_2, ... can differ from zero, and therefore for all values of j, except perhaps one, $j = j'$, the inequality (15) holds, and consequently

$$|\, \Psi(t)\,| \leqslant |\, b_1 s_1(ti)\,| + |\, b_2 s_2(ti)\,| + \ldots$$

$$< b_{j'} |\, s_{j'}(ti)\,| + 2\epsilon.$$

Thus we see that the series (16) is convergent and represents a bounded function. If we now replace each term of the series (16) by its explicit expression as the sum of a finite number of exponential functions, we

obtain a series of exponentials, whose arguments are ti multiplied by a sequence of decreasing constants tending to $-\infty$:

$$\Psi(t) = c_1 e^{-\lambda_1 ti} + c_2 e^{-\lambda_2 ti} + \dots \tag{20}$$

$$(0 < \lambda_1 < \lambda_2 < \dots ; \ \lambda_n \to \infty).$$

We shall next show that the series (20) is also convergent.

Let t_0 be any fixed value of t and let

$$\Psi(t) = \sigma_n(t) + \rho_n(t),$$

where

$$\sigma_n(t) = \sum_{j=1}^{n} c_j e^{-\lambda_j ti}.$$

For any n we have

$$\sigma_n(t) = \sum_{j=1}^{\nu-1} b_j s_j(ti)\, e^{-\mu_j ti} + b_\nu s'_\nu(ti)\, e^{-\mu_\nu ti}, \tag{21}$$

$$\rho_n(t) = b_\nu s''_\nu(ti)\, e^{-\mu_\nu ti} + \sum_{j=\nu+1}^{\infty} b_j s_j(ti)\, e^{-\mu_j ti}, \tag{22}$$

where $s'_\nu(ti)$ is the sum of the terms of $s_\nu(t)$ up to and including some particular term, and $s''_\nu(ti)$ is the sum of the remaining terms (if any) of $s_\nu(ti)$. The series (16) being convergent, the $\rho_n(t)$ of (22) has a definite value for every t, and to prove the convergence of the series (20) it is sufficient to show that to any given pair of numbers t_0, η_0 (η_0 positive) corresponds a number n_0 such that $|\rho_n(t_0)| < \eta_0$ for $n \geqslant n_0$. Let d be a positive number such that

$$\frac{1}{d} < \frac{\eta_0}{2}. \tag{23}$$

Let, as before, (a_j, a_j+1) be the interval nearest to the origin (or one of the intervals) in which $\phi_j(t)$ differs from zero. We know that $|a_j| \to \infty$ as $j \to \infty$. Therefore there exists a number ν_0 such that $\epsilon/2^{\nu_0-1} < \eta/2$ and that, for $\nu \geqslant \nu_0$, $\phi_\nu(t) = 0$ throughout the interval (t_0-d, t_0+d). Let the first term of $b_{\nu_0} s_{\nu_0}(ti)\, e^{-\mu_{\nu_0} ti}$ have the suffix n_0 in the series (20). Then for $n \geqslant n_0$ we shall have $|\rho_n(t_0)| < \eta_0$. For in the first place from $n \geqslant n_0$ we have $\nu \geqslant \nu_0$ and therefore $\phi_\nu(t) = 0$ in the interval (t_0-d, t_0+d), and from (11) we have

$$|s''_\nu(t_0 i)| < \frac{1}{2d},$$

and consequently

$$|\, b_\nu s''_\nu(t_0 i)\, e^{-\mu_\nu t_0 i}\,| < \frac{1}{d} < \frac{\eta_0}{2}$$

by (23). Secondly we have from (15), since $\phi_j(t_0) = 0$,

$$\left| \sum_{j=\nu+1}^{\infty} b_j s_j(ti)\, e^{-\mu_j t i} \right| < \frac{\epsilon}{2^{\nu-1}} \leqslant \frac{\epsilon}{2^{\nu_0-1}} < \frac{\eta_0}{2}\,.$$

Hence $|\,\rho_n(t_0)\,| < \eta_0$. Thus the series (20) is convergent.

We shall now show that the mean value $M\{|\,\Psi(t)\,|^2\}$ exists. We can represent the function $\Psi(t)$ as the sum of two functions:

$$\Psi(t) = \Psi_1(t) + \Psi_2(t),$$

where $\Psi_1(t)$ is defined by

$$\Psi_1(t) = \sum_{j=1}^{n} b_j s_j(ti)\, e^{-\mu_j t i}, \tag{24}$$

when t belongs to the intervals where

$$\sum_{j=1}^{n} \phi_j(t) = \chi(t), \tag{25}$$

and

$$\Psi_1(t) = 0 \tag{26}$$

for all other values of t. n is chosen arbitrarily and will ultimately be infinitely great. Let L_1 be the set of intervals in which (25) holds, and L_2 the complementary set of intervals. $\Psi_1(t)$ is a periodic function with 2^{k_n} for a period. For suppose $\Psi_1(t_0)$ is different from zero. Then for $t = t_0$ the equation (25) holds; but, 2^{k_n} being a period of both sides of the equation (25), it holds also for $t = t_0 + 2^{k_n}$ and consequently $\Psi_1(t_0)$ and $\Psi_1(t_0+2^{k_n})$ are both defined by (24) and are equal, since the expression of the right-hand side of (24) has 2^{k_n} for a period. If $\Psi_1(t_0) = 0$, then neither t_0 nor $t_0+2^{k_n}$ satisfy (25), and thus $\Psi_1(t_0) = \Psi_1(t_0+2^{k_n}) = 0$.

Let us now consider the function $\Psi_2(t)$. In the intervals L_1 we have $\phi_j(t) = 0$ for $j > n$, and therefore, from (15),

$$|\,\Psi_2(t)\,| = \left| \sum_{j=n+1}^{\infty} b_j s_j(ti)\, e^{-\mu_j t i} \right| < \frac{\epsilon}{2^{n-1}}.$$

Now let $(k, k+1)$ (k integer) be any interval of L_2, and let $\phi_l(t)$ $(l > n)$

be the function which differs from zero in this interval. Then

$$\int_k^{k+1} |\, b_1 s_1(ti)\, e^{-\mu_1 ti} + \ldots + b_l s_l(ti)\, e^{-\mu_l ti} \,|^2 dt = 1,$$

and in the interval $(k,\ k+1)$ we have also

$$\left| \sum_{j=l+1}^{\infty} b_j s_j(ti)\, e^{-\mu_j ti} \right| < \frac{\epsilon}{2^{l-1}} \leqslant \frac{\epsilon}{2^{n-1}}.$$

We now define the function $\Omega(t)$ as follows :

$\Omega(t)$ is to be equal to $\Psi_1(t)$ in L_1. Let $(k,\ k+1)$ (k integer) be any interval of L_2 and let ϕ_l be the function which can differ from zero in this interval. We take

$$\Omega(t) = b_1 s_1(ti)\, e^{-\mu_1 ti} + \ldots + b_l s_l(ti)\, e^{-\mu_l ti}$$

for $k < t < k+1$.

Then, for all values of t, we have

$$|\, \Psi(t) - \Omega(t)\,| < \frac{\epsilon}{2^{n-1}}. \tag{27}$$

Let now $g(t)$ be the characteristic function of the set L_2. The function $g(t)$ is periodic. Plainly

$$\Omega(t) = \Psi_1(t) + g(t)\,\Omega(t)$$

and

$$|\,\Omega(t)\,|^2 = |\,\Psi_1(t)\,|^2 + g(t)\,|\,\Omega(t)\,|^2,$$

so that

$$\frac{1}{2T}\int_{-T}^{+T} |\,\Omega(t)\,|^2 dt = \frac{1}{2T}\int_{-T}^{+T} |\,\Psi_1(t)\,|^2 dt + \frac{1}{2T}\int_{-T}^{+T} g(t)\,|\,\Omega(t)\,|^2 dt. \tag{28}$$

$\Psi_1(t)$ is periodic, and hence it possesses a mean value. Therefore the first integral of the right-hand side of (28) tends to a limit as $T \to \infty$. As for the second integral, the function $g(t)\,|\,\Omega(t)\,|^2$ differs from zero only in the intervals L_2, which are disposed periodically, and the value of the integral of this function over any interval of L_2 between two consecutive integers is 1. Therefore the second integral also tends to a limit as $T \to \infty$. Thus the limit

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |\,\Omega(t)\,|^2 dt = C_n$$

exists and, as we can easily see, its value is near to 1.

The function $\Omega(t)$ being bounded, we deduce from (27) that the oscillation of the integral

$$\frac{1}{2T}\int_{-T}^{+T} |\Psi(t)|^2 dt, \tag{29}$$

as T tends to infinity, is of the order $\epsilon/2^n$ and, since n was chosen arbitrarily, we conclude that the expression (29) tends to a limit as $T \to \infty$ and that its value is near to 1.

Let us now consider the Dirichlet series

$$s(z) = b_1 s_1(z)\, e^{-\mu_1 z} + b_2 s_2(z)\, e^{-\mu_2 z} + \ldots \quad (z = \sigma + it) \tag{30}$$

which coincides with the series (16) for $\sigma = 0$, and thus for $\sigma \geqslant 0$ represents an analytic function bounded in the half-plane $\sigma \geqslant 0$ and passing continuously to its values on the boundary $\sigma = 0$. In the half-plane $\sigma > 0$ Parseval's theorem holds for this function, but it does not hold on the boundary $\sigma = 0$. For all the coefficients of the series (30) are among the numbers $b_j b_{\nu,j}$, where

$$j = 1, 2, \ldots \quad \text{and} \quad \nu = \ldots, -2, -1, 0, +1, +2, \ldots,$$

$b_j < 2$, and the $b_{\nu,j}$ are given by (12). Every function ϕ_j being periodic, we have

$$M\{|\phi_j|^2\} = \sum_{\nu=-\infty}^{\infty} |b_{\nu,j}|^2,$$

and from the definition of ϕ_j we have

$$M\{|\phi_j|^2\} = \frac{1}{2^{k_j}},$$

and consequently

$$\sum_{\nu=-\infty}^{\infty} |b_{\nu,j}|^2 = \frac{1}{2^{k_j}}.$$

Thus the sum of squares of the coefficients of (30) is less than

$$4\left(\frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \ldots\right)$$

and so, if $k_1 = 4$, $k_2 = 5$, $k_3 = 6$, ..., the sum is less than $\frac{1}{2}$.

But the mean value of (30) is near to 1; hence Parseval's theorem does not hold for $\sigma = 0$.

SOME FURTHER CONTRIBUTIONS TO THE THEORY OF NULL SERIES AND THEIR CONNEXION WITH NULL INTEGRALS

By C. Fox.

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1. *Introduction.*

In a previous paper, entitled "A Class of Null Series*, it has been shown that if $\phi(x, t)$ satisfies certain conditions, then the equation

$$\tfrac{1}{2}a + \sum_{n=1}^{\infty} \frac{\phi(x, j_n)}{J'_\nu(j_n)}\, j_n^{\nu+2m} = 0 \tag{1}$$

(where j_n is the nth zero of $J_\nu(t)$, m is a positive integer, and a in general is a constant) holds over a certain range of values of x. Special cases included in (1) are the well known results of Nielsen on Schlömilch series having zero as their sum for a whole range of values of the variable involved. The object of this paper is to apply the methods of the previous paper to the study of series such as occur in (1) for values of x outside the null range.

During the course of the investigations integrals of the type

$$\int_0^\infty \phi(x, t)\, J_\nu(at)\, t^{\nu+2m+1}\, dt \tag{2}$$

were encountered. These integrals vanish for a whole range of values of x, provided that m is a positive integer and $\phi(x, t)$ satisfies conditions identical with those required in order to ensure the truth of (1). Many special cases are well known. For example, Nicholson's† integral, the

* *Proc. London Math. Soc.* (2), 24 (1924), 479–493.

† Watson, *Theory of Bessel functions* (1922), 413.

Weber-Schafheitlin* integral, and also certain curious integrals discovered by Ramanujan†.

There is a close connexion between these integrals and the series in (1), and it seems impossible to give a complete theory of one without the other. Nielsen had actually compared his series with the Weber-Schafheitlin integral without going into the question of any connexion.

The paper will be divided into two parts: Part I dealing with the integrals and Part II with the series. I shall deal with the series of (1) only in the cases $\nu = \pm\frac{1}{2}$. The case $\nu = \frac{1}{2}$ will be dealt with first, and corresponding results for $\nu = -\frac{1}{2}$ will then be stated with the minimum of additional proof.

In addition, in Part II, whilst dealing with the case $\nu = \frac{1}{2}$, the exposition of a new type of null series having Legendre functions as their base will be given. These were discovered since the previous paper was written, the simplest example being

$$\tfrac{1}{2}P_{-\frac{1}{2}}(\cos\theta) + \sum_{n=1}^{\infty} (-1)^n P_{n-\frac{1}{2}}(\cos\theta) = 0, \tag{3}$$

provided that $0 < x\theta < \pi$ and $0 < \theta < \pi$.

The theorems which will be proved are stated in §§ 2 and 8, and special cases are considered in §§ 4, 5, and 16–20.

Part I.—The Study of Null Integrals.

2. *Statement of the problem*‡.

The type of theorem which we shall prove first is of the form

$$\int_0^\infty \phi(x, t)\, J_\nu(at)\, t^{2s}\, dt = \cos(s - \tfrac{1}{2}\nu)\pi \int_0^\infty \left\{ \frac{2}{\pi}\, \Phi(x, t)\, K_\nu(at) - F(t) \right\} t^{2s}\, dt, \tag{1}$$

$$\int_0^\infty \phi(x, t)\, Y_\nu(at)\, t^{2s}\, dt = \sin(s - \tfrac{1}{2}\nu)\pi \int_0^\infty \left\{ \frac{2}{\pi}\, \Phi(x, t)\, K_\nu(at) - F(t) \right\} t^{2s}\, dt, \tag{2}$$

where $\phi(x, t)$ is an even integral function of t satisfying certain general conditions to be stated hereafter, $\nu > -1$, and x lies in a certain range.

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* *Loc. cit.*, 398.

† S. Ramanujan, "A class of definite integrals", *Quart. J. of Math.*, 48 (1920), 294.

‡ I use the notation of Watson's *Theory of Bessel functions.*

Also $\Phi(x, t) = \phi(x, it)$ and $F(t)\, t^{2s}$ is that portion of the principal part of $2\pi^{-1}\Phi(x, t)\, K_\nu(at)\, t^{2s}$ at the origin which is necessary for the convergence of the integrals on the right of (1) and (2) at the lower limit.

When the integrals in (1) converge and $2s-\nu$ is a positive odd integer, the left-hand integral becomes one of the null integrals of which we are in search. A similar remark applies to (2) when $2s-\nu$ is an even integer.

When $\nu-2s$ is a positive odd integer, $2m+1$ say, $F(t)$ causes divergence of the integrals on the right of (1) and (2) at their upper limits. It will be shown in this case that, for the same range of values of x, (1) becomes

$$\int_0^\infty \phi(x, t)\, J_\nu(at)\, t^{\nu-2m-1}\, dt = \tfrac{1}{2}\pi i a, \tag{3}$$

where a is the coefficient of t^{-1} in the expansion of $t^{2s}F(t)$, provided, of course, that the integral on the left of (3) converges at its lower limit.

The more complicated theorem which (2) becomes when $\nu-2s$ is a positive odd integer will also be given.

In order that these integrals should converge at their upper limits, the values of ν and s must in general be restricted. It will be shown, however, that, if s is so large that the left-hand integrals of (1) and (2) diverge, they will still be summable (R, t^2, q) for some value of q, their sum (R, t^2, q) being equal to the right-hand integrals of (1) and (2) respectively. The convergence of the latter integrals, excluding the exceptional cases already mentioned, is independent of the value of s.

The actual theorems proved will be more general than those given by (1) and (2), since the case which we shall here consider will allow for $\phi(x, t)$ being regular in the t plane except at poles.

3. *Proof of the main formulae.*

The proof consists in considering the integral

$$\int_U \{\phi(x, t)\, H_\nu^{(1)}(at) - f_1(t)\}\, t^{2s}\, dt, \tag{1}$$

taken round the contour U, where $H_\nu^{(1)}(at)$ is the well known first func-

tion of Hankel*, and where

(i) U is the contour in the t plane which consists of that quadrant of the circle $|t| = R$ for which $0 \leqslant \arg t \leqslant \frac{1}{2}\pi$, together with the parts of the real and imaginary axes extending from the origin to the end points of the quadrant. The axes are indented, if necessary, and R must be chosen so that the shortest distance of U from each of the poles of $\phi(x, t)$ is not less than $\delta(\delta > 0)$.

(ii) $\phi(x, t)$ is an even function of t, meromorphic throughout the t plane, with at most a finite number of poles, and these poles simple.

It follows incidentally that $\phi(x, t)$ must be regular at the origin.

(iii) $f_1(t)$ is a function inserted to secure convergence of the integral at the origin. By definition

$$H_\nu^{(1)}(t) = \frac{J_{-\nu}(t) - e^{-\nu\pi i} J_\nu(t)}{i \sin \nu\pi},$$

and so, from the expansion of $J_\nu(t)$ in powers of t, and the remark is condition (ii) above, it can be seen that the principal part of $\phi(x, t) H_\nu^{(1)}(at) t^{2s}$ at the origin is of the form

$$t^{2s-\nu}(a_0 + a_1 t^2 + \ldots a_p t^{2p}),$$

where p is the greatest positive integer such that $\nu - 2s - 2p > 0$.

It follows then that

$$t^{2s} f_1(t) = t^{2s-\nu}(a_0 + a_1 t^2 + \ldots + a_r t^{2r}), \tag{2}$$

where r is the greatest positive integer such that $\nu - 2s - 2r \geqslant 1$.

(iv) Finally, as $R \to \infty$, it is necessary to ensure convergence of the integral (1) along the quadrant. For this it is sufficient to assume that, when $t = Re^{i\theta}$ and R is sufficiently large,

$$|\phi(x, t)| < CR^p \exp(\lambda R |\sin\theta|) \tag{3}$$

at all points whose distance from any pole of $\phi(x, t)$ exceeds $\frac{1}{2}\delta$, where C is a constant independent of θ, λ is a function of x and $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

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* Watson, *loc. cit.*, 73 *et seq.*

On putting $t = Re^{i\theta}$ in the asymptotic expansion*

$$H_\nu^{(1)}(t) \sim \left(\frac{2}{\pi t}\right)^{\frac{1}{2}} \exp\{i(t-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\}[1+O(t^{-1})]$$

which is valid when $-\pi < \arg t < 2\pi$, and combining this result with (3) above, we get

$$|\phi(x, t) H_\nu^{1)}(at) t^{2s}| < DR^{2s+p-\frac{1}{2}} \exp\{-(a-\lambda)R \sin\theta\} \tag{4}$$

whenever R is sufficiently large, D being a constant independent of θ.

Denote the quadrant of U by γ_U. It follows from Jordan's† lemma that, as $R \to \infty$,

$$\int_{\gamma_U} \phi(x, t) H_\nu^{(1)}(at) t^{2s} dt \to 0,$$

and also that the integrals of this function taken along the axes (with the neighbourhood of the origin excluded) converge, provided that either

$$\lambda < a \quad \text{and} \quad 2s+p-\tfrac{1}{2} < 0, \tag{5}$$

or

$$\lambda \leqslant a \quad \text{and} \quad 2s+p-\tfrac{1}{2} < -1. \tag{6}$$

It remains to deal with

$$\int t^{2s} f_1(t)\, dt$$

taken along γ_U and the axes, with the neighbourhood of the origin excluded.

From (2) condition (iii) above, it follows that on γ_U,

$$t^{2s} f_1(t) = O(t^{2s-\nu+2r}), \tag{7}$$

where r is the greatest positive integer such that $\nu-2s-2r \geqslant 1$.

Hence we must divide the proof into two cases :

Case I. $\nu-2s-2r > 1$ together with (5) or (6) above.

Case II. $\nu-2s-2r = 1$ together with (5) or (6) above.

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* Watson, *loc. cit.*, 198.

† Whittaker and Watson, *Modern analysis*, 115 (1920 ed.).

Thus in *Case* I we have

$$\int_{\gamma_U} t^{2s} f_1(t)\, dt \to 0, \tag{8}$$

as $R \to \infty$, whereas in *Case* II we have

$$\int_{\gamma_U} t^{2s} f_1(t)\, dt \to \tfrac{1}{2}\pi i a_r, \tag{9}$$

since, from (2) condition (iii), a_r is the coefficient of t^{-1} in the integrand on the left of (9).

3.1. *Case* I. $\nu - 2s - 2r > 1$.

Obviously in this case the integral of $t^{2s} f_1(t)$ along either axis (the neighbourhood of the origin excluded) converges, and the integral along γ_U tends to zero as $R \to \infty$, from (8), § 3.

Denote the residues of $\phi(x, t)\, H_\nu^{(1)}(at)\, t^{2s}$ at the poles of $\phi(x, t)$ inside U by R_U and at the poles on U by R'_U. Then the result of taking the integral (1) round U, after making $R \to \infty$ in the prescribed manner (condition i, § 3), is

$$\int_0^\infty \{\phi(x, t)\, H_\nu^{(1)}(at) - f_1(t)\}\, t^{2s}\, dt$$

$$= e^{(s+\frac{1}{2})\pi i} \int_0^\infty \{\phi(x, it)\, H_\nu^{(1)}(at) - f_1(it)\}\, t^{2s}\, dt + 2\pi i \Sigma R_U + \pi i \Sigma R'_U, \tag{10}$$

In the portion of the integral along the imaginary axis, t has been replaced by $te^{\frac{1}{2}\pi i}$. If $\phi(x, t)$ has poles on the axes, the principal values of these integrals must be taken.

The Σ merely denotes the sum of all the residues at poles in or on the quadrant.

Now consider

$$\int_L \{\phi(x, t)\, H_\nu^{(2)}(at) - f_2(t)\}\, t^{2s}\, dt \tag{11}$$

taken round the contour L, where $H_\nu^{(2)}(at)$ is Hankel's* second function and

(v) L denotes the contour which is the reflection of U in the real axis of the t plane.

---

* Watson, *loc. cit.*, 73.

(vi) $f_2(t)$ is a function inserted to secure convergence at the origin. Since $\phi(x, t)$ is an even function of t and

$$H_\nu^{(2)}(at) = \frac{J_{-\nu}(at) - e^{\nu\pi i} J_\nu(at)}{-i \sin \nu\pi},$$

it follows, by comparing this with $H_\nu^{(1)}(at)$ (condition iii, § 3), that, when $\nu > 0$, we must have

$$f_2(t) = -f_1(t). \tag{12}$$

Also, when $-1 < \nu < 0$, since $\phi(x, t)$ is regular at the origin, neither of the functions $f_1(t)$ or $f_2(t)$ exists. Hence we assume as our last restriction

(vii) $\nu > -1$.

The result, then, of taking (11) round L and making $R \to \infty$, is

$$\int_0^\infty \{\phi(x, t) H_\nu^{(2)}(at) + f_1(t)\} t^{2s} dt$$

$$= e^{-(s+\frac{1}{2})\pi i} \int_0^\infty \{\phi(x, -it) H_\nu^{(2)}(-ait) + f_1(-it)\} t^{2s} dt - 2\pi i \Sigma R_L - \pi i \Sigma R'_L, \tag{13}$$

where R_L and R'_L denote residues of $\phi(x, t) H_\nu^{(2)}(at) t^{2s}$ at the poles of $\phi(x, t)$ in and on L respectively. Along the imaginary axis t has been replaced by $te^{-\frac{1}{2}\pi i}$.

Equations (10) and (13) can be simplified by a slight change of notation. Let

$$\phi(x, it) = \phi(x, -it) = \Phi(x, t). \tag{14}$$

From (2) condition (iii), § 3, we have

$$ie^{\frac{1}{2}\nu\pi i} f_1(te^{\frac{1}{2}\pi i}) = F(t), \tag{15}$$

where
$$F(t) = it^{-\nu}\{a_0 - a_1 t^2 + \ldots + (-1)^r a_r t^{2r}\}. \tag{15.1}$$

Also*
$$-ie^{-\frac{1}{2}\nu\pi i} H_\nu^{(2)}(e^{-\frac{1}{2}\pi i} at) = ie^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(e^{\frac{1}{2}\pi i} at)$$

$$= \frac{2}{\pi} K_\nu(at) \tag{16}$$

* Watson, *loc. cit.*, (5) and (6), 75, and (8), 78.

with the help of which, (10) and (13) become respectively

$$\int_0^\infty \{\phi(x,t)\,H_\nu^{(1)}(at)-f_1(t)\}\,t^{2s}\,dt$$

$$= e^{(s-\frac{1}{2}\nu)\pi i}\int_0^\infty \left\{\frac{2}{\pi}\,\Phi(x,t)\,K_\nu(at)-F(t)\right\}t^{2s}\,dt+2\pi i\Sigma R_U+\pi i\Sigma R'_U, \quad (17)$$

$$\int_0^\infty \{\phi(x,t)\,H_\nu^{(2)}(at)+f_1(t)\}\,t^{2s}\,dt$$

$$= e^{-(s-\frac{1}{2}\nu)\pi i}\int_0^\infty \left\{\frac{2}{\pi}\,\Phi(x,t)\,K_\nu(at)-F(t)\right\}t^{2s}\,dt-2\pi i\Sigma R_L-\pi i\Sigma R'_L. \quad (18)$$

Addition and subtraction of these results give respectively*

$$\int_0^\infty \phi(x,t)\,J_\nu(at)\,t^{2s}\,dt$$

$$= \cos(s-\tfrac{1}{2}\nu)\,\pi\int_0^\infty \left\{\frac{2}{\pi}\,\Phi(x,t)\,K_\nu(at)-F(t)\right\}t^{2s}\,dt$$

$$+\pi i\Sigma(R_U-R_L)+\tfrac{1}{2}\pi i\Sigma(R'_U-R'_L), \quad (19)$$

$$\int_0^\infty \{\phi(x,t)\,Y_\nu(at)+if_1(t)\}\,t^{2s}\,dt$$

$$= \sin(s-\tfrac{1}{2}\nu)\,\pi\int_0^\infty \left\{\frac{2}{\pi}\,\Phi(x,t)\,K_\nu(at)-F(t)\right\}t^{2s}\,dt$$

$$+\pi\Sigma(R_U+R_L)+\tfrac{1}{2}\pi\Sigma(R'_U+R'_L). \quad (20)$$

When $\phi(x,t)$ is an even integral function of t, (19) reduces to (1) § 2, and when, in addition $-1<\nu<1$, (20) reduces to (2) § 2.

We have excluded the case in which $\phi(x,t)$ has an infinite number of poles. This can easily be dealt with, provided that they are real and simple, and that the distance between any two of them is greater than δ [δ as in condition (iv), § 3]. Equations (19) and (20) are then still true if the integrals are replaced by their principal values, provided that these values and the infinite sums

$$\Sigma(R'_U-R'_L), \quad \Sigma(R'_U+R'_L)$$

exist. The case in which $\phi(x,t)$ has an infinite number of complex poles I omit to avoid complication.

* If $\phi(x,t)$ has any poles on the positive real axis, the integrals in (19) and (20) must have their principal values.

3 . 2. *Case* II. $\nu-2s$ *an odd positive integer.*

This is the case where $\nu-2s-2r=1$ in (2) § 3, *i.e.* where the term $a_r t^{-1}$ occurs in $t^{2s} f_1(t)$. We cannot then make R (the radius of the quadrant of U) tend to infinity immediately. The result of taking (1), § 3, round U with R finite is

$$\int_0^R \{\phi(x,t)\, H_\nu^{(1)}(at)-f_1(t)\}\, t^{2s}\, dt$$

$$= e^{(s-\frac{1}{2}\nu)\pi i} \int_0^R \left\{\frac{2}{\pi}\, \Phi(x,t)\, K_\nu(at)-F(t)\right\} t^{2s}\, dt$$

$$-\int_{\gamma_U} \{\phi(x,t)\, H_\nu^{(1)}(at)-f_1(t)\}\, t^{2s}\, dt+2\pi i \Sigma R_U+\pi i \Sigma R'_U, \quad (1)$$

where the Σ merely denotes the summation of the residues at the poles in U, and (14) and (15), § 3 . 1, have been used, as in obtaining (17), § 3 . 1, from (10), § 3 . 1.

The result of taking (11), § 3 . 1, round L when R remains finite is

$$\int_0^R \{\phi(x,t)\, H_\nu^{(2)}(at)+f_1(t)\}\, t^{2s}\, dt$$

$$= e^{-(s-\frac{1}{2}\nu)\pi i} \int_0^R \left\{\frac{2}{\pi}\, \Phi(x,t)\, K_\nu(at)-F(t)\right\} t^{2s}\, dt$$

$$+\int_{\gamma_L} \{\phi(x,t)\, H_\nu^{(2)}(at)+f_1(t)\}\, t^{2s}\, dt-2\pi i \Sigma l_L-\pi i \Sigma l'_L. \quad (2)$$

From (4), § 3, and (9), § 3, we have

$$\int_{\gamma_U} \{\phi(x,t)\, H_\nu^{(1)}(at)-f_1(t)\}\, t^{2s}\, dt \to \tfrac{1}{2}\pi i a_r, \quad (3)$$

as $R \to \infty$, and similarly

$$\int_{\gamma_L} \{\phi(x,t)\, H_\nu^{(2)}(at)+f_1(t)\}\, t^{2s}\, dt \to -\tfrac{1}{2}\pi i a_r, \quad (4)$$

where a_r is the coefficient of t^{-1} in $f_1(t)\, t^{2s}$.

Also $\nu-2s$ is an odd positive integer; hence adding (1) and (2) and making $R \to \infty$, we have

$$\int_0^\infty \phi(x,t)\, J_\nu(at)\, t^{2s}\, dt = \tfrac{1}{2}\pi i a_r+\pi i \Sigma(R_U-R_L)+\tfrac{1}{2}\pi i \Sigma(R'_U-R'_L), \quad (5)$$

while on subtracting, and then proceeding to the limit, we have

$$\int_0^\infty [\{\phi(x,t)\,Y_\nu(at)+if_1(t)\}\,t^{2s}-(-1)^r\{\Phi(x,t)\,K_\nu(at)-F(t)\}\,t^{2s}]\,dt$$
$$=\pi\Sigma(R_U+R_I)+\tfrac{1}{2}\pi\Sigma(R'_U+R'_L). \quad (6)$$

The remark at the end of § 3 . 1, admitting the possibility of $\phi(x,t)$ having an infinite number of simple poles on the axes, applies in its entirety to (5) and (6) above.

4. *Special cases.*

The most important special cases, from the point of view of this paper, occur when $\phi(x,t)$, in addition to satisfying conditions (ii) and (iv) of § 3, is an even integral function of t. Equation (19), § 3 . 1, then becomes

$$\int_0^\infty \phi(x,t)\,J_\nu(at)\,t^{\nu+2m+1}\,dt=0 \quad (1)$$

whenever m is a positive integer or zero.

This result holds, of course, only when either of the inequalities (5) or (6) of § 3 are satisfied. For this case they become

$$\lambda<a \quad \text{and} \quad \nu+2m+\tfrac{1}{2}+p<0, \quad (2)$$

or
$$\lambda\leqslant a \quad \text{and} \quad \nu+2m+\tfrac{1}{2}+p<-1. \quad (3)$$

In general, the condition $\lambda\leqslant a$ simply means that (1) above is satisfied for all values of x which lie in a certain range.

From (6), § 3 . 2, we get

$$\int_0^\infty \phi(x,t)\,J_\nu(at)\,t^{\nu-2m-1}\,dt=\tfrac{1}{2}\pi i a_r, \quad (4)$$

where m is a positive integer or zero and a_r is the coefficient of t^{-1} in the principal part of $\phi(x,t)\,H_\nu^{(1)}(at)t^{\nu-2m-1}$ at the origin. The convergence conditions for (4) are obtained by replacing $2m+1$ in (2) and (3) above by $-(2m+1)$. In addition, there are some conditions for convergence of

(4) at its lower limit, but these are easily obtained. Since

$$H_\nu^{(1)}(at) = J_\nu(at) + iY_\nu(at)$$

and for convergence of (4) at the origin $\phi(x, t) J_\nu(at)\, t^{\nu-2m-1}$ cannot have a term of order t^{-1} there, it follows that $-ia_r$ is the coefficient of t^{-1} in $\phi(x, t)\, Y_\nu(at)\, t^{\nu-2m-1}$.

In particular, when $m = 0$ and ν is positive, (4) becomes

$$\int_0^\infty \phi(x, t)\, J_\nu(at)\, t^{\nu-1}\, dt = \tfrac{1}{2}\left(\frac{2}{a}\right)^\nu \Gamma(\nu)\, \phi(x, 0). \tag{5}$$

This includes many well known results as special cases, for example, Gegenbauer's integral*; and, when $\nu = \frac{1}{2}$, it reduces to Bateman's† formula.

From (20), § 3 . 1, we get

$$\int_0^\infty \{\phi(x, t)\, Y_\nu(at) + if_1(t)\}\, t^{\nu+2m}\, dt = 0, \tag{6}$$

where m is a positive or negative integer or zero. In particular

$$\int_0^\infty \left\{\pi Y_{2m}(t) + (\tfrac{1}{2}t)^{-2m} \sum_{n=0}^{m-1} \frac{(2m-n-1)!}{n!}\, (\tfrac{1}{2}t)^{2n}\right\} dt = 0, \tag{7}$$

when m is a positive integer. The finite series on the right of the integrand of (7) is merely the principal part of $\pi Y_{2m}(t)$ at the origin.

Other special cases of interest occur when $\nu = \pm\frac{1}{2}$ in (19) and (20) of § 3 . 1. When $\phi(x, t)$ is an even integral function of t satisfying condition (iv) of § 3 they become

$$\int_0^\infty \begin{matrix}\sin at \\ \cos at\end{matrix} \phi(x, t)\, t^{2u}\, dt = \begin{matrix}\sin(u+\frac{1}{2})\pi \\ \cos(u+\frac{1}{2})\pi\end{matrix} \int_0^\infty e^{-at}\, \Phi(x, t)\, t^{2u}\, dt. \tag{8}$$

In particular

$$\int_0^\infty \sin at\, \phi(x, t)\, t^{2m+1}\, dt = 0, \tag{9}$$

$$\int_0^\infty \cos at\, \phi(x, t)\, t^{2m}\, dt = 0, \tag{10}$$

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* Watson, *loc. cit.*, 419, Eq. (15).

† See G. H. Hardy, "On an integral equation", *Proc. London Math. Soc.* (2), 7 (1909), 445.

whenever m is a positive integer. Since these two results are of fundamental importance in Part II of this paper, the conditions under which they hold are given here. They are

(i) $\phi(x, t)$ is an even integral function of t such that, when $|t|$ is sufficiently large and $t = Re^{i\theta}$, we have

$$|\phi(x, t)| < A\, R^p \exp(\lambda R |\sin\theta|), \tag{11}$$

where A is a constant independent of θ; λ is a function of x and $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(ii) x must lie in such a range and p must be such that, for (9), either

$$\lambda < a \quad \text{and} \quad p+2m+1 < 0, \tag{12}$$

or

$$\lambda \leqslant a \quad \text{and} \quad p+2m+1 < -1, \tag{13}$$

while, for (10), either

$$\lambda < a \quad \text{and} \quad p+2m < 0, \tag{14}$$

or

$$\lambda \leqslant a \quad \text{and} \quad p+2m < -1. \tag{15}$$

When $\lambda < a$, the integrals on the left of (9) and (10) represent null functions, while when $\lambda > a$ they do not. It is of interest to observe how the change occurs as x varies in such a way that λ increases through a. Take, for example, the integral on the left of (9). When

$$p+2m+1 < -1$$

the value of the integral at $\lambda = a$ is zero, and so the change occurs continuously. When $-1 \leqslant p+2m+1 < 0$, the integral diverges at $\lambda = a$, and so the change occurs through an infinite discontinuity.

Similar remarks apply to the integrals on the left of (1), (6), and (10).

The results (1), (6), (9), and (10) are much more easily obtained when s is an integer by taking the integral (1) of § 3 round a semi-circle embracing the upper half of the t plane. This contour has been greatly used by Hankel and Watson* to evaluate similar integrals. The disadvantage of the method, however, is that the reason for the vanishing

* Watson, *loc. cit.*, 424 *et seq.*

of these integrals over a whole range of values of x is not so apparent as in their deduction from (19) and (20), § 3 . 1.

5. *Applications to special functions.*

Before considering special functions, it is worth noting that, in all the equations of § 4, $\phi(x,t)$ can be replaced by a product of similar functions, each of which satisfies the conditions of § 3. The conditions accompanying the new equations are easily found from those already given.

Applications to gamma functions.

Since

$$\log\Gamma(x+t) = (x+t-\tfrac{1}{2})\log t - t + \tfrac{1}{2}\log(2\pi) + O(t^{-1}), \tag{1}$$

provided that $-\pi < \arg t < \pi$, we find, on putting $t = Re^{i\theta}$ in

$$\{\Gamma(x+t)\,\Gamma(x-t)\}^{-1} = \frac{\Gamma(1-x+t)}{\Gamma(x+t)}\,\frac{\sin\pi(x-t)}{\pi}, \tag{2}$$

that

$$|\{\Gamma(x+t)\,\Gamma(x-t)\}^{-1}| < C\,R^{1-2x}\exp(\pi R|\sin\theta|), \tag{3}$$

where C is a constant independent of θ and $-\pi < \theta < \pi$.

Hence $\{\Gamma(x+t)\,\Gamma(x-t)\}^{-1}$ is an even integral function of t of the type $\phi(x,t)$, with $\lambda = \pi$ and $p = 1-2x$.

From (1), § 4, then, when $\nu > -1$ and m is a positive integer,

$$\int_0^\infty \frac{J_\nu(at)\,t^{\nu+2m+1}}{\Gamma(x+t)\,\Gamma(x-t)}\,dt = 0, \tag{4}$$

provided that either

$$\pi < a \quad\text{and}\quad (\nu+2m+1)+\tfrac{1}{2}+(1-2x) < 0 \tag{4.1}$$

or

$$\pi \leqslant a \quad\text{and}\quad (\nu+2m+1)+\tfrac{1}{2}+(1-2x) < -1. \tag{4.2}$$

The cases $\nu = \pm\frac{1}{2}$ in (4) have been given by Ramanujan*; but his method (an application of Fourier's integral theorem) enabled him, in addition, to evaluate these integrals outside the null interval.

* S. Ramanujan, *loc. cit.*, 294.

From (6), § 4, we get

$$\int_0^\infty \frac{Y_\nu(at)\, t^{\nu+2m}}{\Gamma(x+t)\,\Gamma(x-t)}\, dt = 0, \tag{5}$$

the conditions being (4 . 1) and (4 . 2) above, with $2m+1$ replaced by $2m$.

Applications to Legendre functions. It is well known that*

$$P_{t-\frac{1}{2}}(\cos x) = {}_2F_1(t+\tfrac{1}{2},\ -t+\tfrac{1}{2};\ 1;\ \sin^2 \tfrac{1}{2}x), \tag{6}$$

and Watson† has proved that

$$P_{t-\frac{1}{2}}(\cos x) \sim \left\{\frac{4}{(2t-1)\,\pi \sin x}\right\}^{\frac{1}{2}} \cos(tx - \tfrac{1}{4}\pi), \tag{7}$$

provided that x is real, and also that $0 < x < \pi$. On putting $t = Re^{i\theta}$ in (7), we have, when R is sufficiently large,

$$|P_{t-\frac{1}{2}}(\cos x)| < A\, R^{-\frac{1}{2}} \exp(xR|\sin\theta|), \tag{7 . 1}$$

where A is a constant independent of θ and $-\pi < \theta < \pi$. It follows that $P_{t-\frac{1}{2}}(\cos x)$ is an even integral function of t of the form $\phi(x, t)$ with $\lambda = x$ and $p = -\frac{1}{2}$. Hence from (1) and (6), § 4, we can deduce equations for the Legendre function similar to (4) and (5) above.

It is easily proved also that $P^m_{t-\frac{1}{2}}(\cos x)$ is an even integral function of t of the form $\phi(x, t)$ with $\lambda = x$ and $p = m-\frac{1}{2}$.

Applications to Bessel and allied functions.

Equations (1) and (6), § 4, contain as special cases many well known integrals in the theory of Bessel functions. Thus all the integrals which represent null functions in Chapter XIII of Watson's "Theory of Bessel functions" are special cases.

It is easy to deduce from the asymptotic expansion of the Bessel function that

$$\frac{J_\nu\{x\sqrt{(t^2+a^2)}\}}{\{x\sqrt{(t^2+a^2)}\}^\nu} \tag{8}$$

---

* Whittaker and Watson, *loc. cit.*, 305.

† Watson, "Asymptotic expansion of hypergeometric functions", *Trans. Camb. Phil. Soc.*, 22 (1923), 277–308.

is an even integral function of t of the form $\phi(x,t)$ with $\lambda = x$ and § 5, contains such integrals as Nicholson's* integral, Sonine's† integral, and the integral‡ in the problem of random flights as special cases of (1), § 4.

From the asymptotic expansion it is also easily seen that, if $\mathbf{H}_\nu$ denotes Struve's function, then

$$\frac{\mathbf{H}_\nu\{x\sqrt{(t^2+a^2)}\}}{\{x\sqrt{(t^2+a^2)}\}^\nu} \tag{9}$$

is an even integral function of t of the form $\phi(x,t)$, with $\lambda = x$ and $p = -\nu - \frac{3}{2}$ or -2, whichever is algebraically the larger.

Again, when $|t|$ is large, from the Bessel function series

$$J_{\nu-xt}(a) = \frac{(\frac{1}{2}a)^{\nu-xt}}{\Gamma(1+\nu-xt)}\{1+O(t^{-1})\}.$$

Hence, when $|t|$ is large,

$$J_{\nu-xt}(c)\,J_{\nu+xt}(c) = \frac{(\frac{1}{2}c)^{2\nu}}{\Gamma(1+\nu-xt)\,\Gamma(1+\nu+xt)}\{1+O(t^{-1})\}. \tag{10}$$

Hence, using (3), § 5, we easily deduce that $J_{\nu-xt}(c)\,J_{\nu+xt}(c)$ is an even integral function of t of the form $\phi(x,t)$ with $\lambda = \pi x$ and $p = -(2\nu+1)$.

Finally, it is easy to see from an example due to Ramanujan§ that the restrictions for $\phi(x,t)$, in order that equations (9) and (10), § 4, should hold, are sufficient but not necessary. His result is equivalent to the equation

$$\int_0^\infty \frac{J_{\nu+xt}(c_1)}{c_1^{\nu+xt}}\,\frac{J_{\nu-xt}(c_2)}{c_2^{\nu-xt}}\cos at\,dt = 0, \tag{11}$$

provided that either $-\pi x < a < \pi x$ together with $R(2\nu) > -1$, or $-\pi \leqslant a \leqslant \pi$ together with $R(2\nu) > 0$. It can be seen that the function multiplying $\cos at$ is not an even function of t, although it satisfies the other requirements of $\phi(x,t)$, namely (11), § 4. Ramanujan's result, however, can be proved by the methods of this paper, by multiplying out

* *Quart. J. of Math.*, 48 (1920), 325.

† Watson, *Theory of Bessel functions*, 415.

‡ *Ibid.*, 421.

§ Ramanujan, *loc. cit.*, 304. Watson, *loc. cit.*, 449.

the Bessel functions in (11) and using (4), § 5, in the form

$$\int_{-\infty}^{\infty} \frac{e^{iat}}{\Gamma(\alpha+t)\,\Gamma(\beta-t)}\,dt = 0, \tag{12}$$

with suitable conditions.

6. *Summability.*

Equations (19) and (20), § 3 . 1, and the results deduced from them have been proved, provided that λ and a are real, $\nu > -1$, and either

$$\lambda < a \quad \text{and} \quad 2s+p-\tfrac{1}{2} < 0, \tag{1}$$

or

$$\lambda \leqslant a \quad \text{and} \quad 2s+p-\tfrac{1}{2} < -1,$$

where λ is as defined in condition (iv), § 3.

When $\lambda = a$ and $2s+p-\frac{1}{2} \geqslant -1$ the integrals on both sides of (19) and (20), § 3 . 1, diverge, and none of them are summable in the sense defined below. When $\lambda < a$ and $2s+p-\frac{1}{2} > 0$, the integrals on the right of (19) and (20), § 3 . 1, still converge, although the integrals on the left diverge at their upper limits. In this case, however, the integrals on the left are summable under suitable conditions. I use the following well known definitions of summability*.

(i) $\sum_{n=0}^{\infty} u_n$ is summable (R, λ, κ) to the sum C if

$$\lim_{R\to\infty} \sum_{n \leqslant R} u_n \left(1-\frac{\lambda_n}{\lambda_R}\right)^{\kappa} = C,$$

where λ_x is a positive function of $x(>0)$ tending steadily to infinity with x, and $\kappa > 0$.

(ii) $\int_0^{\infty} \psi(t)\,dt$ is summable (R, λ, κ) to the value C if

$$\lim_{R\to\infty} \int_0^R \psi(t) \left(1-\frac{\lambda_t}{\lambda_R}\right)^{\kappa} dt = C,$$

with the same restrictions for λ_x and κ as above.

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* See Hardy and Riesz, *The general theory of Dirichlet series* (Cambridge Tracts in Mathematics, No. 18).

In order to restrict λ_x to be real, it will be assumed that $\phi(x, t)$ has only a finite number of complex poles. With this restriction I shall prove the following theorem :

If $\phi(x, t)$ satisfies conditions (ii) *and* (iv) *of* § 3 *and has at most a finite number of poles, then, when $\lambda < a$ and $2s+p-\frac{1}{2} > 0$, the integrals on the left-hand sides of* (19) *and* (20), § 3 . 1, *and* (6) *and* (7), § 3 . 2, *are summable* $(R, t^2, 2s+p-\frac{1}{2}+\epsilon)$ $(\epsilon \geq 0)$, *respectively to the right-hand sides.*

It is possible to extend the theorem to the case when $\phi(x, t)$ has an infinite number of poles on the positive real axis, but the result is too complicated to be of much interest, and is therefore omitted.

7. *Proof of the theorem of* § 6.

The proof consists in taking the integral

$$\int_U \{\phi(x, t)\, H_\nu^{(1)}(at) - f_1(t)\}\, t^{2s}\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon} dt \qquad (1)$$

round the contour U (defined in § 3), where $\phi(x, t)$ satisfies the conditions of § 3, $\epsilon > 0$, $q = 2s+p-\frac{1}{2}$ and $f_1(t)$ is as defined in § 3 (iii). To exclude the branch point at $t = R$, an indentation of radius ρ is introduced where the quadrant of U meets the real axis.

It is assumed that R is so large that all the poles of $\phi(x, t)$ are inside or on U, and that none are in the neighbourhood of the quadrant.

From (2) and (4), § 3, we have

$$\left| \{\phi(x, t)\, H_\nu^{(1)}(at) - f_1(t)\}\, t^{2s}\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon} \right| < K\rho^{q+\epsilon} R^{-\epsilon},$$

when t is real and large, and so the part of (1) round the indentation will tend to zero for all values of R as $\rho \to 0$ whenever $q \geq 0$.

To deal with (1) along the quadrant of U, divide this quadrant into two arcs, the upper arc containing those points for which

$$R \sin\theta \geq R^{\epsilon_1} \quad (0 < \epsilon_1 < 1)$$

and the lower arc containing the points for which

$$R \sin\theta \leq R^{\epsilon_1}.$$

If I denotes the first part of the integrand of (1), we deduce from (4), § 3, that, when $\lambda < a$ and t is on the upper arc,

$$|I| = \left| \phi(x, t)\, H_\nu^{(1)}(at)\, t^{2s} \left(1 - \frac{t^2}{R^2}\right)^{q+\epsilon} \right|$$

$$< 2^{q+\epsilon}\, D\, R^{2s+p-\frac{1}{2}} \exp\left\{-(a-\lambda)\, R^{\epsilon_1}\right\}.$$

Hence, as $R \to \infty$, for the upper arc

$$\int I\, dt \to 0.$$

On the lower arc, whose length is $O(R^{\epsilon_1})$, we have

$$1 - \frac{t^2}{R^2} = 1 - e^{2i\theta} = O(\theta) = O(R^{\epsilon_1 - 1})$$

and so $$|I| < K\, R^{(q+\epsilon)(\epsilon_1 - 1)}\, R^{2s+p-\frac{1}{2}},$$

where K is a constant independent of θ. Hence the value of $\left|\int I\, dt\right|$ taken along the lower arc cannot exceed

$$O(R^{\epsilon_1 + (q+\epsilon)(\epsilon_1 - 1) + 2s + p - \frac{1}{2}}) = O(R^{\epsilon_1(1+q+\epsilon) - \epsilon}),$$

since $q = 2s + p - \frac{1}{2}$, and for a definite positive ϵ this tends to 0 as $R \to \infty$ provided that ϵ_1 is chosen so that $0 < \epsilon_1 < \epsilon/(1+q+\epsilon)$.

To deal with the remainder of the integrand of (1), we must, as before, divide the proof into two cases :

Case I. $\nu - 2s - 2r > 1$ and $\lambda > a$.

Case II. $\nu - 2s - 2r = 1$ and $\lambda < a$.

From (2), § 3,

$$f_1(t)\, t^{2s} \left(1 - \frac{t^2}{R^2}\right)^{q+\epsilon} = O(t^{2s-\nu+2r}),$$

and so for *Case* I,

$$\int f_1(t)\, t^{2s} \left(1 - \frac{t^2}{R^2}\right)^{q+\epsilon} dt \to 0,$$

as $R \to \infty$, the path being the quadrant of U; while for *Case* II this integral tends to $\frac{1}{2}\pi i a_r$, where a_r is the coefficient of t^{-1} in $f_1(t)\, t^{2s}$.

Continuing with *Case* I, the result of taking (1) round U, and making $R \to \infty$, is

$$\lim_{R\to\infty} \int_0^R \{\phi(x, t)\, H_\nu^{(1)}(at) - f_1(t)\}\, t^{2s}\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon} dt$$

$$= \lim_{R\to\infty} e^{(s-\frac{1}{2}\nu)\pi i} \int_0^R \left\{\frac{2}{\pi}\, \Phi(x, t)\, K_\nu(at) - F(t)\right\} t^{2s}\left(1+\frac{t^2}{R^2}\right)^{q+\epsilon} dt$$

$$+ 2\pi i \Sigma R_U + \pi i \Sigma R_U, \quad (2)$$

where use has been made of (15) and (16), § 3 . 1, and both the Σ signs refer to the finite sum of all the residues in and on the first quadrant.

It can be proved that

$$\lim_{R\to\infty} \int_0^R \left\{\frac{2}{\pi}\, \Phi(x, t)\, K_\nu(at) - F(t)\right\} t^{2s}\left(1+\frac{t^2}{R^2}\right)^{q+\epsilon} dt$$

$$= \int_0^\infty \left\{\frac{2}{\pi}\, \Phi(x, t)\, K_\nu(at) - F(t)\right\} t^{2s}\, dt, \quad (3)$$

but owing to the length of the paper I shall assume this and refer for the proof to my previous paper*, where an almost identical problem is discussed. (2) then becomes

$$\lim_{R\to\infty} \int_0^R \{\phi(x, t)\, H_\nu^{(1)}(at) - f_1(t)\}\, t^{2s}\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon} dt$$

$$= e^{(s-\frac{1}{2}\nu)\pi i} \int_0^\infty \left\{\frac{2}{\pi}\, \Phi(x, t)\, K_\nu(at) - F(t)\right\} t^{2s}\, dt + 2\pi i \Sigma R_U + \pi i \Sigma R'_U. \quad (4)$$

Similarly, by taking the integral

$$\int_L \{\phi(x, t)\, H_\nu^{(2)}(at) - f_2(t)\}\, t^{2s}\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon} dt \quad (5)$$

* *Proc. London Math. Soc.* (2), 24 (1925), 488–489.

round the contour L, where L and $f_2(t)$ are defined as in § 3 . 1, we get

$$\lim_{R\to\infty}\int_0^R \{\phi(x,t)\,H_\nu^{(2)}(at)+f_1(t)\}\,t^{2s}\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon}dt$$

$$= e^{-(s-\frac{1}{2}\nu)\pi i}\int_0^\infty \left\{\frac{2}{\pi}\,\Phi(x,t)\,K_\nu(at)-F(t)\right\}t^{2s}\,dt-2\pi i\Sigma R_L-\pi i\Sigma R'_L. \quad (6)$$

On adding and subtracting (4) and (6) we prove that the left-hand sides of (19) and (20), § 3 . 1, are summable $(R, t^2, p+2s-\frac{1}{2}+\epsilon)$ to their right-hand sides respectively.

If we deal with *Case* II in the same way, we can prove that the integrals on the left of (6) and (7), § 3 . 2, are summable

$$(R, t^2, p+2s-\tfrac{1}{2}+\epsilon)$$

to the right-hand sides respectively.

Part II.—The Study of Null Series Outside the Null Interval.

8. *Preliminary statements.*

We now deal with the series (1) of § 1 when $\nu=\frac{1}{2}$, and we shall prove the following theorems :

Theorem 1.

$$\tfrac{1}{2}a+\sum_{n=1}^{\infty}(-1)^n\,\phi(x,n)\,n^{2m}=2\sum_{n=1}^{\rho}\int_0^\infty \cos(2n-1)\,\pi t\,\phi(x,t)\,t^{2m}\,dt, \quad (1)$$

where

(i) $\phi(x,t)$ *is an even integral function of* t ;

(ii) *when* $|t|$ *is sufficiently large and* $t=Re^{i\theta}$, *we have*

$$|\phi(x,t)|<KR^\rho\exp(\lambda R|\sin\theta|),$$

where λ is a function of x, K is a constant independent of θ and

$$-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi.$$

(iii) *x lies in such a range and p is such that either*

$$(2\rho-1)\,\pi < \lambda < (2\rho+1)\,\pi \quad \text{and} \quad p+2m < 0,$$

or

$$(2\rho-1)\,\pi \leqslant \lambda \leqslant (2\rho+1)\,\pi \quad \text{and} \quad p+2m < -1,$$

where ρ is a positive integer or zero.

(iv) *m is a positive integer or zero.*

When x lies in the range which makes $0 < \lambda < \pi$, the value of the series on the left is zero, the case studied in a previous paper.

THEOREM 2.

$$\sum_{n=1}^{\infty} (-1)^n\, \phi(x, n)\, n^{2s} = 2 \sum_{n=1}^{\rho} \int_0^{\infty} \phi(x, t)\, t^{2s} \cos(2n-1)\,\pi t\, dt$$

$$-2 \sin s\pi \sum_{n=\rho+1}^{\infty} \int_0^{\infty} \Phi(x, t)\, t^{2s}\, e^{-(2n+1)\pi t}\, dt,$$

where conditions (i), (ii), (iii) *of Theorem* 1 *hold, $s > 0$ and*

$$\Phi(x, t) = \phi(x, it).$$

[*m in* (iii) *is replaced by s.*]

THEOREM 3.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\phi(x, n)}{n^2} = -\tfrac{1}{2}\pi a_1 + \rho^2\pi^2 \phi(x, 0)$$

$$+2 \sum_{n=1}^{\rho} \int_0^{\infty} \left\{ \frac{\phi(x, t)}{t^2} \cos(2n-1)\,\pi t - \frac{\phi(x, 0)}{t^2} \right\} dt,$$

where conditions (i), (ii), (iii) *of Theorem* 1 *hold* [*m being replaced by -1 in* (iii)], *and a_1 is the coefficient of t^{-1} in $\phi(x, t)\, t^{2s}$ cosec πt.*

THEOREM 4.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \phi(x, n-\tfrac{1}{2})(n-\tfrac{1}{2})^{2s-1}$$

$$= 2 \sum_{n=1}^{\rho} (-1)^{n-1} \int_0^{\infty} \phi(x, t)\, t^{2s-1} \sin(2n-1)\pi t\, dt$$

$$+ 2 \sin s\pi \sum_{n=\rho+1}^{\infty} (-1)^{n-1} \int_0^{\infty} \Phi(x, t)\, t^{2s-1} e^{-(2n-1)\pi t}\, dt,$$

where conditions (i), (ii), (iii) *of Theorem* 1 *hold* [*with* $2m$ *in* (iii) *replaced by* $2s-1$], *and* $\Phi(x, t)$ *is defined as in Theorem* 2.

THEOREM 5.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \phi(x, n-\tfrac{1}{2})$$

$$= (-1)^{\rho} \tfrac{1}{4}\pi\, \phi(x, 0) + \sum_{n=1}^{\rho} (-1)^n \int_0^{\infty} \phi(x, t)\, \frac{\sin(2n-1)\pi t}{t}\, dt,$$

where (i), (ii), (iii) *of Theorem* 1 *hold* [*with* $2m$ *in* (iii) *replaced by* -1].

Applications to Gamma, Legendre, and Bessel functions, and some remarks on summability, will follow after the proofs. During the proofs many other forms into which these theorems can be thrown will also be given.

9. *The contours* U, L, *and* C.

The two contours U and L of §§ 3 and 3 . 1, conditions (i) and (v) will be required, but there will be no need for indentations in the axes, except possibly at the origin. In addition, we shall require the contour C defined by the semi-circle $|t| = R$ on the right of the imaginary axis, together with the part of the imaginary axis from $(0, -R)$ to $(0, R)$ (with an indentation, if necessary, to exclude the origin). The radius R is the same as the radii of the quadrants of U and L.

As before, R must be made to tend to infinity in such a way that the poles of the functions which are integrated along these contours are avoided by an amount exceeding δ.

10. *Proof of Theorem* 1, § 8.

Consider the integral

$$\int_C \frac{\phi(x,t)}{\sin \pi t}\, t^{2m}\, dt \tag{1}$$

taken round C, where $\phi(x,t)$ satisfies the conditions of § 8 and m is a positive integer or zero. The residues are

$$\frac{1}{\pi} \sum_{n=1}^{[R]} (-1)^n\, \phi(x,n)\, n^{2m}, \tag{2}$$

where $[R]$ is the greatest integer in R.

Denoting the indentation which excludes the origin by γ and its radius by ϵ, we have

$$\lim_{\epsilon \to 0} \int_\gamma \frac{\phi(x,t)}{\sin \pi t}\, t^{2m}\, dt = -i\phi(x,0) \quad (m=0), \qquad = 0 \quad (m>0). \tag{3}$$

Denote the value of this limit in either case by $-ia$. Then (2) and (3) account for the left-hand side of (1), § 8.

From (i), § 8, the integrand of (1) is an odd function of t, and so the values of the integral taken along the positive and negative portions of the imaginary axis included in C cancel each other.

It follows, then, that the value of the left-hand side of (1) depends entirely on the value of the part of integral (1) taken along the semi-circle of C.

On putting $t = Re^{i\theta}$ along this semi-circle, we have, from (2), § 8, when R is sufficiently large,

$$\left| \frac{\phi(x,t)}{\sin \pi t}\, t^{2m} \right| < KR^{p+2m} \exp\{(\lambda - \pi)\, R\, |\sin\theta|\},$$

when $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Hence this part of the integral converges as $R \to \infty$ only if either

$$\lambda < \pi \quad \text{and} \quad p+2m < 0, \tag{4}$$

or

$$\lambda \leqslant \pi \quad \text{and} \quad p+2m < -1. \tag{5}$$

If then we wish to investigate the value of the series on the left-hand side of (1), § 8, for values of x such that $(2\rho-1)\pi < \lambda < (2\rho+1)\pi$,

where ρ is a positive integer other than zero, the integral (1) by itself is not adequate. For this reason it is necessary to introduce two geometrical progressions, with the help of which convergence can be obtained.

10.1. *Introduction of $U(t)$ and $L(t)$.*

Let σ denote any positive integer not less than ρ. Then

$$\frac{1}{\sin \pi t}+2ie^{\pi it}+2ie^{3\pi it}+\ldots+2ie^{(2\sigma-1)\pi it}=\frac{e^{2\sigma\pi it}}{\sin \pi t}. \tag{1}$$

If we denote either side of (1) by $U(t)$, and put $t=Re^{i\theta}$, we have from (2), § 8,

$$|\phi(x, t)\, t^{2m}\, U(t)| < KR^{p+2m} \exp\{(\lambda-\pi) R|\sin\theta|-2\sigma\pi R \sin\theta\}.$$

Hence, when the path of integration is the upper half of the large semi-circle of C, on which $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, *i.e.* the quadrant γ_U of U, it follows from Jordan's lemma that

$$\lim_{R\to\infty} \int_{\gamma_U} \phi(x, t)\, t^{2m}\, U(t)\, dt = 0 \tag{2}$$

provided that $\sigma > \rho$ and $p+2m < 0$. When σ is taken equal to ρ, the conditions are either $\lambda < (2\rho+1)\pi$ and $p+2m > 0$, or $\lambda \leqslant (2\rho+1)\pi$ and $p+2m < -1$.

Similarly

$$\frac{1}{\sin \pi t}-2ie^{-\pi it}-2e^{-3\pi it}-\ldots-2ie^{-(2\sigma-1)\pi} = \frac{e^{-2\sigma\pi it}}{\sin \pi t}, \tag{4}$$

and, if either side of (4) is denoted by $L(t)$, and $t=Re^{i\theta}$, we have, on the large semi-circle of C,

$$|\phi(x, t)\, t^{2m}\, L(t)| < KR^{p+2m} \exp\{(\lambda-\pi) R|\sin\theta|+2\sigma\pi R \sin\theta\}.$$

Hence when the path of integration is the lower half of the large semi-circle of C, on which $-\frac{1}{2}\pi \leqslant \theta \leqslant 0$, *i.e.* the quadrant γ_L of L, we have

$$\lim_{R\to\infty} \int_{\gamma_L} \phi(x, t)\, t^{2m}\, L(t)\, dt = 0 \tag{5}$$

under the same conditions which follow (2) above.

Consider now the system of integrals

$$\int_C \frac{\phi(x,t)}{\sin \pi t}\, t^{2m}\, dt + 2i \sum_{n=1}^{\sigma} \int_U \phi(x,t)\, t^{2m}\, e^{(2n-1)\pi i t}\, dt$$

$$-2i \sum_{n=1}^{\sigma} \int_L \phi(x,t)\, t^{2m}\, e^{-(2n-)\pi i t}\, dt, \qquad (6)$$

where the contour over which each integral is to be taken is denoted by the suffix at the foot of the integral.

If we break up the large semi-circle of C into the two quadrants γ_U and γ_L of U and L respectively, the sum total of the system of integrals (6) taken over the large arcs of C, U, and L is

$$\int_{\gamma_U} \phi(x,t)\, t^{2m}\, U(t)\, dt + \int_{\gamma_L} \phi(x,t)\, t^{2m}\, L(t)\, dt,$$

and, from (2) and (5), these tend to zero as $R \to \infty$.

Along the imaginary axis, since $\phi(x, t)$ is an even function of t, the first integral of (6) vanishes.

Again, since $\phi(x,t)$ is an even function of t,

$$\int_{Ri}^{0} \phi(x,t)\, t^{2m}\, e^{(2n-1)\pi i t}\, dt = \int_{0}^{-Ri} \phi(x,t)\, t^{2m}\, e^{-(2n-1)\pi i t}\, dt, \qquad (7)$$

and so the n-th integral of the first sum of (6) cancels with the n-th integral of the second sum.

Hence the total contribution of all the integrals along the imaginary axis is zero.

There only remains, then, the contribution of the first integral over the indentation excluding the origin, namely $-ia$, and the sums of the other integrals along the real axis.

Since only the first integrand of (6) has any poles, it now follows from Cauchy's theorem that

$$-ia + 2i \sum_{n=1}^{\sigma} \int_0^{\infty} \phi(x,t)\, t^{2m}\, e^{(2n-1)\pi i t}\, dt - 2i \sum_{n=1}^{\sigma} \int_{\infty}^{0} \phi(x,t)\, t^{2m}\, e^{-(2n-1)\pi i t}\, dt$$

$$= 2i \sum_{n=1}^{\infty} (-1)^n\, \phi(x,n)\, n^{2m}.$$

Hence

$$\tfrac{1}{2}a+\sum_{n=1}^{\infty}(-1)^n\,\phi(x,n)\,n^{2m}=2\sum_{n=1}^{\sigma}\int_0^{\infty}\cos(2n-1)\pi t\,\phi(x,t)\,t^{2m}\,dt, \qquad (8)$$

provided that x is such that either

$$\left.\begin{aligned}&(2\rho-1)\pi<\lambda<(2\rho+1)\pi \quad\text{and}\quad p+2m<0,\\ \text{or}\quad &(2\rho-1)\pi\leqslant\lambda\leqslant(2\rho+1)\pi \quad\text{and}\quad p+2m<-1\end{aligned}\right\}, \qquad (9)$$

σ is any integer not less than ρ and m is a positive integer or zero. When x is such that $0<\lambda<\pi$, then the value of the series on the left of (18) is zero. This case has been studied in my previous paper.

Now the integrals on the right of (8) are of the type studied in Part I, and it follows from equation (10), § 4, that

$$\int_0^{\infty}\cos(2n-1)\pi t\,\phi(x,t)\,t^{2m}\,dt=0, \qquad (10)$$

whenever $\lambda<(2n-1)\pi$. Since $\lambda<(2\rho+1)\pi$, it follows that the integrals on the right of (8) for which $n\geqslant\rho+1$ all vanish. Hence the summation on the right of (8) need be taken only as far as ρ.

In fact, (8) can be written in either of the following two forms :

$$\tfrac{1}{2}a+\sum_{n=1}^{\infty}(-1)^n\,\phi(x,n)\,n^{2m}=2\sum_{n=1}^{\rho}\int_0^{\infty}\cos(2n-1)\pi t\,\phi(x,t)\,t^{2m}\,dt, \qquad (11)$$

$$=\int_0^{\infty}\frac{\sin 2\rho\pi t}{\sin\pi t}\,\phi(x,t)\,t^{2m}\,dt. \qquad (12)$$

Thus the sum of the series on the left of (11) can be expressed as a finite number of integrals. As x varies so that λ increases through an odd multiple of π, a new integral is added to this sum. These integrals are of the type studied in § 5, Part I, equation (12). From the remarks which follow this equation it can be seen that the new integral enters the right-hand side of (11) above through a zero when $p+2m<-1$ as λ increases through an odd multiple of π, and enters through an infinite discontinuity when $-1<p+2m<0$ and λ increases through an odd multiple of π.

In other words, when $p+2m<-1$, the series on the left of (11) has a continuous sum, but, when $-1<p+2m<0$, the sum of the series on

the left of (11) has poles whenever x is such that λ is an odd multiple of π. When $p+2m>0$ the series diverges, but, as will be seen later, it is frequently summable according to the definitions of § 6.

11. *Proof of Theorem* 2, § 8.

For the proof of this theorem consider the system of integrals

$$\int_C \frac{\phi(x,t)}{\sin \pi t}\, t^{2s}\, dt + 2i \sum_{n=1}^{\rho} \int_U \phi(x,t)\, t^{2s}\, e^{(2n-1)\pi i t}\, dt$$

$$-2i \sum_{n=1}^{\rho} \int_L \phi(x,t)\, t^{2s}\, e^{-(2n-1)\pi i t}\, dt, \qquad (1)$$

where the contour round which each integral is to be taken is denoted by the suffix at the foot of the integral. The contours are defined in § 9, and the conditions in the enunciation § 8.

From the consideration of the very similar system of integrals in (6), § 10 . 1, the part of (1) over the curved portions of U, L, and C vanish in the limit as $R \to \infty$ whenever either

$$\left.\begin{aligned} &(2\rho-1)\pi < \lambda < (2\rho+1)\pi \quad \text{and} \quad p+2s<0 \\ \text{or} \quad &(2\rho-1)\pi \leqslant \lambda \leqslant (2\rho+1)\pi \quad \text{and} \quad p+2s<-1 \end{aligned}\right\}. \qquad (2)$$

The only difference between the system (1) above and (6), § 10 . 1, occurs along the imaginary axis, for, unless s is a positive integer, the integrals of (1) along the imaginary axis do not cancel one another.

If we replace t by $te^{\frac{1}{2}\pi i}$ along the upper portion of the imaginary axis and by $te^{-\frac{1}{2}\pi i}$ along the lower portion, and write, as before,

$$\phi(x, it) = \Phi(x, t), \qquad (3)$$

it is not difficult to see that the result of considering (1) is, in the limit when $R \to \infty$,

$$\sum_{n=1}^{\infty} (-1)^n\, \phi(x,n)\, n^{2s} = -\sin s\pi \int_0^{\infty} \Phi(x,t)\, t^{2s} \left\{ \frac{1}{\sinh \pi t} - 2 \sum_{n=1}^{\rho} e^{-(2n-1)\pi t} \right\} dt$$

$$+2 \sum_{n=1}^{\rho} \int_0^{\infty} \phi(x,t)\, t^{2s} \cos(2n-1)\pi t\, dt. \qquad (4)$$

The series on the left is due to the residues of the first integral of (1), the only integral of (1) to have poles. The first integral on the right

is due to the part of (1) along the imaginary axis, while the series of integrals is due to the part of (1) along the real axis.

It is easy to see that (4) can be written in either of the forms

$$\sum_{n=1}^{\infty} (-1)^n \phi(x, n) n^{2s} = -\sin s\pi \int_0^{\infty} \Phi(x, t) \frac{t^{2s} e^{-2\rho\pi t}}{\sinh \pi t} dt$$

$$+2 \sum_{n=1}^{\rho} \int_0^{\infty} \phi(x, t) t^{2s} \cos(2n-1)\pi t \, dt, \qquad (5)$$

$$\sum_{n=1}^{\infty} (-1)^n \phi(x, n) n^{2s} = 2 \sum_{n=1}^{\rho} \int_0^{\infty} \phi(x, t) t^{2s} \cos(2n-1)\pi t \, dt$$

$$-2 \sin s\pi \sum_{n=\rho+1}^{\infty} \int_0^{\infty} \Phi(x, t) t^{2s} e^{-(2n-1)\pi t} dt. \qquad (6)$$

Now $\phi(x, t)$ satisfies all the conditions required in order that (8), § 4, should hold. Hence (6) can be written

$$\sum_{n=1}^{\infty} (-1)^n \phi(x, n) n^{2s} = 2 \sum_{n=1}^{\infty} \int_0^{\infty} \phi(x, t) t^{2s} \cos(2n-1)\pi t \, dt. \qquad (7)$$

This form has the advantage that it is true for all values of x, unless $-1 < p+2s < 0$, when values of x for which λ is an odd multiple of π make both sides infinite. The forms (4), (5), and (6) hold only under conditions (2).

The disadvantage of (7), however, is that integrals of the type

$$\int_0^{\infty} \cos at \, \phi(x, t) t^{2s} dt, \qquad (8)$$

with $\phi(x, t)$ satisfying the conditions for (7), change their form as x varies in such a way that λ increases through a. Thus the integrals on the right of (7) for which $(2n-1)\pi > \lambda$ have a form different from those for which $(2n-1)\pi < \lambda$. This will be apparent in the illustration given in § 17, equations (8), (9), and (10).

12. *The series* $\sum_{n=1}^{\infty} (-)^n \phi(x, n) n^{-2s}$ $(s > 0)$ *and the proof of Theorem* 3, § 8.

For this investigation C, U, and L must be modified by inserting quadrantal indentations to exclude the origin. Denote the modified contours by C_1, U_1, and L_1. If $\phi(x, t)$ and λ satisfy the conditions of

Theorem 3, § 8, the series can be studied by considering

$$\int_{C_1} \left\{ \frac{\phi(x,t)}{\sinh \pi t} \frac{1}{t^{2s}} - p(t) \right\} dt + 2i \sum_{n=1}^{\rho} \int_{U_1} \left\{ \frac{\phi(x,t)}{t^{2s}} e^{(2n-1)\pi i t} - p_n(t) \right\} dt$$

$$- 2i \sum_{n=1}^{\rho} \int_{L_1} \left\{ \frac{\phi(x,t)}{t^{2s}} e^{-(2n-1)\pi i t} - q_n(t) \right\} dt, \quad (1)$$

where the contour round which an integral is taken is at the foot of the integral.

The function $p(t)$ is that portion of the principal part of $\dfrac{\phi(x,t)}{t^{2s}\sinh \pi t}$ necessary for the convergence of the first integral at the origin. Similar definitions exist for $p_n(t)$ and $q_n(t)$, and these ensure the convergence at the origin of those integrals of (1) in which they occur.

The arguments of § 10 . 1 serve to show that the contribution of (1) taken over the circular arcs of C_1, U_1, and L_1 tends to zero as $R \to \infty$ (where R is the radius of these arcs), provided that $p - 2s < 0$ [or in the case $(2\rho-1)\pi \leqslant \lambda \leqslant (2\rho+1)\pi$ provided that $p - 2s < -1$].

The residues due to the first integral are

$$\frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \phi(x,n)\, n^{-2s}, \quad (2)$$

and so this series can be expressed in terms of a finite series of integrals along the real and imaginary axes. The result, however, is extremely complicated.

When s is a positive integer, the arguments of § 10 . 1 show that all the integrals along the imaginary axis cancel one another. There is a further simplification in this case owing to the fact that the various integrals along the indentations, excluding the origin, are expressible in terms of the coefficient of t^{-1} in their integrands. The case $s = 1$ only will be considered in detail here.

Since $\phi(x,t)$ is an even function of t near the origin

$$\frac{\phi(x,t)}{t^2 \sinh \pi t} = \frac{a_0}{t^3} + \frac{a_1}{t} + \text{positive powers of } t. \quad (3)$$

Hence, in the limit, as the radius of the semi-circular indentation of C which excludes the origin tends to zero, the first integral of (1) contributes the amount $-\pi i a_1$ towards the total value of (1).

Again, since $\phi(x, t)$ is an even integral function of t, we have, near the origin,

$$\frac{\phi(x, t)}{t^2}\, e^{(2n-1)\pi i t}$$

$$= \frac{\phi(x, 0)}{t^2} + \frac{(2n-1)\pi i\, \phi(x, 0)}{t} + \text{a constant and positive powers of } t, \quad (4)$$

and also

$$\frac{\phi(x, t)}{t^2}\, e^{-(2n-1)\pi i t}$$

$$= \frac{\phi(x, 0)}{t^2} - \frac{(2n-1)\pi i\, \phi(x, 0)}{t} + \text{a constant and positive powers of } t. \quad (5)$$

Hence
$$p_n(t) = q_n(t) = \frac{\phi(x, 0)}{t^2}, \quad (6)$$

and, when the path of integration is the quadrantal indentation of U_1 which excludes the origin, we have, in the limit when the radius of this indentation tends to zero,

$$2i \int \left\{ \frac{\phi(x, t)}{t^2}\, e^{(2n-1)\pi i t} - \frac{\phi(x, 0)}{t^2} \right\} dt = -\tfrac{1}{2}\pi i \{ 2i(2n-1)\, \pi i\, \phi(x, 0) \}. \quad (7)$$

Also for the corresponding indentation of L_1 we have, in the limit,

$$-2i \int \left\{ \frac{\phi(x, t)}{t^2}\, e^{-(2n-1)\pi i t} - \frac{\phi(x, 0)}{t^2} \right\} dt$$

$$= -\tfrac{1}{2}\pi i(-2i) \{ -(2n-1)\, \pi i\, \phi(x, 0) \}. \quad (8)$$

We have thus accounted for the contributions of (1) due to all the portions of the contours C_1, U_1, and L_1, except the real axis. Substituting these values in (1) and summing the series of terms due to (7) and (8) we get

$$\sum_{n=1}^{\infty} (-1)^n \frac{\phi(x, n)}{n^2} = -\tfrac{1}{2}\pi a_1 + \rho^2 \pi^2\, \phi(x, 0)$$

$$+ 2 \sum_{n+1}^{\rho} \int_0^{\infty} \left\{ \frac{\phi(x, t)}{t^2} \cos(2n-1)\, \pi t - \frac{\phi(x, 0)}{t^2} \right\} dt, \quad (9)$$

where a_1 is the coefficient of t in $\phi(x, t)\, t^{-2s} \operatorname{cosec} \pi t$, and either

$$\left.\begin{aligned} &(2\rho-1)\pi < \lambda < (2\rho+1)\pi \quad \text{and} \quad p-2 < 0 \\ \text{or} \quad &(2\rho-1)\pi \leqslant \lambda \leqslant (2\rho+1)\pi \quad \text{and} \quad p-2 < -1 \end{aligned}\right\}, \tag{10}$$

ρ being any positive integer. When $0 < \lambda < \pi$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\phi(x, n)}{n^2} = -\tfrac{1}{2}\pi a_1. \tag{11}$$

In addition, it is not difficult to see that (9) can be written

$$\sum_{n=1}^{\infty} (-1)^n \frac{\phi(x, n)}{n^2} = -\tfrac{1}{2}\pi a_1 + \sigma^2\pi^2\phi(x, 0) + 2\sum_{n=1}^{\sigma} \int_0^{\infty} \left\{\frac{\phi(x, t)}{t^2} \cos(2n-1)\pi t - \frac{\phi(x, 0)}{t^2}\right\} dt \tag{12}$$

whenever $(2\rho-1)\pi < \lambda < (2\rho+1)\pi$, and σ is any positive integer not less than ρ.

For, by considering the integral

$$\int \left\{\frac{\phi(x, t)}{t^2}\, e^{(2n-1)\pi i t} - \frac{\phi(x, 0)}{t^2}\right\} dt \tag{13}$$

taken round an infinite semi-circle in the upper half of the t plane, and along the real axis with an indentation at the origin, it is easy to prove that, when $\lambda < (2n-1)\pi$,

$$2\int_0^{\infty} \left\{\frac{\phi(x, t)}{t^2} \cos(2n-1)\pi t - \frac{\phi(x, 0)}{t^2}\right\} dt = -(2n-1)^2\pi^2\phi(x, 0). \tag{14}$$

Hence (9) is true for all values of x such that $0 < \lambda < (2\rho+1)\pi$ (excluding, sometimes, values of x such that λ is an odd multiple of π).

13. *The proof of Theorem* 4, § 8.

For the proof of this theorem we consider the system of integrals

$$\int_C \frac{\phi(x, t)}{\cos \pi t}\, t^{2s-1}\, dt - 2\sum_{n=1}^{\rho} (-1)^{n-1} \int_U \phi(x, t)\, e^{(2n-1)\pi i t}\, t^{2s-1}\, dt - 2\sum_{n=1}^{\rho} (-1)^{n-1} \int_L \phi(x, t)\, e^{-(2n-1)\pi i t}\, t^{2s-1}\, dt, \tag{1}$$

where C, U, and L are the contours of § 9 . 1, and $\phi(x, t)$, λ, and p satisfy the conditions in the enunciation of the theorem.

The arguments of § 10 . 1 show that as R, the radius of the large arcs of C, U, and L, tends to infinity, the sum of the portions of the integrals of (1) taken over these arcs tends to zero whenever λ and p satisfy the conditions in the enunciation.

Thus the only relevant parts of these integrals are along the axes and over the indentation of C which excludes the origin. The expression of these in terms of the residues of the first integral of (1) gives

$$\sum_{n=1}^{\infty} (-1)^n \phi(x, n-\tfrac{1}{2})(n-\tfrac{1}{2})^{2s-1}$$

$$= -\sin s\pi \int_0^{\infty} \Phi(x, t)\, t^{2s} \left\{ \operatorname{sech} \pi t - 2 \sum_{n=1}^{p} (-1)^{n-1} e^{-(2n-1)\pi t} \right\} dt$$

$$-2 \sum_{n=1}^{p} (-1)^{n-1} \int_0^{\infty} \phi(x, t)\, t^{2s-1} \sin (2n-1)\pi t\, dt. \qquad (2)$$

where, as before, $$\phi(x, it) = \Phi(x, t). \qquad (3)$$

When we expand $\operatorname{sech} \pi t$ in powers of $e^{-\pi t}$, (2) assumes the form given in the enunciation of the theorem in § 8.

Now $\phi(x, t)$ satisfies the conditions for the truth of (8), § 4. Using this in (2), after expanding $\operatorname{sech} \pi t$ in powers of $e^{-\pi t}$, we get

$$\sum_{n=1}^{\infty} (-1)^{n-1} \phi(x, n-\tfrac{1}{2})(n-\tfrac{1}{2})^{2s-1}$$

$$= 2 \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^{\infty} \phi(x, t)\, t^{2s-1} \sin (2n-1)\pi t\, dt. \qquad (4)$$

Both (2) and the form of the theorem given in the enunciation hold whenever x is such that $(2\rho-1)\pi < \lambda < (2\rho+1)\pi$, while (4) holds for all values of x. (4), however, has disadvantages similar to those of (7), § 11, and the remarks following that equation apply entirely to this one, provided that $\cos at$ in (8), § 11, is replaced by $\sin at$.

When s is a positive integer, m say, other than zero, (2) evidently becomes

$$\sum_{n=1}^{\infty} (-1)^{n-1} \phi(x, n-\tfrac{1}{2})(n-\tfrac{1}{2})^{2m-1}$$

$$= 2 \sum_{n=1}^{p} (-1)^{n-1} \int_0^{\infty} \phi(x, t)\, t^{2m-1} \sin (2n-1)\pi t\, dt, \qquad (5)$$

provided that either $\lambda < (2\rho+1)\pi$ and $p+2m-1<0$, or $\lambda \leqslant (2\rho+1)\pi$ and $p+2m<0$.

14. *The series* $\sum_{n=1}^{\infty} (-1)^n \phi(x, n-\frac{1}{2})(n-\frac{1}{2})^{-2s-1}$ $(s \geqslant 0)$, *and the proof of Theorem* 5, § 8.

To investigate this series, consider the system of integrals

$$\int_C \left\{\frac{\phi(x,t)}{t^{2s+1}\cos \pi t} - p(t)\right\} dt - 2\sum_{n=1}^{\rho} (-1)^{n-1} \int_U \left\{\frac{\phi(x,t)\, e^{(2n-1)\pi i t}}{t^{2s+1}} - p_n(t)\right\} dt$$

$$-2\sum_{n=1}^{\rho} (-1)^{n-1} \int_L \left\{\frac{\phi(x,t)\, e^{-(2n-1)\pi i t}}{t^{2s+1}} - q_n(t)\right\} dt, \quad (1)$$

where C, U, and L are defined as in § 8, and $\phi(x,t)$, λ, and p satisfy the conditions in the enunciation of Theorem 5, § 8.

The functions $p_n(t)$ and $q_n(t)$ simply ensure the convergence of the integrals in which they occur at the origin (as in § 12).

The result is then obtained by using the methods of §§ 10 and 17, but it is extremely complicated, even when s is an integer. The result for the case $s = 0$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \phi(x, n-\tfrac{1}{2})$$

$$= (-1)^{\rho} \tfrac{1}{4}\pi \phi(x,0) + \sum_{n=1}^{\rho} (-1)^n \int_0^{\infty} \phi(x,t) \sin(2n-1)\pi t \frac{dt}{t}, \quad (2)$$

whenever either

$$(2\rho-1)\pi < \lambda < (2\rho+1)\pi \quad \text{and} \quad p<1,$$

or

$$(2\rho-1)\pi \leqslant \lambda \leqslant (2\rho+1)\pi \quad \text{and} \quad p<0.$$

It is worth observing that ρ can be replaced in (2) by any positive integer σ not less than ρ, when λ is as above, and the result still remains true. This follows from the result that, when $(2n-1)\pi > \lambda$,

$$\int_0^{\infty} \phi(x,t) \sin(2n-1)\pi t \frac{dt}{t} = \tfrac{1}{2}\pi \phi(x,0), \quad (3)$$

a special case of (5), § 4. Thus (2) is true as it stands for all values of ρ such that either $0 < \lambda < (2\rho+1)\pi$ and $p<1$, or $0 \leqslant \rho \leqslant (2\rho+1)\pi$ and $p<0$.

It must be observed also that, when $0 < \lambda < \pi$, (2) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \phi(x, n-\tfrac{1}{2}) = \tfrac{1}{4}\pi \phi(x, 0). \tag{4}$$

15. *Summability.*

I use the definitions of summability given in (i) and (ii) of § 6. In Theorem 2, § 8, for example, the series are never summable whenever x is such that λ is an odd multiple of π, and both sides then diverge. But when $(2\rho-1)\pi < \lambda < (2\rho+1)\pi$ and $p+2s > 0$, *i.e.* when the series on the left are divergent, they are summable $(R, n^2, p+2s+\epsilon)$ $(\epsilon > 0)$, provided that those integrals on the right-hand side which are of the type

$$\int_0^{\infty} \cos at \, \phi(x, t) \, t^{2s} \, dt \tag{1}$$

are summable $(R, t^2, p+2s+\epsilon)$. *If this is so, then Theorem* 2, § 8, *still remains true when* $(2\rho-1)\pi < \lambda < (2\rho+1)\pi$ *and* $p+2s > 0$, *provided that the series on the left is replaced by its sum* $(R, n^2, p+2s+\epsilon)$ *and the integrals on the right of the type* (1) *are replaced by their values when summed* $(R, t^2, p+2s+\epsilon)$.

The case when s is a positive integer m, gives the modification necessary for the truth of Theorem 1, § 8, when $p+2m > 0$.

If the integrals on the right of Theorem 3, § 8, are summable $(R, t^2, p-2+\epsilon)$, then the series on the left are summable $(R, n^2, p-2+\epsilon)$ whenever $p > 2$, *and Theorem* 3 *still remains true, provided that the integrals and series concerned therein are replaced by their sums obtained according to the mode of summability to be adopted.*

The modification of Theorem 4, § 8, for the case when $p+2s-1 > 0$ is the same as for Theorem 2, except that $2s$ must be replaced by $2s-1$ and integrals of the type

$$\int_0^{\infty} \sin at \, \phi(x, t) \, t^{2s-1} \, dt \tag{2}$$

must be summable $(R, t^2, p+2s-1+\epsilon)$.

Theorem 5 is also true when $p > 1$ and integrals of the type (2) are summable $(R, t^2, p-1+\epsilon)$, provided that the series are replaced by their sums $(R, n^2, p-1+\epsilon)$ and the integrals by their sums $(R, t^2, p-1+\epsilon)$.

The proofs of all these theorems are obtained by multiplying all the integrands used to prove Theorems 1 to 5, § 8, by the factor $\left(1-\frac{t^2}{R^2}\right)^{q+\epsilon}$, considering the new systems of integrals round C, U, and L (indented so as to avoid the point $t=R$), and then using the methods and arguments of §§ 6 and 7. The value of q for the extension of Theorem 2 is $p+2s$, and it is similarly found for the extensions of the other theorems.

I may add that if $\phi(x, t)$ satisfies a restriction somewhat narrower than (ii), § 9, namely that, when $|t|$ is sufficiently large,

$$\phi(x, t) = t^p \left\{ \sum_{n=1}^{\infty} \frac{a_n \cos(\lambda t + \alpha_n)}{t^n} + O\left(\frac{1}{t^m}\right) \right\}, \tag{3}$$

all the integrals mentioned above are summable $(R, t^2, q+\epsilon)$, where $q = p+2s$ for Theorem 2, &c.

16. *Applications to Gamma functions.*

From (3), § 5, $\{\Gamma(\alpha+xt)\,\Gamma(\alpha-xt)\}^{-1}$ is a function of the form $\phi(x, t)$ with $\lambda = x\pi$ and $p = 1-2\alpha$.

Now Ramanujan* has shown that

$$2\int_0^{\infty} \frac{\cos \beta t}{\Gamma(\alpha+xt)\,\Gamma(\alpha-xt)}\,dt = \frac{[2\cos(\beta/2x)]^{2\alpha-2}}{x\Gamma(2\alpha-1)}, \tag{1}$$

if $2\alpha > 1$ and $\beta < x\pi$. If $\beta > x\pi$, the value of the integral is zero.

Combining this with Theorem 1, § 8, we get

$$\frac{\frac{1}{2}}{\{\Gamma(\alpha)\}^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(\alpha+nx)\,\Gamma(\alpha-nx)} = \sum_{n=1}^{\rho} \frac{2^{2\alpha-2}}{x\Gamma(2\alpha+1)} \left\{\cos \frac{2n-1)\,\pi}{2x}\right\}^{2\alpha-2}, \tag{2}$$

provided that x is positive and either

$$2\rho-1 < x < 2\rho+1 \quad \text{and} \quad 1-2\alpha < 0,$$

or

$$2\rho-1 \leqslant x \leqslant 2\rho+1 \quad \text{and} \quad 1-2\alpha < -1,$$

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* *Quart. J. of Math.*, 48 (1920), 294-310.

ρ being a positive integer. When $0<x<1$, the series on the left of (2) has zero as its sum.

If α is an integer $m+1$, (2) becomes

$$\tfrac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi x}{nx\{1-(n^2x^2/1^2)\}\{1-(n^2x^2/2^2\}\dots\{1-(n^2x^2/m^2)\}}$$

$$=\frac{(m!)^2}{2m!}\frac{2^{2m}}{x}\sum_{n=1}^{\rho}\left\{\cos\frac{(2n-1)\pi}{2x}\right\}^{2m}, \qquad (3)$$

whenever either

$$2\rho-1<x<2\rho+1 \quad \text{and} \quad 2m+1>0,$$

or

$$2\rho-1\leqslant x\leqslant 2\rho+1 \quad \text{and} \quad m>0.$$

From (11), § 12, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2\Gamma(\alpha-nx)\,\Gamma(\alpha+nx)}=\frac{1}{2\{\Gamma(\alpha)\}^2}\left\{\frac{\pi^2}{6}-x^2\zeta(2,\alpha)\right\}, \qquad (4)$$

if either $0\leqslant x<1$ and $2\alpha>-1$, or $0\leqslant x\leqslant 1$ and $2\alpha>0$.

Differentiating (1) with respect to β, and combining the result with Theorem 4, § 8, when $s=1$, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{\Gamma\{\alpha+(2n-1)x\}\,\Gamma\{\alpha-(2n-1)x\}}$$

$$=\frac{1}{2x^2\Gamma(2\alpha-2)}\sum_{n=1}^{\rho}(-1)^{n-1}\left\{2\cos\frac{(2n-1)\pi}{4x}\right\}^{2\alpha-3}\sin\frac{(2n-1)\pi}{4x} \qquad (5)$$

if either $(2\rho-1)<2x<(2\rho+1)$ and $2-2\alpha<0$,

or $(2\rho-1)\leqslant 2x\leqslant 2\rho+1$ and $3-2\alpha<0$.

When $0<2x<1$, the series on the left has zero as its sum.

From (4), § 13, we get

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)\,\Gamma\{\alpha+(2n-1)x\}\,\Gamma\{\alpha-(2n-1)x\}}=\frac{\pi}{4\{\Gamma(\alpha)\}^2}, \qquad (6)$$

provided that either $0<2x<1$ and $\alpha>0$, or $0\leqslant 2x\leqslant 1$ and $2\alpha>1$. When α is a positive integer, the left-hand sides of (4), (5), and (6) assume forms similar to the left-hand side of (3).

It is easily deducible from (1) and (2), § 5, that

$$\{\Gamma(\alpha-xt)\,\Gamma(\alpha+xt)\}^{-1}$$

satisfies (3), § 14. Hence, when α is such that the series on the left of the above results diverge, they are summable $(R, n^2, q+\epsilon)$ for some value of q $(\epsilon > 0)$ to their right-hand sides. Thus the series in (2) is summable $(R, n^2, 1-2\alpha+\epsilon)$ to the right-hand side whenever $1-2\alpha \geq 0$, provided that x is not an odd integer. Similarly for the other equations.

17. *Applications to Bessel functions.*

From the asymptotic expansion of the Bessel function we can see that

$$\frac{J_\nu\{x\sqrt{(t^2+a^2)}\}}{\{\frac{1}{2}x\sqrt{(t^2+a^2)}\}^\nu} \tag{1}$$

satisfies the conditions of § 8, and is a function of the form $\phi(x, t)$ with $\lambda = x$ and $p = -\frac{1}{2}-\nu$. Also*

$$\int_0^\infty \cos bt\,\frac{J_\nu\{x\sqrt{(t^2+a^2)}\}}{(t^2+a^2)^{\frac{1}{2}\nu}}\,dt = \frac{(\frac{1}{2}\pi)^{\frac{1}{2}}}{x^\nu}\left(\frac{x^2-b^2}{a^2}\right)^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}}\{a\sqrt{(x^2-b^2)}\}, \tag{2}$$

when $x > b$. When $x < b$ the value of the integral is zero. Hence, from Theorem 1, § 8, we have

$$\frac{\frac{1}{2}}{\Gamma(\nu+1)} + \sum_{n=1}^{\infty}(-1)^n\,\frac{J_\nu\{x\sqrt{(n^2+a^2)}\}}{\{\frac{1}{2}x\sqrt{(n^2+a^2)}\}^\nu}$$

$$= \frac{(2\pi)^{\frac{1}{2}}}{x^\nu}\sum_{n=1}^{\rho}\left\{\frac{x^2-(2n-1)^2\pi^2}{a^2}\right\}^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}}[a\sqrt{\{x^2-(2n-1)^2\pi^2\}}]. \tag{3}$$

provided that either

$$(2\rho-1)\pi < x < (2\rho+1)\pi \quad \text{and} \quad 2m-\tfrac{1}{2}-\nu < 0,$$

or $$(2\rho-1)\pi \leqslant x \leqslant (2\rho+1)\pi \quad \text{and} \quad 2m+\tfrac{1}{2}-\nu < 0.$$

If $(2\rho-1)\pi < x < (2\rho+1)\pi$ and $2m-\frac{1}{2}-\nu > 0$, the series on the left is summable $(R, n^2, 2m-\frac{1}{2}-\nu+\epsilon)$ to the right-hand side.

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* This result is due to Sonine. See Watson, *loc. cit.*, 415, Equation (1).

The special case $a = 0$ gives a result due to Nielsen*.

Combining Schafheitlin's theorem†

$$\int_0^\infty J_{2m+\frac{1}{2}}(x) \cos \mu x \frac{dx}{\sqrt{x}} = (-1)^m (\tfrac{1}{2}\pi)^{\frac{1}{2}} P_{2m}(\mu) \tag{4}$$

(which holds when m is a positive integer or zero) with Theorem 1, § 8, we get, when m is a positive integer other than zero,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} J_{2m+\frac{1}{2}}(nx) = (-1)^m \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} \sum_{n=1}^{\rho} P_{2m}\left\{\frac{(2n-1)\pi}{x}\right\}, \tag{5}$$

if $(2\rho-1)\pi < x < (2\rho+1)\pi$. When $0 < x < \pi$ the series on the left has zero as its sum. The functions on the right are the usual Legendre functions. When $m = 0$, the value of $\frac{1}{2}a$ in the enunciation of Theorem 1, § 8, is $\frac{1}{2}x$, and this must then be added to the left-hand side.

From (10), § 5, it can be seen that $J_{\nu+xt}(a) J_{\nu-xt}(a)$ is a function of the form $\phi(x, t)$ with $\lambda = \pi x$ and $p = -2\nu-1$, provided that x is real. Now Ramanujan‡ has shown that

$$\int_0^\infty J_{\nu+xt}(a) J_{\nu-xt}(a) \cos bt \, dt = \frac{1}{2x} J_{2\nu}\left(2a \cos \frac{b}{2x}\right), \tag{6}$$

whenever $-\pi x < b < \pi x$. If $|b| > \pi |x|$ the integral is zero. From Theorem 1, § 8, we deduce that, when $x > 0$,

$$\tfrac{1}{2}\{J_\nu(a)\}^2 + \sum_{n=1}^{\infty} (-1)^n J_{\nu+nx}(a) J_{\nu-nx}(a) = \sum_{n=1}^{\rho} \frac{1}{2x} J_{2\nu}\left\{2a \cos \frac{(2n-1)\pi}{2x}\right\}, \tag{7}$$

if either $(2\rho-1) < x < (2\rho+1)$ and $\nu > -\frac{1}{2}$, or $(2\rho-1) \leqslant x \leqslant (2\rho+1)$ and $\nu > 0$. If $(2\rho-1) < x < (2\rho+1)$ and $2\nu+1 < 0$, the series on the left is summable $(R, n^2, -2\nu-1+\epsilon)$ to the right-hand side. When $0 < x < 1$, the series on the left has zero as its sum, a result which has recently been discovered by Wilton§. Dividing through by $a^{2\nu}$ and making $a = 0$, we get (2), § 16.

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* Watson, *loc. cit.*, 636, Equation (4).

† See Whittaker, *Proc. London Math. Soc.* (1), 35 (1903), 198–206.

‡ S. Ramanujan, *loc. cit.*, 304, Equation (5 . 41).

§ *Proc. London Math. Soc.* (2), 23 (1925), Records. Wilton's null series is more general than the case above when $0 < x < 1$.

We shall apply Theorem 2, § 8, in the form (7), § 11, in order to illustrate the remarks that follow the latter equation.

From its asymptotic expansion, $J_\nu(xt)(\frac{1}{2}xt)^{-\nu}$ is a function of the form $\phi(x,t)$ with $\lambda = x$ and $p = -\nu - \frac{1}{2}$. Also it is known that*

$$\int_0^\infty \cos at \frac{J_\nu(xt)}{(\frac{1}{2}xt)^\nu} t^{2s}\, dt = \left(\frac{2}{x}\right)^{2s} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(\nu-s+\frac{1}{2})} {}_2F_1\left(s+\tfrac{1}{2},\ s-\nu+\tfrac{1}{2};\ \tfrac{1}{2};\ \frac{a^2}{x^2}\right) \quad (8)$$

when $a < x$; while, when $a > x$, the value of the integral is

$$-\frac{\sin s\pi}{a^{2s+1}} \frac{\Gamma(2s+1)}{\Gamma(\nu+1)} {}_2F_1\left(s+\tfrac{1}{2},\ s+1;\ \nu+1;\ \frac{x^2}{a^2}\right). \quad (9)$$

Combining these with (7), § 11, we get

$$\sum_{n=1}^{\infty} (-1)^n \frac{J_\nu(nx)}{(\frac{1}{2}nx)^\nu} n^{2s}$$

$$= 2\left(\frac{2}{x}\right)^{2s} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(\nu-s+\frac{1}{2})} \sum_{n=1}^{\rho} {}_2F_1\left(s+\tfrac{1}{2},\ s-\nu+\tfrac{1}{2};\ \tfrac{1}{2};\ \frac{(2n-1)^2\pi^2}{x^2}\right)$$

$$- \frac{\sin s\pi}{(2n-1)\pi^{2s+1}} \frac{\Gamma(2s+1)}{\Gamma(\nu+1)}$$

$$\times \sum_{n=\rho+1}^{\infty} {}_2F_1\left(s+\tfrac{1}{2},\ s+1;\ \nu+1;\ \frac{x^2}{(2n-1)^2\pi^2}\right), \quad (10)$$

if either $(2\rho-1)\pi < x < (2\rho+1)\pi$ and $2s-\nu-\frac{1}{2} < 0$,

or $(2\rho-1)\pi \leqslant x \leqslant (2\rho+1)\pi$ and $2s-\nu-\frac{1}{2} < -1$,

When $(2\rho-1)\pi < x < (2\rho+1)\pi$ and $2s-\nu-\frac{1}{2} > 0$, the series on the left is summable $(R, n^2, 2s-\frac{1}{2}+\epsilon)$ to the right-hand side.

I shall illustrate Theorem 3, § 8, only in the simplest case. $J_0(xt)$ is a function of the form $\phi(x,t)$ with $\lambda = x$ and $p = -\frac{1}{2}$. Since†

$$\int_0^\infty J_0(xt) \frac{\sin bt}{t}\, dt = \arcsin \frac{b}{x} \quad (b < x)$$

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* Watson, *loc. cit.*, Equations (2) and (3), 401.

† *Ibid.*, Equation (2), 405.

is an integral uniformly convergent with respect to b, on integrating under the integral sign we get

$$-\int_0^\infty \left(\frac{\cos bt}{t^2}-\frac{1}{t^2}\right) J_0(xt)\,dt = b \arcsin\frac{b}{x}+(x^2-b^2)^{\frac{1}{2}}-x, \qquad (11)$$

when $b<x$. Again* $$\int_0^\infty \frac{J_1(xt)}{t}\,dt = 1,$$

when $x>0$, an integral uniformly convergent with respect to x. Integrating with respect to x, we get

$$-\int_0^\infty \left\{\frac{J_0(xt)}{t^2}-\frac{1}{t^2}\right\} dt = x. \qquad (12)$$

Adding (11) and (12) we find that, when $b<x$,

$$\int_0^\infty \left\{\frac{\cos bt}{t^2}\,J_0(xt)-\frac{1}{t^2}\right\} dt = -b \arcsin\frac{b}{x}-(x^2-b^2)^{\frac{1}{2}}. \qquad (13)$$

Combining this with Theorem 3, § 8, we get

$$\sum_{n=1}^{\infty}(-1)^n\frac{J_0(nx)}{n^2} = \frac{x^2}{4}-\frac{\pi^2}{12}+\rho^2\pi^2-2\sum_{n=1}^{\rho}\left\{(2n-1)\pi \arcsin\frac{(2n-1)\pi}{n}\right.$$

$$\left.+\sqrt{\left(1-\frac{(2n-1)^2\pi^2}{x^2}\right)}\right\}, \qquad (14)$$

whenever $(2\rho-1)\pi \leqslant x \leqslant (2\rho+1)\pi$.

To illustrate Theorem 4, § 8, we use the formula†

$$\int_0^\infty \sin bt\,\frac{J_\nu\{x\sqrt{(t^2+a^2)}\}}{(t^2+a^2)^{\frac{1}{2}\nu}}\,t\,dt$$

$$= (\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{b}{x^\nu}\left(\frac{x^2-b^2}{a^2}\right)^{\frac{1}{2}\nu-\frac{3}{4}} J_{\nu-\frac{3}{2}}\{a\sqrt{(x^2-b^2)}\}, \qquad (15)$$

when $x>b$ (if $x<b$ the value of the integral is zero), and we get from

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* Watson, *loc. cit.*, Equation (2), 405.

† *Ibid.*, Equation (1), 415.

Theorem 4, § 8,

$$\sum_{n=1}^{\infty}(-1)^{n-1}(n-\tfrac{1}{2})\frac{J_\nu[x\sqrt{\{a^2+(n-\frac{1}{2})^2\}}]}{\{a^2+(n-\frac{1}{2})^2\}^{\frac{1}{2}\nu}}$$

$$=\sqrt{(2\pi)}\frac{1}{x^\nu}\sum_{n=1}^{\rho}(-1)^{n-1}(2n-1)\,\pi\left\{\frac{x^2-(2n-1)^2\pi^2}{a^2}\right\}^{\frac{1}{4}\nu-\frac{1}{2}}$$

$$\times J_{\nu-\frac{1}{2}}[a\sqrt{\{x^2-(2n-1)^2\pi^2\}}], \quad (16)$$

if $(2\rho-1)\pi<x<(2\rho+1)\pi$ and $2\nu>1$, or $(2\rho-1)\pi\leqslant x\leqslant(2\rho+1)\pi$ and $2\nu>3$. Whenever $(2\rho-1)\pi<x<(2\rho+1)\pi$ and $2\nu<1$, the series on the left is summable $(R, n^2, 1-2\nu+\epsilon)$ to the right-hand side.

On differentiating Ramanujan's result (6) above with respect to b and using Theorem 4, § 8, we find that

$$\sum_{n=1}^{\infty}(-1)^{n-1}(n-\tfrac{1}{2})\,J_{\nu+(n-\frac{1}{2})x}(a)\,J_{\nu-(n-\frac{1}{2})x}(a)$$

$$=\frac{a}{2x^2}\sum_{n=1}^{\rho}(-1)^{n-1}\sin\frac{(2n-1)\pi}{2x}$$

$$\times\left\{J_{2\nu-1}\left(2a\cos\frac{(2n-1)\pi}{x}\right)-J_{2\nu+1}\left(2a\cos\frac{(2n-1)\pi}{x}\right)\right\} \quad (17)$$

with conditions as for (7) above.

An illustration to Theorem (5), § 8, is

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\,J_{\nu+(n-\frac{1}{2})x}(a)\,J_{\nu-(n-\frac{1}{2})x}(a)=\tfrac{1}{4}\pi J_{\nu-\frac{1}{2}x}(a)\,J_{\nu+\frac{1}{2}x}(a), \quad (18)$$

if either $0<x<1$ and $2\nu>1$, or $0\leqslant x\leqslant 1$ and $\nu>1$. If $0<x<1$ and $2\nu<1$, the series on the left is summable $(R, n^2, 1-2\nu+\epsilon)$ to the right-hand side.

18. *Applications to Legendre functions.*

From (6) and (7 . 1), § 5, it can be seen that $P_{xt-\frac{1}{2}}(\cos\phi)$ is a function of the form $\phi(x,t)$ with $\lambda=x\phi$ and $p=-\frac{1}{2}$. An application of

Fourier's integral transform to the well known Mehler-Dirichlet* integral for the Legendre function gives

$$\int_{-\infty}^{\infty} P_{t-\frac{1}{2}}(\cos\phi)\cos at\,dt = \frac{1}{\{2(\cos a-\cos\phi)\}^{\frac{1}{2}}}, \tag{1}$$

when $a<\phi$. When $a>\phi$ the value of the integral is zero. Hence from Theorem 1, § 8, we get, when $0<\phi<\pi$,

$$\tfrac{1}{2}P_{-\frac{1}{2}}(\cos\phi)+\sum_{n=1}^{\infty}(-1)^n P_{nx-\frac{1}{2}}(\cos\phi) = \frac{1}{x\sqrt{2}}\sum_{n=1}^{\rho}\left\{\cos\frac{(2n-1)\pi}{x}-\cos\phi\right\}^{-\frac{1}{2}} \tag{2}$$

provided that $(2\rho-1)\pi<x\phi<(2\rho+1)\pi$. When $0<x\phi<\pi$, the series on the left has zero as its sum and is the null series (3) of the introduction.

Similarly it can be shown that

$$\tfrac{1}{2}P^m_{-\frac{1}{2}}(\cos\phi)+\sum_{n=1}^{\infty}(-1)^n P^m_{nx-\frac{1}{2}}(\cos\phi)=0, \tag{3}$$

if $0<\phi<\pi$, $0<x\phi<\pi$ and $m<\frac{1}{2}$.

Differentiating (1) above with respect to b and using Theorem 4, § 8, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)\,P_{x(n-\frac{1}{2})-\frac{1}{2}}(\cos\phi) = \frac{1}{x^2\sqrt{2}}\sum_{n=1}^{\rho}\frac{(-1)^n\sin\{(2n-1)/x\}}{[\cos\{(2n-1)\pi/x\}-\cos\phi]^{\frac{3}{2}}}, \tag{4}$$

the divergent series on the left being summable $(R,\ n^2,\ \frac{1}{2}+\epsilon)$ to the right-hand side if $(2\rho-1)\pi<x\phi<(2\rho+1)\pi$.

Finally, from Theorem 5, § 8, we get

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{P_{x(n-\frac{1}{2})-\frac{1}{2}}(\cos\phi)}{2n-1} = \tfrac{1}{4}\pi P_{-\frac{1}{2}}(\cos\phi), \tag{5}$$

whenever $0<\phi<\pi$ and $0\leqslant x\phi<\pi$.

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* Whittaker and Watson, *loc. cit.*, 315.

19. *A general deduction.*

If $\phi(x, t)$ is given by the formula

$$\phi(x, t) = \int_a^b f(\xi) \cos(xt\xi)\, d\xi, \tag{1}$$

an application of Fourier's integral theorem shows that

$$\frac{1}{x} f\left(\frac{\xi}{x}\right) = \frac{2}{\pi} \int_{-\infty}^{\infty} \phi(x, t) \cos(\xi t)\, dt, \tag{2}$$

provided that $ax < \xi < bx$. If x and ξ are positive and ξ lies outside these limits, the value of the integral on the right of (2) is zero.

If $f(\xi)$ is of limited total fluctuation throughout the interval (a, b), it is not difficult to prove that, if $t = Re^{i\theta}$,

$$|\phi(x, t)| < KR^{-1} \exp(bxR|\sin\theta|), \tag{3}$$

when R is sufficiently large, where K is a constant independent of θ, and $-\pi < \theta < \pi$. Hence $\phi(x, t)$ satisfies the conditions of § 8 with $\lambda = bx$ and $p = -1$. Hence from Theorem 1, § 8, we get*

$$\tfrac{1}{2}\phi(0) + \sum_{n=1}^{\infty} (-1)^n \phi(nx) = \frac{\frac{1}{2}\pi}{x} \sum_{n=\sigma}^{\rho} f\left((2n-1)\frac{\pi}{x}\right), \tag{4}$$

where ρ and σ are the smallest positive integers such that

$$bx < (2\rho+1)\pi \quad \text{and} \quad ax < (2\sigma-1)\pi.$$

20. *A converse of these methods.*

If $\phi(x, t)$ satisfies conditions (i) and (ii) of § 8, obviously a product of such functions will also do so. Hence from (6), (7), and (7 . 1), § 5,

$$P_{t-\frac{1}{2}}(\cos\alpha)\, P_{t-\frac{1}{2}}(\cos\beta)\, P_{t-\frac{1}{2}}(\cos\gamma)$$

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* This includes a result given by Nielsen as a special case, *Annali di Mat.* (3), 6 (1901), 307.

is of the form $\phi(x, t)$ when α, β, and γ lie between 0 and π, with

$$\lambda = \alpha+\beta+\gamma \quad \text{and} \quad p = -\tfrac{3}{2}.$$

Now Dougall* has shown that

$$\pi \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)\, P_{n-1}(\cos\alpha)\, P_{n-1}(\cos\beta)\, P_{n-1}(\cos\gamma)$$

$$= \{-\cos\tfrac{1}{2}(\alpha+\beta+\gamma)\cos\tfrac{1}{2}(\alpha+\beta-\gamma)$$

$$\times \cos\tfrac{1}{2}(\alpha-\beta+\gamma)\cos\tfrac{1}{2}(-\alpha+\beta+\gamma)\}^{-\frac{1}{2}}, \quad (1)$$

provided that the expression on the right is real. Otherwise the sum of the series on the left is zero.

We can also express the sum of the series on the left of (1) by Theorem 4, § 8, with $s = 1$. Hence we get

$$4\pi \int_0^{\infty} P_{t-\frac{1}{2}}(\cos\alpha)\, P_{t-\frac{1}{2}}(\cos\beta)\, P_{t-\frac{1}{2}}(\cos\gamma)\, t \sin\pi t\, dt$$

$$= \{-\cos\tfrac{1}{2}(\alpha+\beta+\gamma)\cos\tfrac{1}{2}(\alpha+\beta-\gamma)$$

$$\times \cos\tfrac{1}{2}(\alpha-\beta+\gamma)\cos\tfrac{1}{2}(-\alpha+\beta+\gamma)\}^{-\frac{1}{2}}, \quad (2)$$

provided that α, β, and γ lie between 0 and π, $\pi < \alpha+\beta+\gamma < 3\pi$, and the expression on the right is real. If $0 < \alpha+\beta+\gamma < \pi$, or if the right-hand side of (2) is not real, the value of the integral is zero.

In particular, when $\alpha = \beta = \gamma = \frac{1}{2}\pi$, (2) becomes

$$\int_0^{\infty} \frac{t \sin\pi t}{\{\Gamma(\frac{3}{4}+\frac{1}{2}t)\,\Gamma(\frac{3}{4}-\frac{1}{2}t)\}^3}\, dt = \frac{1}{2\pi^{\frac{5}{2}}}. \quad (3)$$

21. *The expansion of zero as a Schlömilch series.*

If it is required to expand $f(x)$ in the interval 0 to π as a Schlömilch series in the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n J_0(nx), \quad (1)$$

* *Proc. Edin. Math. Soc.*, 37 (1919), 33–47.

the coefficients a_n are given by

$$a_0 = 2f(0)+\frac{2}{\pi}\int_0^{\pi}\int_0^{\frac{1}{2}\pi} uf'(u\sin\phi)\,d\phi\,du, \tag{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\int_0^{\frac{1}{2}\pi} uf'(u\sin\phi)\cos nu\,d\phi\,du, \tag{3}$$

and the expansion is valid in the interval $(0, \pi)$, provided that $f'(x)$ is continuous and has limited total fluctuations in the closed interval $(0, \pi)$. Nielsen's well known expansion of zero as a Schlömilch series

$$0 = \tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^n J_0(nx) \tag{4}$$

has been obtained, however, by indirect methods, such as integrating trigonometrical series. The object of this paragraph is to show that (4) can be obtained directly by the use of (1), (2), and (3), and a limiting process.

The method* consists in showing from (2) and (3) that

$$\tfrac{1}{2}\frac{1}{\Gamma(\nu+1)}+\sum_{n=1}^{\infty}(-1)^n\frac{J_\nu(ny)}{(\frac{1}{2}ny)^\nu}J_0(nx) \tag{5}$$

is its own Schlömilch expansion. In my previous paper I have shown that, if $0<x+y<\pi$, the sum of this series is zero. Hence, on making $y\to 0$, (5) has the value zero in the range $0<x<\pi$. Also on making $y\to 0$, (5) becomes the right-hand side of (4) divided by $\Gamma(\nu+1)$.

On expanding (5) as a Schlömilch series, we find, from (3), that the coefficients are given by

$$a_n = -\frac{2}{\pi}\int_0^{\pi}\int_0^{\frac{1}{2}\pi} u\cos nu\sum_{n=1}^{\infty}(-1)^n\frac{J_\nu(ny)}{(\frac{1}{2}ny)^\nu}J_1(nu\sin\phi)\,n\,d\phi\,du, \tag{6}$$

the last series being uniformly convergent with respect to ϕ in the range $(0, \frac{1}{2}\pi)$, if, as will be assumed, $\nu>1$. But

$$\int_0^{\frac{1}{2}\pi} J_1(nu\sin\phi)\,d\phi = \frac{1-\cos nu}{nu}. \tag{7}$$

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* I am much indebted to a referee for suggesting the use of (5) instead of the sum of (5) as expressed by Theorem 1, § 8, which was my original method.

Hence, changing the order of integration and summation,

$$a_n = -\frac{2}{\pi}\int_0^\pi u\cos nu \sum_{n=1}^{\infty}(-1)^n \frac{J_\nu(ny)}{(\frac{1}{2}ny)^\nu}\frac{1-\cos nu}{nu}\, n\, du \qquad (8)$$

$$= (-1)^n \frac{J_\nu(ny)}{(\frac{1}{2}ny)^\nu}.$$

$$a_0 = \frac{1}{\Gamma(\nu+1)} + 2\sum_{n=1}^{\infty}(-1)^n \frac{J_\nu(ny)}{(\frac{1}{2}ny)^\nu} - \frac{2}{\pi}\int_0^\pi u \sum_{n=1}^{\infty}(-1)^n \frac{J_\nu(ny)}{(\frac{1}{2}ny)^\nu}\frac{1-\cos nu}{nu}\, n\, du$$

from (2) and (8), and so

$$a_0 = \frac{1}{\Gamma(\nu+1)}.$$

Hence (5) is its own Schlömilch series, and, since it is uniformly convergent for all values of y (zero included when $\nu > 1$), (4) follows if we make $y \to 0$.

THE THIRD ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION POSSESSING REAL, DISTINCT, AND LINEAR CHARACTERISTICS

By C. A. STEWART

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1. It has been found necessary in connexion with the subject of the present article to introduce an equation of the third order in three independent variables of the following type :—

$$(1.1)\quad V_{uvw}+A_1V_{vw}+A_2V_{wu}+A_3V_{uv}+B_1V_u+B_2V_v+B_3V_w+CV=0,$$

where A_r, B_s, C are functions of u, v, w.

By Cauchy's existence theorem, this equation possesses a unique solution in the vicinity of a point Q of a given boundary surface $F(u, v, w) = 0$, if we are given the values of $V(=E_{000}^{(u,\,v,\,w)})$, $V_u(=E_{100}^{(u,\,v,\,w)})$, $V_{uu}(=E_{200}^{(u,\,v,\,w)})$ at Q.* The conditions under which the theorem is valid are of two kinds :—(i) conditions associated with the functions A_r, B_s, C, F, E_t; (ii) conditions associated with the characteristics. With regard to conditions (i), it will be sufficient for our present purpose to assume that the functions concerned are analytic in the neighbourhood of Q.

2. *Conditions* (ii).

A characteristic of (1.1) is $G(u, v, w) = 0$, if G satisfies the equation $G_uG_vG_w = 0$, and therefore the characteristics consist of the three aggregates of cylinders, whose generators are parallel respectively to the coordinate axes. We shall therefore restrict Q to be any point in a portion σ of the boundary, within which none of the functions F_u, F_v, F_w vanishes. Let $P(u_0, v_0, w_0)$ be a point not on the boundary, but near Q

* The theorem is usually stated with reference to a plane boundary. A punctual transformation gives the above form, due regard being paid to conditions (ii).—J. Hadamard, *Lectures on Cauchy's problem* (1915), ch. 1, § 3.

(Fig. 1). Draw lines through P parallel to the axes to meet the boundary at A, B, C. P can be taken sufficiently near Q to ensure that A, B, C lie on σ. Since F_u, F_v, F_w do not vanish in σ, the points A, B, C so specified are unique*. The planes PAB, PBC, PCA are characteristics. If we know the values of $V(=E_{000})$, $V_u(=E_{100})$, $V_{uu}(=E_{200})$, we can calculate the remaining first and second order derivatives $V_v, \ldots, V_{vw}$. Denote these by $E_{010}, \ldots, E_{002}$. Then the following equations (a, b, c, d) must be

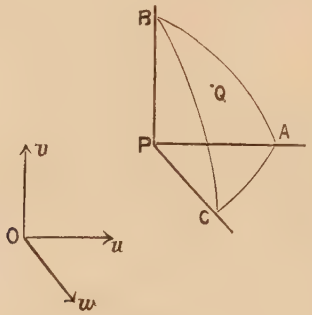

Fig 1.

satisfied throughout A, B, C on the boundary for all du, dv, dw satisfying equation (e).

$$(2.1)\quad \begin{cases} (a)\ \left(\dfrac{\partial E_{000}}{\partial u}-E_{100}\right)du+\left(\dfrac{\partial E_{000}}{\partial v}-E_{010}\right)dv+\left(\dfrac{\partial E_{000}}{\partial w}-E_{001}\right)dw=0 \\ (b)\ \left(\dfrac{\partial E_{100}}{\partial u}-E_{200}\right)du+\left(\dfrac{\partial E_{100}}{\partial v}-E_{110}\right)dv+\left(\dfrac{\partial E_{100}}{\partial w}-E_{101}\right)dw=0 \\ (c)\ \left(\dfrac{\partial E_{010}}{\partial u}-E_{110}\right)du+\left(\dfrac{\partial E_{010}}{\partial v}-E_{020}\right)dv+\left(\dfrac{\partial E_{010}}{\partial w}-E_{011}\right)dw=0 \\ (d)\ \left(\dfrac{\partial E_{001}}{\partial u}-E_{101}\right)du+\left(\dfrac{\partial E_{001}}{\partial v}-E_{011}\right)dv+\left(\dfrac{\partial E_{001}}{\partial w}-E_{002}\right)dw=0 \\ (e)\ F_u\ du+F_v\ dv+F_w\ dw=0. \end{cases}$$

From the eight equations determined from (2.1), seven, suitably chosen, will determine E_{010}, E_{001}, E_{110}, E_{101}, E_{020}, E_{011}, E_{002} in terms of E_{000}, E_{100}, E_{200}; and the remaining equation is identically satisfied. If, then, we are *given* the *ten* functions $E_{000}, \ldots, E_{002}$ and they satisfy (2.1), we can say that the problem is "correctly set"†.

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* By the Implicit Function Theorem.

† J. Hadamard, *Lectures on Cauchy's problem* (1915), ch. 1, § 1.

3. *A particular set of conditions.*

Let the boundary be the cylinder

$$\psi(x_0+u+v+w,\quad y_0+k_1u+k_2v+k_3w)=0, \tag{3.1}$$

where x_0, y_0, k_1, k_2, k_3 are real constants, and $(k_2-k_3)(k_3-k_1)(k_1-k_2)\neq 0$. This last condition ensures that no generator is parallel to a coordinate axis. The further conditions $\psi_v(=\psi_x+k_1\psi_y)\neq 0$, $\psi_v(=\psi_x+k_2\psi_y)\neq 0$, $\psi_w(=\psi_x+k_3\psi_y)\neq 0$ (where $x=x_0+u+v+w$, $y=y_0+k_1u+k_2v+k_3w$) ensure that no tangent plane in ABC is parallel to a coordinate plane. Again, $\psi(x, y)=0$ represents a curve in an (x, y) plane, and our conditions become: (*a*) $y=k_1x$, $y=k_2x$, $y=k_3x$ are real distinct lines; (*b*) at no point of the portion of the curve under consideration is the tangent parallel to any of the lines given in (*a*).

If possible, let the *ten* boundary functions be

$$E_{000}=\phi_{000}(x, y).$$

$$E_{100}=\phi_{10}(x, y)+k_1\phi_{01}(x, y);\ E_{010}=\phi_{10}+k_2\phi_{01};\ E_{001}=\phi_{10}+k_3\phi_{01}.$$

$$E_{200}=\phi_{20}(x, y)+2k_1\phi_{11}(x, y)+k_1^2\phi_{02}(x, y);\ E_{110}=\phi_{20}+(k_1+k_2)\phi_{11}+k_1k_2\phi_{02},$$

with obvious expressions for E_{101}, E_{020}, E_{011}, E_{002}.

These functions satisfy (2.1) if

$$(a)\quad \left(\frac{\partial\phi_{00}}{\partial x}-\phi_{10}\right)dx+\left(\frac{\partial\phi_{00}}{\partial y}-\phi_{01}\right)dy=0,$$

$$(b)\quad \left\{\left(\frac{\partial\phi_{10}}{\partial x}-\phi_{20}\right)dx+\left(\frac{\partial\phi_{10}}{\partial y}-\phi_{11}\right)dy\right\}+k_1\left\{\left(\frac{\partial\phi_{01}}{\partial x}-\phi_{11}\right)dx+\left(\frac{\partial\phi_{01}}{\partial y}-\phi_{02}\right)dy\right\}=0,$$

$$(c)\quad \left\{\left(\frac{\partial\phi_{10}}{\partial x}-\phi_{20}\right)dx+\left(\frac{\partial\phi_{10}}{\partial y}-\phi_{11}\right)dy\right\}+k_2\left\{\left(\frac{\partial\phi_{01}}{\partial x}-\phi_{11}\right)dx+\left(\frac{\partial\phi_{01}}{\partial y}-\phi_{02}\right)dy\right\}=0,$$

$$(d)\quad \left\{\left(\frac{\partial\phi_{10}}{\partial x}-\phi_{20}\right)dx+\left(\frac{\partial\phi_{10}}{\partial y}-\phi_{11}\right)dy\right\}+k_3\left\{\left(\frac{\partial\phi_{01}}{\partial x}-\phi_{11}\right)dx+\left(\frac{\partial\phi_{01}}{\partial y}-\phi_{02}\right)dy\right\}=0,$$

$$(e)\quad \psi_x dx+\psi_y dy=0,$$

and these are obviously satisfied if ϕ_{00}, ϕ_{10}, ϕ_{01}, ϕ_{20}, ϕ_{11}, ϕ_{02} are the *six* boundary functions of a correctly set problem belonging to a third order equation in the two variables x, y with a boundary $\psi(x, y) = 0$.

If, in addition, the coefficients A_1, A_2, A_3, B_1, B_2, B_3, C of equation (1 . 1) contain the variables in the particular combinations

$$x_0+u+v+w, \quad y_0+k_1u+k_2v+k_3w,$$

we can show that the required solution V also contains them in these combinations. For, by the existence theorem, the third order equation

$$\begin{aligned}(3\,.\,2)\quad &\left(\frac{\partial}{\partial x}+k_1\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+k_2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+k_3\frac{\partial}{\partial y}\right)W\\ &+A_1(x, y)\left(\frac{\partial}{\partial x}+k_2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+k_3\frac{\partial}{\partial y}\right)W\\ &+A_2(x, y)\left(\frac{\partial}{\partial x}+k_3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+k_1\frac{\partial}{\partial y}\right)W\\ &+A_3(x, y)\left(\frac{\partial}{\partial x}+k_1\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+k_2\frac{\partial}{\partial y}\right)W\\ &+B_1(x, y)\left(\frac{\partial}{\partial x}+k_1\frac{\partial}{\partial y}\right)W+B_2(x, y)\left(\frac{\partial}{\partial x}+k_2\frac{\partial}{\partial y}\right)W\\ &+B_3(x, y)\left(\frac{\partial}{\partial x}+k_3\frac{\partial}{\partial y}\right)W+C(x, y)\,W=0\end{aligned}$$

possesses a unique solution $W(x, y)$ if, on the boundary $\psi(x, y) = 0$, $W = \phi_{00}(x, y)$; $W_x = \phi_{10}$; $W_y = \phi_{01}$; $W_{xx} = \phi_{20}$; $W_{xy} = \phi_{11}$; $W_{yy} = \phi_{02}$; and therefore $W(x_0+u+v+w, \ y_0+k_1u+k_2v+k_3w)$ is the required unique solution of (1 . 1), since on

$$\psi(x_0+u+v+w, \ y_0+k_1u+k_2v+k_3w) = 0,$$

$$W = \phi_{00}, \quad W_u = \phi_{10}+k_1\phi_{01}; \quad \&c.$$

The solution we are seeking is then a *common* solution of the equations

$$(1\,.\,1)\quad V_{uvw}+A_1V_{vw}+A_2V_{wu}+A_3V_{uv}+B_1V_u+B_2V_v+B_3V_w+CV=0,$$

$$(3\,.\,3)\qquad (k_2-k_3)V_u+(k_3-k_1)V_v+(k_1-k_2)V_w=0.$$

The second of these equations, however, is automatically satisfied because A_1, ..., C and the boundary conditions contain the variables in the combinations $x_0+u+v+w$, $y_0+k_1u+k_2v+k_3w$. It can be used, in

combination with (1 . 1), if necessary, to simplify the analysis of the actual integration.

The solution which will be taken finally is the value of V at 0, 0, 0. It is now a function of x_0, y_0 only and satisfies (3 . 2) for $x = x_0$, $y = y_0$ and corresponding boundary conditions on the curve $\psi(x_0, y_0) = 0$.*

4. *The integration of* (1 . 1).

A method of integrating (1 . 1) will now be established which is analogous to that of Riemann for the second order hyperbolic equation.

$$\text{If } A \equiv \frac{\partial^3(\lambda V)}{\partial u \partial v \partial w} - \frac{\partial^2}{\partial v \partial w}\{(\lambda_u - A_1\lambda)V\} - \frac{\partial^2}{\partial w \partial u}\{(\lambda_v - A_2\lambda)V\}$$

$$- \frac{\partial^2}{\partial u \partial v}\{(\lambda_w - A_3\lambda)V\} + \frac{\partial}{\partial u}(\lambda_1 V) + \frac{\partial}{\partial v}(\lambda_2 V) + \frac{\partial}{\partial w}(\lambda_3 V),$$

where
$$\lambda_1 = \lambda_{vw} - \frac{\partial}{\partial v}(A_3\lambda) - \frac{\partial}{\partial w}(A_2\lambda) + B_1\lambda;$$

$$\lambda_2 = \lambda_{wu} - \frac{\partial}{\partial w}(A_1\lambda) - \frac{\partial}{\partial u}(A_3\lambda) + B_2\lambda;$$

$$\lambda_3 = \lambda_{uv} - \frac{\partial}{\partial u}(A_2\lambda) - \frac{\partial}{\partial v}(A_1\lambda) + B_3\lambda,$$

and, if $L \equiv V_{uvw} + A_1 V_{vw} + A_2 V_{wu} + A_3 V_{uv} + B_1 V_u + B_2 V_v + B_3 V_w + CV$,

$$M \equiv \lambda_{uvw} - \frac{\partial^2}{\partial v \partial w}(A_1\lambda) - \frac{\partial^2}{\partial w \partial u}(A_2\lambda) - \frac{\partial^2}{\partial u \partial v}(A_3\lambda)$$

$$+ \frac{\partial}{\partial u}(B_1\lambda) + \frac{\partial}{\partial v}(B_2\lambda) + \frac{\partial}{\partial w}(B_3\lambda) - C\lambda,$$

then
$$A = \lambda L + VM;$$

and therefore $A = 0$ if $L = 0$ and $M = 0$.

The point $P(x_0, y_0)$ is chosen to be sufficiently near the curve $\psi(x, y) = 0$ to ensure that the characteristic conditions are satisfied (Fig. 2). P is the origin of the u, v, w coordinates, and we may integrate $A(= 0)$ throughout the volume determined by the planes PAB, PBC, PCA, and the cylinder $\psi(x_0 + u + v + w,\ y_0 + k_1 u + k_2 v + k_3 w) = 0$. λ is a function

* x_0, y_0 are to be regarded throughout as parameters, unaffected by u, v, w.

of u, v, w to be determined later. The first condition is that it should be a solution of the adjoint equation $M = 0$.

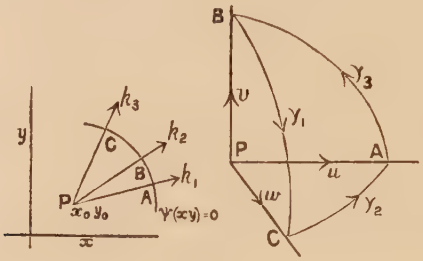

Fig. 2.

The integration of the first term of A gives

$$\iint_S \left\{\frac{\partial^2(\lambda V)}{\partial u\,\partial v}\right\} du\,dv + \int_{\gamma_3} \frac{\partial}{\partial u}(\lambda V)\,du + (\lambda V)_A - (\lambda V)_P,$$

where S is the area on the surface bounded by the curvilinear triangle ABC, and γ_3 the curve $\overrightarrow{AB}$. The next three terms give

$$-\iint_S \left[\frac{\partial}{\partial v}\{(\lambda_u - A_1\lambda)V\} + \frac{\partial}{\partial u}\{(\lambda_v - A_2\lambda)V\}\right] du\,dv$$

$$-\iint_S \frac{\partial}{\partial u}\{(\lambda_w - A_3\lambda)V\}\,du\,dw - \iint_{\gamma_3} (\lambda_u - A_1\lambda)V\,du$$

$$+\int_{\gamma_3} (\lambda_v - A_2\lambda)V\,dv - \int_{\gamma_2} (\lambda_w - A_3\lambda)V\,dw,$$

where γ_2 is the curve $\overrightarrow{CA}$:—provided that λ satisfies the conditions

(4.1) $\lambda_u - A_1\lambda = 0$ when $v = w = 0$; $\lambda_v - A_2\lambda = 0$ when $w = u = 0$;

$\lambda_w - A_3\lambda = 0$ when $u = v = 0$.

The remaining terms give

$$\iint_S V(T_1\,dv\,dw + T_2\,dw\,du + T_3\,du\,dv),$$

where

$$T_1 = \lambda_{vw} - \frac{\partial}{\partial v}(A_3\lambda) - \frac{\partial}{\partial w}(A_2\lambda) + B_1\lambda,$$

$$T_2 = \lambda_{wu} - \frac{\partial}{\partial w}(A_1\lambda) - \frac{\partial}{\partial u}(A_3\lambda) + B_2\lambda,$$

$$T_3 = \lambda_{uv} - \frac{\partial}{\partial u}(A_2\lambda) - \frac{\partial}{\partial v}(A_1\lambda) + B_3\lambda,$$

provided that

(4.2) $T_1 = 0$ when $u = 0$; $T_2 = 0$ when $v = 0$; $T_3 = 0$ when $w = 0$.

If λ can be found to satisfy these conditions (4 . 1), (4 . 2), the value of V at P is known. A symmetrical expression may be obtained by taking the mean of the different expressions for the volume integrals. It is

$$(\lambda V)_P = \tfrac{1}{3}\{(\lambda V)_A + (\lambda V)_B + (\lambda V)_C\} + \iint_S (lP + mQ + nR)\, dS$$

$$- \int_{\gamma_3} (P_1\, du - Q_1\, dv) - \int_{\gamma_1} (Q_1\, dv - R_1\, dw) - \int_{\gamma_2} (R_1\, dw - P_1\, du),$$

where $P = \tfrac{1}{3}\lambda V_{vw} - \tfrac{1}{6}(\lambda_w - 3A_3\lambda) V_v - \tfrac{1}{6}(\lambda_v - 3A_2\lambda) V_w$

$$+ \left(\tfrac{1}{3}\lambda_{vw} - \tfrac{1}{2}\frac{\partial}{\partial w}(A_2\lambda) - \tfrac{1}{2}\frac{\partial}{\partial v}(A_3\lambda) + B_1\lambda\right) V,$$

$$P_1 = \tfrac{1}{6}\lambda V_u - \tfrac{1}{6}(2\lambda_u - 3A_1\lambda) V,$$

with corresponding expressions for Q, R, Q_1, R_1; l, m, n are the direction cosines of the normal to S drawn in the positive direction.

5. *The function* λ.

The success of the method depends on the determination of the required solution λ of the adjoint equation $M = 0$. The simplest case is when the coefficients are constants, and the solution for this will now be given.

The adjoint equation is

$$(5\,.\,1)\quad \lambda_{uvw} - A_1\lambda_{vw} - A_2\lambda_{wu} - A_3\lambda_{uv} + B_1\lambda_u + B_2\lambda_v + B_3\lambda_w - C\lambda = 0,$$

with "initial" conditions

$$(5\,.\,2)\quad A_u = A_1\lambda \text{ when } v = w = 0, \quad \lambda_v = A_2\lambda \text{ when } w = u = 0,$$

$$\lambda_w = A_3\lambda \text{ when } u = v = 0.$$

$$(5\,.\,3)\quad \begin{cases} \lambda_{vw} - A_3\lambda_v - A_2\lambda_w + B_1\lambda = 0 & \text{when } u = 0 \\ \lambda_{wu} - A_1\lambda_w - A_3\lambda_u + B_2\lambda = 0 & \text{when } v = 0 \\ \lambda_{uv} - A_2\lambda_u - A_1\lambda_v + B_3\lambda = 0 & \text{when } w = 0. \end{cases}$$

A comparison with the known results for the second order hyperbolic equation gives us immediately

$$(5\,.\,4)\quad \begin{cases} \lambda(0, v, w) = e^{A_2 v + A_3 w} J_0\{2\rho_1\sqrt{(vw)}\} \\ \lambda(u, 0, w) = e^{A_3 w + A_1 u} J_0\{2\rho_2\sqrt{(wu)}\} \\ \lambda(u, v, 0) = e^{A_1 u + A_2 v} J_0\{2\rho_3\sqrt{(uv)}\}, \end{cases}$$

where $\rho_1^2 = B_1 - A_2A_3$; $\rho_2^2 = B_2 - A_3A_1$; $\rho_3^2 = B_3 - A_1A_2$; and the above functions are one-valued and real even when some of the numbers ρ_1^2, ρ_2^2, ρ_3^2 are negative.

To obtain the required solution for the third order equation, it is more suggestive, however, to write these functions as contour integrals in the t-plane. Thus

$$\lambda(0, v, w) = \frac{e^{A_2v+A_3w}}{2\pi i}\int^{0+} e^{\rho_1(vt-w/t)}\,\frac{dt}{t},$$

and the contour may be taken as $|t| = R$ (R finite); for this suggests that the solution will be expressible as a double integral of the complex variables, associated with an integrand of the form

$$U = e^{A_1u+A_2v+A_3w+\rho_1(vt_1-w/t_1)+\rho_2(wt_2-u/t_2)+\rho_3(ut_3-v/t_3)}.$$

On substituting this in (5 . 1), we find that it is a solution if

$$\rho_1\rho_2\rho_3\left(t_1t_2t_3 - \frac{1}{t_1t_2t_3}\right) = C_1, \tag{5 . 5}$$

where $$C_1 = C - A_1A_2A_3 - A_1\rho_1^2 - A_2\rho_2^2 - A_3\rho_3^2.$$

Omitting for the moment the case when some of the numbers ρ_1, ρ_2, ρ_3 are zero, we see that $t_1t_2t_3$ must be a root of the quadratic equation

$$\theta^2 - \frac{C_1}{\rho_1\rho_2\rho_3}\theta - 1 = 0. \tag{5 . 6}$$

An inspection of this result shows that our required solution is the term independent of t_1, t_2 in the "Laurent" expansion* of U, and therefore

$$\lambda = -\frac{1}{4\pi^2}\iint\frac{U dt_1 dt_2}{t_1t_2} \tag{5 . 7}$$

taken over the "surface"† specified by $|t_1| = R_1$, $|t_2| = R_2$ (R_1, R_2 finite). The function λ is unaltered by writing $-1/t_1$ for t_1, $-1/t_2$ for t_2, $-1/\theta_1$ for θ_1, and therefore, since $\theta_1\theta_2 = -1$ from (5 . 6), either root may be taken. Actually, by taking the products of the exponential expansions, we find

$$\lambda = e^{A_1u+A_2v+A_3w}\sum_{n=-\infty}^{\infty}\theta_1^n J_n\{2\rho_1\sqrt{(vw)}\}\,J_n\{2\rho_2\sqrt{(wu)}\}\,J_n\{2\rho_3\sqrt{(uv)}\} \tag{5 . 8}$$

$$= e^{A_1u+A_2v+A_3w}\left\{J_0(\sigma_1)J_0(\sigma_2)J_0(\sigma_3) + \sum_{n=1}^{\infty}(\theta_1^n+\theta_2^n)J_n(\sigma_1)J_n(\sigma_2)J_n(\sigma_3)\right\},$$

* Forsyth, *Functions of two complex variables* (1914), § 59.

† Picard, *Traité d'analyse*, 2 (1893), ch. 9, § 17.

where $\sigma_1 = 2\rho_1\sqrt{(vw)}$, &c.; since $\theta_1\theta_2 = -1$. All the terms in the above series are still real, whether ρ_1^2, ρ_2^2, ρ_3^2 are positive or negative, since the expression $\rho_1\rho_2\rho_3(\theta_1+\theta_2)$ is real and $\iota^n J_n(\iota a)$ is real if a is real.

The cases when some of the numbers ρ_1, ρ_2, ρ_3 are zero can be deduced from the above by a limiting process, but it is better to return to the original equation and to take an appropriate trial integrand. The following are the results :—

(i) $\rho_1 \neq 0, \quad \rho_2 \neq 0, \quad \rho_3 = 0$:

$$(5.9) \qquad \lambda = -\frac{e^{A_1u+A_2v+A_3w}}{4\pi^2}\iint e^{\rho_1(vt_1-w/t_1)+\rho_2(ut_2-w/t_2)+(C_1w/\rho_1\rho_2t_1t_2)}\,\frac{dt_1\,dt_2}{t_1t_2},$$

$$= e^{A_1u+A_2v+A_3w}\sum_{n=0}^{\infty}\left(\frac{C_1w\sqrt{(uv)}}{\rho_1\rho_2}\right)^n \frac{J_n\{2\rho_1\sqrt{(vw)}\}\,J_n\{2\rho_2\sqrt{(wu)}\}}{n\,!},$$

where $B_3 = A_1A_2, \quad C_1 = C+A_1A_2A_3-A_1B_1-A_2B_2$.

(ii) $\rho_1 \neq 0, \quad \rho_2 = 0, \quad \rho_3 = 0$:

$$(5.11) \qquad \lambda = -\frac{e^{A_1u+A_2v+A_3w}}{4\pi^2}\iint e^{\rho_1(vt_1-w/t_1)+ut_2+(wC_1/\rho_1t_1t_2)}\,\frac{dt_1\,dt_2}{t_1t_2},$$

$$= e^{A_1u+A_2v+A_3w}\sum_{n=0}^{\infty}\left(\frac{C_1u\sqrt{(vw)}}{\rho_1}\right)^n \frac{J_n\{2\rho_1\sqrt{(vw)}\}}{(n\,!)^2},$$

where $B_3 = A_1A_2, \quad B_2 = A_1A_3, \quad C_1 = C-A_1B_1$.

(iii) $\rho_1 = 0, \quad \rho_2 = 0, \quad \rho_3 = 0$:

$$(5.12) \qquad \lambda = -\frac{e^{A_1u+A_2v+A_3w}}{4\pi^2}\iint e^{ut_1+vt_2+(wC_1/t_1t_2)}\,\frac{dt_1\,dt_2}{t_1t_2},$$

$$= e^{A_1u+A_2v+A_3w}\sum_{n=1}^{\infty} C_1^n\frac{(uvw)^n}{(n\,!)^3},$$

where $B_3 = A_1A_2, \quad B_2 = A_1A_3, \quad B_1 = A_2A_3, \quad C_1 = C-A_1A_2A_3$.

In particular, for $\lambda_{uvw} = C\lambda$,

$$\lambda = 1+\frac{Cuvw}{(1\,!)^3}+\frac{C^2u^2v^2w^2}{(2\,!)^3}+\ldots.$$

6. *The third order equation in two independent variables.*

The solution of (1 . 1) with the appropriate boundary conditions, satisfies the equation :

$$(6\,.\,1)\qquad \left(\frac{\partial}{\partial x_0}+k_1\frac{\partial}{\partial y_0}\right)\left(\frac{\partial}{\partial x_0}+k_2\frac{\partial}{\partial y_0}\right)\left(\frac{\partial}{\partial x_0}+k_3\frac{\partial}{\partial y_0}\right)V$$
$$+A_1\left(\frac{\partial}{\partial x_0}+k_2\frac{\partial}{\partial y_0}\right)\left(\frac{\partial}{\partial x_0}+k_3\frac{\partial}{\partial y_0}\right)V$$
$$+\ldots+\ldots+B_1\left(\frac{\partial}{\partial x_0}+k_1\frac{\partial}{\partial y_0}\right)V+\ldots+\ldots+CV=0,$$

for $u=v=w=0$, and this is the same as a *given* equation

$$(6\,.\,2)\qquad \frac{\partial^3 V}{\partial x_0^3}+a\frac{\partial^3 V}{\partial x_0^2\partial y_0}+\beta\frac{\partial^3 V}{\partial x_0\partial y_0^2}+\gamma\frac{\partial^3 V}{\partial y_0^3}+a\frac{\partial^2 V}{\partial x_0^2}+2h\frac{\partial^2 V}{\partial x_0 dy_0}$$
$$+b\frac{\partial^2 V}{\partial y_0^2}+2g\frac{\partial V}{\partial x_0}+2f\frac{\partial V}{\partial y_0}+cV=0$$

(where a, β, γ are constant) if

(i) k_1, k_2, k_3 are the roots of $\theta^3-a\theta^2+\beta\theta-\gamma=0$*

(supposed here to be real and distinct) ;

(ii) $A_1+A_2+A_3=a$, $A_1(k_2+k_3)+A_2(k_3+k_1)+A_3(k_1+k_2)=2h$,
$$A_1k_2k_3+A_2k_3k_1+A_3k_1k_2=b,$$

so that A_1, A_2, A_3 are determined uniquely since the determinant of the coefficients [viz., $-(k_2-k_3)\,(k_3-k_1)\,(k_1-k_2)$] is not zero.

(iii) $B_1+B_2+B_3=2g$, $B_1k_1+B_2k_2+B_3k_3=2f$, $C=c$;

these provide values for B_1, B_2, B_3, one of which appears to be arbitrary. Taking this to be B_3, we find that

$$B_1=\frac{2gk_2-f-B_3(k_2-k_3)}{k_2-k_1},\quad B_2=\frac{2f-2gk_1-B_3(k_3-k_1)}{k_2-k_1}.$$

Substitution of these values in (1 . 1) gives for the terms of the first order

$$\frac{2gk_2-f}{k_2-k_1}V_u+\frac{2f-2gk_1}{k_2-k_1}V_v+\frac{B_3}{k_1-k_2}\{(k_2-k_3)V_u+(k_3-k_1)V_v+(k_1-k_2)V_w\},$$

thus bringing into evidence the significance of the arbitrariness of B_3 ; for the solution which we are seeking is such that the coefficient of B_3

* The equation is to be taken as $\frac{\partial}{\partial y_0}\left(\frac{\partial}{\partial x_0}+k_2\frac{\partial}{\partial y_0}\right)\left(\frac{\partial}{\partial x_0}+k_3\frac{\partial}{\partial y_0}\right)V+\ldots$, and $a=1$ if there is no term in $\partial^3V/\partial x_0^3$.

above is zero. B_3 should be chosen in such a way as to make the determination of λ as simple as possible.

Thus for the equation

$$\frac{\partial^3 V}{\partial x_0^2 \partial y_0} + \frac{\partial V}{\partial x_0 \partial y_0^2} + \frac{6V}{(x_0+y_0)^3} = 0,$$

we should take $x = x_0+u+w$, $y = y_0+v+w$, $A_1 = A_2 = A_3 = 0$, $B_1 = B_2 = B_3 = 0$, and deal with the equation

$$V_{uvw} + \frac{6V}{(x_0+y_0+u+v+2w)^3} = 0,$$

with adjoint

$$\lambda_{uvw} - \frac{6\lambda}{(x_0+y_0+u+v+2w)^3} = 0.$$

It should be noted that λ need not contain the variables in any particular combinations (like those in V), and that it has not been proved that a function λ exists in the general case satisfying the conditions specified (these not being Cauchy conditions).

7. *The integration.*

With regard to the actual integration, it would appear to be simpler (*a*) to use an unsymmetrical form of the result given in § 4 and (*b*) to retain the function λ in the form of a double complex integral until the other integrations have been performed. The procedure may be illustrated by the following example :—

Find the solution of

$$\frac{\partial^3 V}{\partial x^2 \partial y} + \frac{\partial^3 V}{\partial x \partial y^2} + 2V = 0, \tag{7.1}$$

if we are given that, on the line $x+y = 1$,

$$V = \phi_0(x), \quad V_x = \phi_1(x), \quad V_{xx} = \phi_2(x).$$

We deduce immediately that

$$V_y = \phi_1 - \phi_1', \quad V_{xy} = \phi_2 - \phi_1', \quad V_{yy} = \phi_2 - 2\phi_1' + \phi_0''.$$

Take

$$x = x_0+u+w, \quad y = y_0+v+w.$$

The new boundary is $x_0+y_0+u+v+2w = 1$, and our new boundary conditions are

$$V = \phi_0(x_0+u+w), \quad V_u = \phi_1(x_0+u+w), \quad V_v = \phi_1 - \phi_0', \quad V_w = 2\phi_1 - \phi_0',$$

$$V_{uv} = \phi_2(x_0+u+w) - \phi_1'(x_0+u+w), \quad V_{vw} = 2\phi_2 - 3\phi_1' + \phi_0'', \quad V_{ww} = 2\phi_2 - \phi_1'.$$

The adjoint equation is $\lambda_{uvw} = 2\lambda$, and so*

$$\lambda = -\frac{1}{4\pi^2}\iint e^{u/t_1+v/t_2+2wt_1t_2}\frac{dt_1\,dt_2}{t_1t_2}, \tag{7.2}$$

the "surface" being specified by $|t_1| = R_1$, $|t_2| = R_2$.

By changing the differential elements $du\,dw$, $dv\,dw$ to $\frac{1}{2}du\,dv$ and $\frac{1}{2}du\,dv$, and noting that $\lambda(0, v, w) = 1$, &c., we find the required solution to be

$$V(x_0, y_0) = \phi_0(1-y_0) - \int_0^{1-x_0-y_0} \phi_1(x_0+u)\,du + \iint_S \{\lambda(\phi_2-\phi_1') - \tfrac{1}{2}\lambda w\,\phi_1 + \tfrac{1}{2}\lambda vw\,\phi_0\}\,du\,dv, \tag{7.3}$$

where, in the double integral, the arguments of the ϕ functions are x_0+u+w, and where, throughout the integrand, we substitute $\frac{1}{2}(1-x_0-y_0-u-v)$ for w.

The completion of the work can now be sufficiently illustrated by taking $\phi_0 = 0$, $\phi_1 = 0$, $\phi_2 = e^{a(x_0+u+w)}$. Then

$$V = -\frac{1}{4\pi^2}\iint_A \left[\iint e^{(u/t_1)+(v/t_2)+t_1t_2(1-x_0-y_0-u-v)}\,\frac{dt_1\,dt_2}{t_1t_2}\right] \times e^{(a/2)(1+x_0-y_0+u-v)}\,du\,dv, \tag{7.4}$$

A being specified by $0 \leqslant u+v \leqslant 1-x_0-y_0$.

Now, since the integrand is continuous, we can integrate with respect to u, v first, and obtain

$$V = -\frac{1}{4\pi^2}\iint (I_1+I_2+I_3+I_4)\,e^{(a/2)(1+x_0-y_0)}\,dt_1\,dt_2,$$

where

$$I_1 = \frac{t_2 e^{(1-x_0-y_0)(1/t_1+a/2)}}{(t_2-t_1+at_1t_2)\{1-t_1t_2^2-(a/2)t_2\}};$$

$$I_2 = -\frac{e^{(1-x_0-y_0)(1/t_2-a/2)}}{(t_2-t_1+at_1t_2)\{1-t_1t_2^2-(a/2)t_2\}};$$

$$I_3 = -\frac{e^{(1-x_0-y_0)(1/t_1+a/2)}}{\{1-t_1^2t_2+(a/2)t_1\}\{1-t_1t_2^2-(a/2)t_2\}};$$

$$I_4 = \frac{e^{(1-x_0-y_0)t_1t_2}}{\{1-t_1^2t_2+(a/2)t_1\}\{1-t_1t_2^2-(a/2)t_2\}}.$$

* A slight modification of 5.12.

Using the known results of the theory of double integrals of two complex variables*, we see that the integral for V can be broken up into four parts (corresponding respectively to I_1, I_2, I_3, I_4), provided that the surface of integration is chosen so as not to pass through the singularities of the integrands. The singularities which give rise to residues are, for I_1 and I_2, the places determined by

$$\text{(i)} \qquad t_2 - t_1 + a t_1 t_2 = 0, \quad 1 - t_1 t_2^2 - \frac{a}{2} t_2 = 0,$$

and, for I_3 and I_4, the places given by

$$\text{(ii)} \qquad 1 - t_1^2 t_2 + \frac{a}{2} t_1 = 0, \quad 1 - t_1 t_2^2 - \frac{a t_2}{2} = 0.$$

Equations (i) and (ii) give the same values, viz. those determined from

$$\frac{1}{t_1} + \frac{a}{2} = \frac{1}{t_2} - \frac{a}{2} = t_1 t_2.$$

so that $t_1 t_2$ is a root of the cubic equation $m^3 - (a^2/4)\, m - 1 = 0$.

For simplicity, take these roots to be different (*i.e.* $a^6 \neq 432$), say m_1, m_2, m_3, and let the corresponding values of t_1 and t_2 be δ_1, δ_2, δ_3 and ϵ_1, ϵ_2, ϵ_3. The original surface of integration is given by

$$|t_1| = R_1, \quad |t_2| = R_2.$$

Take R_1 to be any number greater than the $\max |\delta_r|$ and R_2 greater than $\max |\epsilon_r|$. Then a simple formal analysis will show that I_1, I_2, I_3 possess no isolated singularities for values of t_1, t_2 such that

$$|t_1| > R_1, \quad |t_2| > R_2,$$

including infinity, and therefore the corresponding integrals are zero.

Therefore

$$V = -\frac{e^{(a/2)(1+x_0-y_0)}}{4\pi^2} \iint \frac{e^{(1-x_0-y_0)t_1 t_2}\, dt_1\, dt_2}{\left(1 - t_1^2 t_2 + (a t_1/2)\right)\left(1 - t_1 t_2^2 - (a t_2/2)\right)}.$$

The Jacobian of the denominator is $3t_1^2 t_2^2 - (a^2/4)$ and therefore the solution is

$$(7.6) \qquad V = e^{(a/2)(1+x_0-y_0)} \left\{ \frac{e^{m_1(1-x_0-y_0)}}{3m_1^2 - (a^2/4)} + \frac{e^{m_2(1-x_0-y_0)}}{3m_2^2 - (a^2/4)} + \frac{e^{m_3(1-x_0-y_0)}}{3m_3^2 - (a^2/4)} \right\}.$$

* Picard, *Traité d'analyse*, 2 (1893), ch. 9.

The case when ϕ_2 is a polynomial can now be deduced from (7 . 6). If $\phi_2(x) = x^n$ the required solution is the coefficient of $a^n/n!$ in the expression for V, since the functions of a which occur can be expanded, in the neighbourhood of a, in a uniformly and absolutely convergent series of powers of a.

Thus

$$m_1 = 1 + \frac{a^2}{12} + \ldots, \quad m_2 = \omega + \frac{a^2\omega^2}{12} + \ldots, \quad m_3 = \omega^2 + \frac{a^2\omega}{12} + \ldots,$$

and we find, after simplification, that for

(i) $\phi_2 = 1$, $\quad V = \sum_{r=0}^{\infty} \frac{(1-x_0-y_0)^{3r+2}}{(3r+2)!}$;

(ii) $\phi_2 = x$, $\quad V = \frac{1}{6}(1+x_0-y_0) \sum_{r=0}^{\infty} \frac{(1-x_0-y_0)^{3r+2}}{(3r+2)!}$;

(iii) $\phi_2 = x^2$, $\quad V = \frac{(1+x_0-y_0)^2}{4} \sum_{r=0}^{\infty} \frac{(1-x_0-y_0)^{3r+2}}{(3r+2)!}$

$$+ \frac{(1-x_0-y_0)}{6} \sum_{r=0}^{\infty} \frac{(1-x_0-y_0)^{3r}}{(3r)!} - \frac{1}{6} \sum_{r=0}^{\infty} \frac{(1-x_0-y_0)^{3r+1}}{(3r+1)!}.$$

DERIVATIVES OF A GENERAL MASS

By P. J. DANIELL.

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THE question of the ordinary derivative of a general mass-distribution or additive function of sets—that is the derivative with respect to ordinary measure (length, area, etc.)—is well known, and one solution is given by de la Vallée Poussin in a book* with which the reader is assumed to be acquainted. The more general problem in which differentiation takes place with respect to a second general mass has been handled in the case of one dimension by W. H. Young† and by the writer‡. The fundamental lemmas used in these papers have no evident extension to several dimensions, and in a later paper§ the writer contented himself with the use of net systems. [A second kind of quasi-derivative was also suggested in a third paper||.] But there remains the serious objection that the number of possible net systems is more than countable, even if the vertices of the nets are restricted to have rational coordinates. The net derivative is therefore, to some extent, arbitrary, and a similar criticism can be made about the second quasi-derivative.

The present paper suggests a method of defining the derivative by means of *core sets* in a manner dependent only on the nature of the masses considered. We also obtain the fundamental theorem (3 . 92) connecting the mass M with the integral of its derivative. The defined derivative is not directly related to the limit of ratios of masses of enclosing intervals; but it is shown that the derivative of the integral of a continuous function, at least, is equal to that function, except on the

* de la Vallée Poussin, *Intégrales de Lebesgue* (Paris, 1916).

† W. H. Young, *Proc. London Math. Soc.* (2), 15 (1916), 35.

‡ P. J. Daniell, *Trans. Amer. Math. Soc.*, 19 (1918), 353. See also E. C. Francis, *Proc. Camb. Phil. Soc.*, 22 (1925), 924–934.

§ P. J. Daniell, *Bull. Amer. Math. Soc.*, 26 (1920), 444.

|| P. J. Daniell, *Annals of Math.*, 25 (1924), 193.

desert. This desert is the sum of the open intervals whose mass is zero when the mass is that with respect to which differentiation takes place, and it must create some difficulty in any case.

We consider points

$$x = (x_1, x_2, \ldots, x_p)$$

in a space of p dimensions, and we say that

$$x < y \quad \text{or} \quad x \leqslant y$$

when $$x_r < y_r \quad \text{or} \quad x_r \leqslant y_r \quad (r = 1, 2, \ldots, p).$$

An open interval a, b is the collection of points x, such that

$$a < x < b.$$

A closed interval a, b is the collection of points x, such that

$$a \leqslant x \leqslant b.$$

A single point is a special case of a closed interval (taking $a = b$).

By a set of points we shall mean a set which is measurable in the sense of Borel. Two sets are *disjoint* if they contain no point in common.

A mass-distribution, $M(e)$, is an absolutely additive function of sets. It is called *positive* if it is never negative. In order that a mass-distribution may be positive it is sufficient for $M(\delta)$ to be non-negative for all intervals δ, or for all closed intervals, or for all open intervals.

To $M(e)$ there correspond two positive masses P, N, such that

$$M(e) = P(e) - N(e).$$

The sum $$|M|(e) = P(e) + N(e)$$

is called the modular mass associated with M.

A *dense net-system*, N_n, is a sequence of divisions by $(p-1)$-spreads perpendicular to the axes, such that an interval D or the whole space is divided into a finite number of meshes $\delta_{1,s}$ of order 1. Each of these meshes, again, is divided into a finite number of meshes $\delta_{2,s}$ of order 2, and so on, such that, if x, y belong to D and

$$x < y,$$

there is a stage in the division at which x, y belong to distinct meshes of that order.

Most of the meshes will be intervals, but some may extend to infinity in one or more directions.

There can only be a finite or countably infinite number of $(p-1)$-spreads perpendicular to the axes, such that $m(e)$ is positive for the set e consisting of the points of one of these spreads. Relative to a positive mass $m(e)$ a net-system is said to be *smooth* if all of the dividing $(p-1)$-spreads have zero mass in m.

Smooth dense net-systems can be laid on any interval D whose bounding surfaces have zero mass. If the net-system is smooth we can include the common surface of two intervals in each of them without disturbing their masses. If these boundaries are included the meshes will be closed sets even when they are not bounded.

In what follows $m(e)$ will denote throughout a basic positive mass-distribution.

1.1. *Definition.* Given a closed set, e, we define the *core of* e, $n(e)$, as the complement of the logical sum of all open intervals δ, such that

$$m(\delta \,.\, e) = 0.$$

Let N_n be a smooth dense net-system laid on the whole p-space. Define

$$e_n = \Sigma \delta_{n,s},$$

the sum of all the meshes of order n for which

$$m(e \,.\, \delta_{n,s}) > 0.$$

Evidently e_n includes e_{n+1} and the sets e_1, e_2, ... are closed. Their product (limit) is either closed or the null-set. Denote it by $n'(e)$. Then if δ is an interval (a bounded mesh) of the net (*e.g.* of order m) not containing a point of $n'(e)$,

$$m(e \,.\, \delta) = 0,$$

for if this mass were positive there would be intervals δ', δ'', ... of orders $m+1$, $m+2$, ... contained in δ, for each of which

$$m(e \,.\, \delta) > 0, \quad m(e \,.\, \delta') > 0, \quad m(e \,.\, \delta'') > 0, \quad \ldots$$

and δ, δ', δ'', ... would define a point within all of them belonging to $n'(e)$.

Any open interval δ containing no point of $n'(e)$ consists of at most a countable infinity of net-intervals, and, therefore, again

$$m(e \,.\, \delta) = 0.$$

But, also, any open interval δ which contains a point of $n'(e)$ contains

a net-interval δ' which contains a point of $n'(e)$, so that

$$m(e \,.\, \delta) \geqslant m(e \,.\, \delta') > 0.$$

Hence $n'(e)$ is exactly the complement of the sum of all open intervals δ, for which $m(e \,.\, \delta) = 0$. It therefore coincides with $n(e)$, and is independent of the net-system chosen.

1 . 2. The core $n(e)$ of a closed set e is either the null set or a closed set contained in e.

We have already seen that $n(e)$ is closed or null, and if δ is any open interval belonging to Ce, $m(e \,.\, \delta)$ is 0, so that δ also belongs to $Cn(e)$. Hence Ce is included in $Cn(e)$ and e includes $n(e)$.

1 . 3. The mass of the part of e not contained in $n(e)$ is zero.

For suppose it positive, then returning to the notation of (1 . 1), since $n(e)$ is the limit of e_n, there is a set e_r, let us say, for which

$$m(e \,.\, Ce_r) > 0.$$

Ce_r consists of a finite sum of intervals from which the boundaries can be removed without altering the masses. Hence there is one open interval δ, at least, belonging to $Cn(e)$, such that $m(e \,.\, \delta)$ is positive, which contradicts the definition of $n(e)$.

1 . 4. If e_1, e_2 are closed, and if e_1 is included in e_2,

$$n(e_1) < n(e_2).$$

For, if δ is an open interval, for which $m(e_2 \,.\, \delta) = 0$,

$$m(e_1 \,.\, \delta) \leqslant m(e_2 \,.\, \delta) = 0,$$

$$Cn(e_2) < Cn(e_1),$$

$$n(e_1) < n(e_2).$$

1 . 5. If e_1, e_2 are closed and differ only by a set of zero mass,

$$m(e_1 \,.\, Ce_2 + e_2 \,.\, Ce_1) = 0,$$

then their cores coincide.

For, if δ is any open interval, the equations

$$m(e_1 \,.\, \delta) = 0, \quad m(e_2 \,.\, \delta) = 0,$$

logically imply each other.

1 . 61. The core of $n(e)$ is $n(e)$ itself.

1 . 62. The core of a closed set of zero mass is the null set.

1 . 7. If e_1 is closed, contained in a closed set e, and of the same mass as e, e_1 includes the core of e.

For, by (1 . 5), the core of e_1 is $n(e)$, and this is included in e_1 by (1 . 2).

1 . 71. It follows that we could also define the core $n(e)$ as *the product of all closed set* e_1 contained in e of the same mass as e.

For this product will include $n(e)$ by (1 . 7), and $n(e)$ is itself one of these closed sets e_1.

1 . 81. If e_1, e_2 are closed sets, e_1+e_2 is closed, and

$$n(e_1+e_2) = n(e_1)+n(e_2).$$

First, e_1+e_2 is known to be closed and contains e_1 and e_2. Hence

$$n(e_1) < n(e_1+e_2), \quad n(e_2) < n(e_1+e_2),$$

$$n(e_1)+n(e_2) < n(e_1+e_2).$$

Secondly, if δ is an open interval, such that both

$$m(\delta \,.\, e_1) = 0, \quad m(\delta \,.\, e_2) = 0,$$

then

$$m(\delta \,.\, e_1+e_2) = 0.$$

Hence

$$Cn(e_1) \,.\, Cn(e_2) < Cn(e_1+e_2),$$

$$n(e_1+e_2) < n(e_1)+n(e_2).$$

1 . 82. If e_1, e_2 are closed sets,

$$n(e_1 \,.\, e_2) = n(e_1) \,.\, n(e_2).$$

For $n(e_1 \,.\, e_2)$ is included in both $n(e_1)$ and $n(e_2)$, and therefore in their product. On the other hand, if δ is an open interval, such that either

$$m(\delta \,.\, e_1) = 0 \quad \text{or} \quad m(\delta \,.\, e_2) = 0,$$

$$m(\delta \,.\, e_1 \,.\, e_2) = 0,$$

so that $Cn(e_1)+Cn(e_2)$ is included in $Cn(e_1 \,.\, e_2)$.

1.9. If e_1, e_2, e_3, e_4 are closed sets, and if

$$e_1 < e_2, \quad e_3 < e_4, \quad e_2 - e_1 = e_4 - e_3,$$

then

$$n(e_2) - n(e_1) = n(e_4) - n(e_3).$$

Let

$$f = e_2 - e_1 = e_4 - e_3.$$

$$e_2 . e_4 = (e_1 + f)(e_3 + f) = e_1 . e_3 + f,$$

and f contains no point of e_1, e_3. Hence, if we can prove the theorem in the special case in which e_1 is included in e_3 and e_2 in e_4, we can substitute $e_1 . e_3$ for e_1, $e_2 . e_4$ for e_2, and prove that

$$n(e_2 . e_4) - n(e_1 . e_3) = n(e_4) - n(e_3).$$

But, by the same reasoning,

$$n(e_2 . e_4) - n(e_1 . e_3) = n(e_2) - n(e_1)$$

and our theorem is proved.

Now, in the special case, e_1 is included in e_2 and e_3, while e_2 and e_3 are included in e_4.

$$e_2 . e_3 . Ce_1 = e_3 . e_4 . Ce_3$$

is the null set, and e_1 is included in e_2 and e_3. Hence

$$e_1 = e_2 . e_3.$$

Again

$$\begin{aligned} e_4 &= e_3 + e_2 . Ce_1 = e_3 + e_2 . (Ce_2 + Ce_3) \\ &= e_3 + e_2 . Ce_3 = e_2 + e_3. \end{aligned}$$

$$\begin{aligned} n(e_2) . Cn(e_1) &= n(e_2) . [Cn(e_2) + Cn(e_3)] \\ &= n(e_2) . Cn(e_3), \end{aligned}$$

$$\begin{aligned} n(e_4) . Cn(e_3) &= n(e_2) . Cn(e_3) + n(e_3) . Cn(e_3) \\ &= n(e_2) . Cn(e_3) \end{aligned}$$

and the special case (and therefore the general case) is proved.

2.1. *Definition.* If f is a *difference set*, that is a set

$$f = e_1 . Ce_2,$$

where e_1, e_2 are closed sets, we define the core of f as

$$n(f) = n(e_1) - n(e_1 . e_2).$$

By the previous theorem this core is unique if we replace e_2 by $e_1 . e_2$ (which does not alter f).

It is to be noted that the core of f is *not* necessarily contained wholly in f.

2 . 11. The core of a difference set of zero mass is the null set. This follows from (1 . 5).

2 . 2. If f, g are difference sets, so also is $f . g$, and

$$n(f . g) = n(f) . n(g).$$

For if f is $e_1 . Ce_2$, g is $e_3 . Ce_4$,

$$f . g = (e_1 . e_3) . C(e_2 + e_4),$$

$$n(f . g) = n(e_1) . n(e_3) . C[n(e_2) + n(e_4)]$$

$$= [n(e_1) . Cn(e_2)] . [n(e_3) . Cn(e_4)].$$

2 . 21. If f, g are difference sets and $f . g$ has zero mass, $n(f)$, $n(g)$ are disjoint.

2 . 31. If a difference set f is divided by means of a closed set e in the form

$$f = e . f + f . Ce,$$

$$n(f) = n(e . f) + n(f . Ce).$$

For let f be $e_1 . Ce_2$, then

$$n(f) = n(e_1) . Cn(e_2),$$

$$n(e . f) = n(e_1 . e) . Cn(e_2) = n(e_1) . n(e) . Cn(e_2),$$

$$n(Ce . f) = n(e_1) . Cn(e + e_2) = n(e_1) . Cn(e) . Cn(e_2).$$

2 . 32. If a difference set f is divided by means of a finite number of closed sets $e_1, e_2, \ldots, e_n$, the sum of the cores of the partial sets is the core of f. It follows that this sum is independent of the choice of $e_1, e_2, \ldots, e_n$.

2 . 33. If two sums of finite numbers of difference sets are identical the sums of the corresponding cores are identical.

It is only necessary to find a closed set e, containing all the sets, and to divide by means of the closed sets $e_1, e_1', e_2, e_2', \ldots$ which appear in the definitions of the difference sets concerned.

Note.—To divide e by means of $f_1, f_2, \ldots, f_s$, is to express e as a sum, the result of expanding

$$e \cdot (f_1 + Cf_1) \cdot (f_2 + Cf_2) \ldots (f_s + Cf_s).$$

Each term of this expansion is disjoint from every other term.

2.4. *Definition.* A finite linear combination of characteristics of closed sets is called a *function of class C*. Such a function is expressible in the form

$$f(x) = c_1\phi(e_1, x) + c_2\phi(e_2, x) + \ldots + c_n\phi(e_n, x),$$

where $e_1, e_2, \ldots, e_n$ are closed sets, and

$$\phi(e, x) = 1, \quad \text{if } x \text{ belongs to } e,$$

$$= 0, \quad \text{otherwise.}$$

2.41. If

$$c_1\phi(e_1, x) + c_2\phi(e_2, x) + \ldots + c_r\phi(e_r, x) = 0$$

everywhere, then

$$c_1\phi[n(e_1), x] + c_2\phi[n(e_2), x] + \ldots + c_r\phi[n(e_r), x] = 0$$

everywhere.

Subdivide the set $e_1 + e_2 + \ldots + e_r$ by means of the sets $e_1, e_2, \ldots, e_r$ and their complements. The same process can be applied to the sets $n(e_1), n(e_2), \ldots, n(e_r)$, and their complements, and the formulae will be the same. Hence, if the former expression of the hypothesis is zero on each of the former partial sets, the latter expression is zero on each of the latter partial sets.

2.5. *Definition.* If $f(x)$ is a function of class C,

$$f(x) = c_1\phi(e_1, x) + \ldots + c_r\phi(e_r, x),$$

we define its core function or *core* as

$$nf(x) = g(x) = c_1\phi[n(e_1), x] + \ldots + c_r\phi[n(e_r), x].$$

By the previous theorem this core is unique for a given $f(x)$.

2.6. *Definition.* $N(e)$ is said to be a mass of class C if it can be expressed in the form

$$N(e) = c_1 m(e_1 . e) + c_2 m(e_2 . e) + \ldots + c_r m(e_r . e),$$

where $e_1, e_2, \ldots, e_r$ are closed sets.

Evidently, if

$$f(x) = c_1 \phi(e_1, x) + \ldots + c_r \phi(e_r, x),$$

$$N(e) = \int_e f(x)\, dm(e).$$

2.61. If two functions of class C are "nearly equal" their cores are identical.

Let

$$f(x) = c_1 \phi(e_1, x) + \ldots + c_r \phi(e_r, x),$$

$$g(x) = c_1' \phi(e_1', x) + \ldots + c_s' \phi(e_s', x)$$

be the two functions, and let e be the set (of zero mass) where they are not equal.

$$f_1(x) = c_1 \phi(e_1 . Ce, x) + \ldots + c_r \phi(e_r . Ce, x),$$

$$g_1(x) = c_1' \phi(e_1' . Ce, x) + c_s' \phi(e_s' . Ce, x)$$

are identically equal, and their cores are identical by 2.41. The core of f differs from that of f_1 by the sum of terms of type

$$c_r \phi[n(e_r . e), x]$$

which are all zero, since $e_r . e$ is of zero mass and its core is the null set. Similarly, the cores of g_1 and g are identical and the theorem is proved.

2.7. *Definition.* If $N(e)$ is a mass of class C,

$$N(e) = c_1 m(e_1 . e) + \ldots + c_r m(e_r . e),$$

define its *derivative* as

$$DN = c_1 \phi[n(e_1) . x + \ldots + c_r \phi[n(e_r), x].$$

By 2.61, this derivative is unique.

2.7c. If $N(e)$ is a mass of class C, and if c is a constant,

$$D(cN) = c . DN.$$

2.7A. If $N_1(e)$, $N_2(e)$ are masses of class C,

$$D(N_1 + N_2) = DN_1 + DN_2.$$

2.7P. If $N(e)$ is a *positive* mass of class C,

$$DN \geqslant 0.$$

Let $$N(e) = c_1 m(e_1 . e) + \ldots + c_s m(e_s . e),$$

where now each set $e_1, e_2, \ldots, e_s$ is replaced by its core. By subdividing by means of $e_1, \ldots, e_s$ we can express $N(e)$ in the form

$$N(e) = k_1 m(f_1 . e) + \ldots + k_r m(f_r . e),$$

where $f_1, f_2, \ldots, f_r$ are disjoint difference core sets.

If this is non-negative for $e = f_1, f_2, \ldots, f_r$, it follows that $k_1, k_2, \ldots, k_r$ are non-negative, and

$$DN = k_1 \phi(f_1, x) + \ldots + k_r \phi(f_r, x)$$

is also non-negative.

Conversely, if N is of class C, and if DN is non-negative everywhere, N is a "positive" mass.

2.8. By similar reasoning, if $N(e)$ is a mass of class C and E is a core difference set, such that for all e,

$$N(E . e) \geqslant 0,$$

then the derivative of N is non-negative at points of E.

Conversely, if $N(e)$ is a mass of class C and E is a core difference set, such that the derivative of N is non-negative everywhere on E,

$$N(E . e) \geqslant 0.$$

2.9. *Definition.* The *desert* e_0 is the set of points belonging to open intervals of zero mass.

The part of a set not contained in e_0 may be called the *useful part* of the set.

The complement of the desert Ce_0 is the core of J, the whole p-space, which is a closed set. Since every set is contained in J, every core difference set is contained in Ce_0 and is disjoint from e_0.

2.91. Given b, the boundary of a closed interval $d = (p, q)$, where $p < q$, define b_0 as the set of points of b contained in open intervals δ, such that $m(d . \delta) = 0$.

Evidently b_0 includes $b . e_0$.

2 . 92. The core of a closed interval $d = (p, q)$, where $p < q$, is

$$n(d) = d \,.\, Ce_0 - b_0,$$

where b_0 refers to the set defined in (2 . 91).

If the basic mass $m(e)$ is such that no open intervals possess zero mass, e_0 and b_0 are null sets, and the core of a solid closed interval (one including an open interval) is the closed interval itself.

Again, if the boundary of an interval is of zero mass, its core is the null set, and, in this case, the core of the interval will be the same whether it be open or closed at any of its surfaces in any manner.

2 . 93. The derivative of $m(e)$ with respect to m is

$$\phi(Ce_0, x)$$

where e_0 is the desert.

3 . 0. *Definition.* If E is a core difference set and $M(e)$ a general mass, and if a mass $N(e)$ of class C exists such that, for all e,

$$M(E \,.\, e) \leqslant N(E \,.\, e),$$

define the *upper derivative* of M (with respect to m) *relative to* E at each point of E as the lower bound (including $-\infty$ as a possible value) of the derivatives of such masses N of class C. We denote it by UDM.

3 . 1. If E_1, E_2 are core difference sets, of which E_1 includes E_2, and if UDM relative to E_1 exists, UDM relative to E_2 exists and coincides with the former at all points of E_2. For if, for all e,

$$N(E_1 \,.\, e) \geqslant M(E_1 \,.\, e),$$

$$N(E_2 \,.\, e) \geqslant M(E_2 \,.\, e),$$

and, varying N, at points of E_2, UDM relative to E_2 does not exceed that relative to E_1.

Again, if N is any mass of class C, such that

$$N(E_2 \,.\, e) \geqslant M(E_2 \,.\, e),$$

choose a particular N_1, such that

$$N_1(E_1 \,.\, e) \geqslant M(E_1 \,.\, e),$$

and define $$N'(e) = N(E_2 \,.\, e) + N_1(CE_2 \,.\, e).$$

Then $N'(e)$ is of class C, and its derivative on E_2 coincides with that of N, while

$$N'(E_1 . e) \geqslant M(E_1 . e).$$

Varying N we see that, at points of E_2, UDM relative to E_1 does not exceed that relative to E_2.

3.11. If E_1, E_2 are core difference sets containing points in common and possessing upper derivatives relative to E_1 and E_2, these upper derivatives coincide on $E_1 . E_2$.

3.12. The upper derivative defined in (3.0) is unique (independent of E). Also if, for a particular point x, there can be found a core difference set E containing x and a mass $N(e)$ of class C, such that, for all e,

$$M(E . e) \leqslant N(E . e),$$

then the upper derivative of M at x is the lower bound of the derivatives of N at x for all such N.

From the remark at the end of (2.9) no such x can belong to the desert e_0.

3.13. *Definition.* If for the point x no such combination of a core difference set E (containing x) with a mass N (of class C) can be found satisfying the inequality of (3.12), the upper derivative of M is said to be $+\infty$ at x.

[*Exception.* When x is a point belonging to the deserts of both m and P, where $P(e)$ is the positive mass associated with $M(e)$, we define the upper derivative of M as 0.]

3.14. *Definition.* The *lower derivative* of a mass $M(e)$ is defined as minus the upper derivative of $-M(e)$. It is denoted by LDM.

3.15. *Definition.* Where the upper and lower derivatives coincide, we call their common value the derivative of M and denote it by DM.

3.16. If M is a mass of class C its upper and lower derivatives coincide with its derivative as defined before.

We need only prove this for the upper derivative and, evidently, over any core difference set E, M is itself a possible N, and the derivative of any other possible N (satisfying the inequality) is not less than that of

M. Hence on Ce_0 the upper derivative coincides with the derivative in the sense of (2 . 7).

Again, since $|M|(e_0) = 0$, if M is of class C, it follows that the upper derivative is 0 on e_0, by the exceptional rule in 3 . 13. But 0 is the value of the derivative from (2 . 7), since e_0 is disjoint from any core closed set.

3 . 17. Over a core difference set the derivative of $cm(e)$ is c if c is a constant.

3 . 21. If E is a core difference set, and if for all e,

$$M(E \cdot e) \leqslant 0,$$

then UDM is non-positive over E.

3 . 22. If E is a core difference set, and if for all e,

$$M(E \cdot e) \geqslant 0,$$

then LDM is non-negative over E.

3 . 3. If M_1, M_2 are masses

$$UD(M_1+M_2) \leqslant UDM_1+UDM_2.$$

If either UDM_1 or UDM_2 is $+\infty$, their sum is either $+\infty$ or meaningless $(\infty - \infty)$, so that the inequality will be true in a certain negative sense.

Otherwise, even if either or both is $-\infty$, x belongs to a core difference set E_1 and a core difference set E_2, and, therefore, to their product E (which cannot be null). If $N_1(e)$, $N_2(e)$ are *any* two masses of class C, such that

$$M_1(E \cdot e) \leqslant N_1(E \cdot e), \quad M_2(E \cdot e) \leqslant N_2(E \cdot e),$$

N_1+N_2 is a mass of class C and

$$(M_1+M_2)(E \cdot e) \leqslant (N_1+N_2)(E \cdot e).$$

If we vary N_1, N_2 independently and use (2 . 7A) the theorem is proved. There remains the exceptional case in which x belongs to the deserts of m, P_1, and P_2, where P_1, P_2 are the positive masses associated with M_1, M_2. But then x belongs to the desert of the positive mass associated with M_1+M_2, and all three terms are 0.

3 . 31. $$UDM \geqslant LDM.$$

For
$$0 = UD(M-M) \leqslant UDM+UD(-M) = UDM-LDM.$$

If UDM is $+\infty$ or LDM is $-\infty$, the theorem obviously stands. All other cases fall correctly under (3 . 3).

3 . 32. If M_1, M_2 are masses,

$$UD(M_1+M_2) \geqslant UDM_1+LDM_2.$$

For if
$$M = M_1+M_2,$$

$$UDM_1 = UD(M-M_2) \leqslant UDM-LDM_2.$$

The only difficulty arises when $UD\ (M_1+M_2)$ is $+\infty$, when the theorem stands; or when LDM_2 is $-\infty$, when the right-hand side is either $-\infty$ or meaningless.

3 . 41. If M_1, M_2 are masses and M_2 has a derivative at x,

$$UD(M_1+M_2) = UDM_1+DM_2.$$

For
$$UDM_1+DM_2 = UDM_1+LDM_2 \leqslant UD(M_1+M_2)$$
$$\leqslant UDM_1+UDM_2 = UDM_1+DM_2.$$

The only exception is that in which the right-hand expression is meaningless.

3 . 42. If M_1, M_2 are masses and possess derivatives at x, M_1+M_2 has a derivative at x and

$$D(M_1+M_2) = DM_1+DM_2.$$

Again, the case where the right-hand expression is meaningless must be excepted.

3 . 43. If M_1, M_2 are masses, and if E is a core difference set such that for all e,

$$M_1(E\,.\,e) \leqslant M_2(E\,.\,e),$$

then at all points of E,

$$UDM_1 \leqslant UDM_2, \quad LDM_1 \leqslant LDM_2.$$

These follow directly from the definition of UDM and LDM.

3.5. *Definition.* A general mass $M(e)$ is said to be *based on* m if $M(e)$ is zero whenever $m(e)$ is zero.

3.51. If $M(e)$ is a general mass, and if for a set E

$$M(E) < 0,$$

then there exists a closed set E' contained in E, such that for all open intervals δ, having points in common with E',

$$M(E' . \delta) < 0,$$

and that for all sets e $\qquad M(E' . e) \leqslant 0.$

If, in addition, M is based on m, the set E' can be taken to be a core closed set.

Proof. If $|M|(e)$ is the modular mass associated with $M(e)$, we can find a set E_1 in E of such a type that

$$(m+|M|)(E_1) = (m+|M|)(E)$$

and such that E_1 is the limit of an expanding sequence of closed sets. It follows that

$$M(E_1) = M(E) < 0.$$

Moreover, M is a mass, and therefore $M(E_1)$ is the limit of the masses in M of the sets of its defining sequence. Hence we can choose a closed set E_2 from this sequence contained in E, such that

$$M(E_2) < 0.$$

Introduce a dense net-system N_n, smooth with respect to $m+|M|$, and let

$$e_1 = \Sigma \delta_{1s}$$

be the sum of the closed meshes of the net N_1, for which

$$M(\delta_{1s} . E_2) < 0.$$

Then e_1 is a closed set and

$$M(e_1 . E_2) \leqslant M(E_2).$$

Again, let $\qquad e_2 = \Sigma \delta_{2s}$

be the sum of the closed meshes of N_2 *contained in* e_1, for which

$$M(\delta_{2s} . E_2) < 0.$$

Then e_2 is a closed set contained in e_1 and

$$M(e_2 . E_2) \leqslant M(e_1 . E_2) \leqslant M(E_2).$$

Proceeding in this manner, we have a contracting sequence of closed sets $e_n . E_2$, for each of which

$$M(e_n . E_2) \leqslant M(E_2).$$

Hence the limit set E'

$$E' = \lim e_n . E_2$$

is a closed set and

$$M(E') \leqslant M(E_2) < 0.$$

If δ is an interval disjoint from E', $M(E' . \delta) = 0$.

If δ is an interval of the net-system of order m containing one or more points of E', it is one of the intervals δ_m for which

$$M(\delta_m . E_2) < 0.$$

By the same reasoning as before, for all n greater than m,

$$M(\delta_n . e_n . E_2) \leqslant M(\delta . E_2),$$

and in the limit, $$M(\delta . E') \leqslant M(\delta . E_2) < 0.$$

Any open interval is the sum of at most a countable infinity of net intervals, and any common boundaries have zero mass in both m and M. Therefore, if δ is any open interval containing one or more points of E',

$$M(\delta . E') < 0.$$

It follows, by limiting processes if necessary, that for all intervals and therefore for all sets e,

$$M(E' . e) \leqslant 0.$$

If, in addition, M is based on m, E' can be replaced by its core set which is contained within it, and as the difference has zero mass in m it will have zero mass in M. Moreover, since $M(E') < 0$, the core of E' cannot be the null set in this case.

We may add that, if δ is any closed interval containing points of E' strictly interior to it, it consists of an open interval δ_1, containing points of E' and a boundary b, and

$$M(\delta . E') = M(\delta_1 . E') + M(b . E')$$

$$\leqslant M(\delta_1 . E') < 0.$$

3 . 6. If $M(e)$ is a general mass *based on* m, and if for a set E,

$$M(E) < 0,$$

there exists a core closed set E'' contained in E, over which the upper derivative of M with respect to m is negative.

Choose E', as in (3 . 51), replacing it by its core set. Since this is not a null set it has a non-zero mass in m. Hence there is a positive number η, such that

$$M(E') = -2\eta m(E'),$$

$$M(E') + \eta m(E') < 0.$$

Replacing M in (3 . 51) by $M + \eta m$ and E by E', there exists in E' a closed core set E'', such that, for any set e,

$$M(E'' . e) + \eta m(E'' . e) \leqslant 0,$$

$$M(E'' . e) \leqslant -\eta m(E'' . e).$$

But $-\eta m(E'' . e)$ is a mass of class C, and therefore over E'' the upper derivative of M is not greater than $-\eta$.

3 . 61. *If* $M(e)$ is a general mass *based on* m, and if over a set E the upper derivative is non-negative, then for any set

$$M(E . e) \geqslant 0.$$

For, if not, there is a set E_1 in E for which $M(E_1)$ is negative, and, by (3 . 6), there is at least a point in E_1 at which the upper derivative is negative. This contradicts the hypothesis.

3 . 7. Given a general mass $M(e)$ a set g_0 can be found such that, for all e,

$$M(Cg_0 . e) \geqslant 0, \quad M(g_0 . e) \leqslant 0.$$

And if, for a set E, $M(E) < 0$,

$E . g_0$ is not the null set.

Proceed by means of a dense net system N_n, smooth with respect to $M|(e)$, the modular mass corresponding to $M(e)$. Then let

$$e_r = \Sigma \delta_{rs}$$

the sum of the intervals δ of order r for which $M < 0$. Let

$$f_r = e_r \cdot e_{r+1} \dots,$$

$$g_0 = \lim f_r.$$

If δ is a mesh of order m and if r is greater than m, either f_r is disjoint from δ or else $e_r \cdot \delta$ is a sum of meshes of order r, for which

$$M(e_r \cdot \delta) < 0.$$

In this case, since $e_r \cdot \delta - e_r \cdot e_{r+1} \dots e_{r+p} \cdot \delta$ is a sum of meshes of various stages such that their mass in M is not negative,

$$M(e_r \cdot e_{r+1} \dots e_{r+p} \cdot \delta) \leqslant M(e_r \cdot \delta),$$

$$M(f_r \cdot \delta) \leqslant M(e_r \cdot \delta) < 0.$$

In either case $M(f_r \cdot \delta)$ is non-positive and, in the limit,

$$M(g_0 \cdot \delta) \leqslant 0.$$

Again, as s increases indefinitely,

$$\delta \cdot Cf_r = \lim \delta \cdot (Ce_r + Ce_{r+1} + \dots + Ce_{r+s}),$$

and, at each partial stage, the right-hand expression is either null or the sum of a finite number of meshes, for each of which M is non-negative. Hence $M(\delta \cdot Cf_r)$ is non-negative and, in the limit,

$$M(Cg_0 \cdot \delta) \geqslant 0.$$

Since the net system is dense and smooth for $|M|$ it follows that $M(g_0 \cdot e)$ is a non-positive mass and $M(Cg_0 \cdot e)$ is a non-negative mass for all sets e.

The last statement in the theorem follows immediately from

$$M(E \cdot g_0) = M(E) - M(E \cdot Cg_0) \leqslant M(E) < 0.$$

3.71. Let $g(n)$ be the set g_0, defined as in (3.7), but with $M + nm$ in place of M where n is a positive integer, and let $h(n)$ be the set Cg_0, where g_0 is defined as in (3.7), but with $M - nm$ in place of M. Then, for any interval δ,

$$M(\delta) + nm(\delta) < 0 \quad \text{implies} \quad M(\delta) + (n-1)\, m(\delta) < 0,$$

$$M(\delta) - nm(\delta) < 0 \quad \text{implies} \quad M(\delta) - (n+1)\, m(\delta) < 0.$$

Therefore $g(n)$ is included in $g(n-1)$ and $h(n)$ in $h(n-1)$.

Let g be the limit of $g(n)$, h that of $h(n)$ as n increases indefinitely Since g is included in $g(n)$,

$$M(\delta \,.\, g)+nm(\delta \,.\, g) \leqslant 0,$$

and, for any positive integer n,

$$m(\delta \,.\, g) \leqslant |M|(\delta) \div n.$$

Varying n, we see that for all intervals δ the mass in m of $\delta \,.\, g$ is zero. Thus the mass in m of g, and similarly that of h, is 0.

Again, for any set e,

$$\begin{aligned} -nm[e \,.\, Cg(n) \,.\, Ch(n)] &\leqslant M[e \,.\, Cg(n) \,.\, Ch(n)] \\ &\leqslant +nm[e \,.\, Cg(n) \,.\, Ch(n)], \end{aligned}$$

so that, if e is of zero mass in m,

$$M[e \,.\, Cg(n) \,.\, Ch(n)] = 0.$$

When we make n increase indefinitely,

$$M[e \,.\, Cg \,.\, Ch] = 0,$$

and $M[e \,.\, Cg \,.\, Ch]$ is based on m.

3 . 72. A general mass $M(e)$ is the sum of a mass based on m and a mass which is zero except on a set of zero mass in m. For, from (3 . 71),

$$M(e) = M[e \,.\, Cg \,.\, Ch]+M[e \,.\, (g+h)].$$

3 . 8. If M is based on m, the set E, where

$$UDM \geqslant l$$

is such that, for all e,

$$M(E \,.\, e) \geqslant lm(E \,.\, e).$$

For $M-lm$ is also based on m, and by (3 . 41) and (2 . 93),

$$\begin{aligned} UD(M-lm) &= UDM-l\phi(Ce_0, x) \\ &= UDM-l+l\phi(e_0, x). \end{aligned}$$

Hence, except on $E \,.\, e_0$, $UD(M-lm)$ is non-negative and, by (3 . 61),

$$M(E \,.\, e)-M(E \,.\, e_0 \,.\, e) \geqslant lm(E \,.\, e)-lm(E \,.\, e_0 \,.\, e).$$

But the desert is of zero mass in m, since it is the difference between J and its core. Also M is based on m, so that

$$M(E \,.\, e) \geqslant lm(E \,.\, e).$$

3 . 81. If M is based on m, the set E_i, where

$$l_i < UDM \leqslant l_{i+1}$$

is such that $$l_i m(E_i \,.\, e) \leqslant M(E_i \,.\, e) \leqslant l_{i+1} m(E_i \,.\, e).$$

For the set where UDM is greater than l is included in the set where it is not less than l_i, and the set where it is not greater than l_{i+1} is included in the set where LDM is not greater than l_{i+1}, that is, where $UD(-M_i)$ is not less than $-l_{i+1}$.

3 . 82. Laying a dense net system on the one-dimensional scale of l we see, from (3 . 8) and (3 . 81), that UDM is integrable with respect to m and that for any set e,

$$M(e) = \int_e UDM \,.\, dm(e),$$

provided that M is based on m.

3 . 83. If M is based on m, LDM is summable with respect to m and

$$M(e) = \int_e LDM \,.\, dm(e).$$

Replace M by $-M$ in (3 . 82).

3 . 9. If M is based on m its derivative exists, except at most on a set of zero mass in m, and

$$M(E) = \int_e DM \,.\, dm(e).$$

For $$\int_e (UDM - LDM) \,.\, dm(e) = 0$$

and the integrand is non-negative, so that it can differ from zero only over a set of zero mass in m.

3 . 91. If E is of zero mass in m, and if $M(e)$ is a general mass not necessarily based on m, but zero for all sets disjoint from E, there exists a set, $\lambda(E)$, the *extension* of E of zero mass in m, containing E, and such that, except on $\lambda(E)$, the derivative of $M(e)$ with respect to m is zero.

Since E is of zero mass in m, we can find a contracting sequence of sets, f_r, each containing E, each a complement of a closed set, and such that their limit set is of zero mass in m. The sets

$$Ce_0 \,.\, Cf_r = C(f_r + e_0)$$

will be closed sets disjoint from E and forming an expanding sequence, where e_0 is the desert for the mass m. Their core sets

$$n(Cf_r) \,.\, n(Ce_0) = Ce_0 \,.\, n(Cf_r)$$

will also form an expanding sequence of core closed sets disjoint from E and of the same mass as Cf_r. Define $\lambda(E)$, so that

$$C\lambda(E) = \lim Ce_0 \,.\, n(Cf_r).$$

Then the mass of $\lambda(E)$ is also zero and $\lambda(E)$ contains E. Furthermore, any point on $C\lambda(E)$ belongs to some core closed set $e_r = Ce_0 \,.\, n(Cf_r)$, and for this set, if e is any set,

$$M(e \,.\, e_r) = 0,$$

since e_r is disjoint from E. Therefore, by (3 . 21), (3 . 22), at every point of $C\lambda(E)$ the derivative of M with regard to m is zero and the theorem is proved.

3 . 92. If $m(e)$ is a positive mass, $M(e)$ a general mass not necessarily based on m, the derivative of M with respect to m exists, except possibly on a set E_1+E_2, where E_1 is of zero mass in both m and $|M|$, and E_2 is of zero mass in m; this derivative is integrable, and

$$M(e) = \int_e DM \,.\, dm(e) + M(E_2 \,.\, e).$$

[NOTE.—$|M|$ is the modular mass associated with m.]

From (3 . 72),

$$M(e) = M_1(e) + M_2(e),$$

$$M_1(e) = M(e \,.\, Cg \,.\, Ch), \quad M_2(e) = M[e \,.\, (g+h)].$$

Replace $g+h$ by $E_2 = \lambda(g+h)$,

using (3 . 91). Then

$$M(e) = M(CE_2 \,.\, e) + M(E_2 \,.\, e).$$

But CE_2 is contained in $Cg \,.\, Ch$, $M_1(e)$ is based on m, and CE_2 differs from $Cg \,.\, Ch$ by a set of zero mass in m, so that

$$M_1(e) = M(CE_2 \,.\, e).$$

It follows that $$M_2(e) = M(E_2 \,.\, e).$$

Again, on CE_2, the derivative of M_2 is zero and the derivative of M is equal to that of M_1, that is, it exists except possibly on a set E_1 of zero mass in m. But M_1 is based on m, so that, since E_1 is contained in CE_2,

$$M(E_1 \,.\, e) = M_1(E_1 \,.\, e) = 0.$$

Therefore E_1 is of zero mass in both m and $|M|$. Also, by (3 . 9) again,

$$\int_e DM \,.\, dm(e) = \int_e DM_1 \,.\, dm(e) = M_1(e)$$

$$= M(e) - M(E_2 \,.\, e).$$

4 . 1. If d is a solid closed interval (containing strictly interior points), and if $M(e)$ is a general mass, such that when e is any set $M(e \,.\, d)$ is zero, the derivative of M is everywhere 0 on the interior open interval δ.

For the hypothesis remains true when d is replaced by its core, and, on this core, by (3 . 21), (3 . 22), the derivative of M is everywhere 0. By (2 . 92) this core contains all of δ, except $\delta \,.\, e_0$.

On the other hand, $\delta \,.\, e_0$ is a part of the desert for both m and $|M|$, and on $\delta \,.\, e_0$ the derivative is 0 by the exceptional rule.

4 . 2. Let $f(x)$ be continuous within a solid closed interval, and on the boundary let it be continuous along it and into the interior, but not necessarily outwards. Let $f(x) = 0$ outside d. Then the derivative of the integral of $f(x)$ is

$$DM = D\left[\int_e f(x)\, dm(e)\right] = f(x)\, \phi(Ce_0,\, x)$$

everywhere, except possibly on the boundary of d.

If x is outside d, it can be enclosed in an open interval which has no part of its boundary in common with d, so that, by (4 . 1), the derivative is 0 everywhere outside d.

On d lay a dense smooth net system N_r, and consider the meshes as closed. Let μ_i, ν_i be the upper and lower bounds of $f(x)$ on a mesh δ_i

of order r. Then

$$\Sigma \nu_i m(\delta_i . e) \leqslant M(e) \leqslant \Sigma \mu_i m(\delta_i . e).$$

The upper and lower derivatives lie between

$$\Sigma \nu_i \phi[n(\delta_i), x], \quad \Sigma \mu_i \phi[n(\delta_i), x].$$

At a point which is not on a mesh boundary at any stage

$$\phi[n(\delta_i), x] = \phi[\delta_i . Ce_0, x],$$

and, as r increases indefinitely,

$$\lim \Sigma \nu_i \phi[\delta_i . Ce_0, x] = \lim \Sigma \mu_i \phi[\delta_i . Ce_0, x]$$
$$= f(x)\, \phi(Ce_0, x).$$

Now consider a point x on the common boundary of only two meshes δ_i, δ_j, of order r. Since the net is smooth the mass of the common boundary is 0, and the cores of δ_i, δ_j are disjoint. Moreover,

$$n(\delta_i) + n(\delta_j) = n(\delta_i + \delta_j)$$

and the latter contains all points not on the boundary of $\delta_i + \delta_j$ except points of e_0. Hence, if x belongs to e_0, it does not belong to $n(\delta_i)$ or $n(\delta_j)$, and

$$\Sigma \nu_i \phi[n(\delta_i), x] = \Sigma \mu_i \phi[n(\delta_i), x] = f(x)\, \phi(Ce_0, x).$$

If x does not belong to e_0, it belongs to one or other of $n(\delta_i)$ or $n(\delta_j)$, but not both, so that the above sums are equal to ν_i or ν_j and μ_i or μ_j respectively. Now, as the order of the mesh increases, these all approach $f(x)$, so that in this case, also,

$$\lim \Sigma \nu_i \phi[\delta . Ce_0, x] = \lim \Sigma \mu_i \phi[\delta . Ce_0, x]$$
$$= f(x)\, \phi(Ce_0, x).$$

If x belongs to four or more contiguous meshes, similar reasoning can be employed, so that the theorem is proved. It is to be noted that the theorem states nothing about the derivative on the boundary of the original interval d. But here, evidently,

$$D\left[\int_e f(x)\, dm(e)\right] = f(x)\, \phi(Cb_0, x),$$

where b_0 is the set defined in (2 . 91).

4 . 3. If $f(x)$ is continuous and bounded throughout J, then $D(M)$. the derivative of the integral of $f(x)$, is equal to

$$D\left[\int_e f(x)\,dm(e)\right] = 0 \quad \text{on } e_0,$$

$$= f(x) \quad \text{on } Ce_0.$$

For any point x can be enclosed in an open interval which, with its boundary, makes a closed interval d. By (4 . 2), the derivative of

$$M_1(e) = M(e \,.\, d) = \int_{e\,.\,\delta} f(x)\,dm(e),$$

is

$$DM_1 = f(x)\,\phi(Ce_0,\, x)$$

at x; while, by (4 . 1), the derivative of

$$M_2(e)\; M(e \,.\, Cd)$$

is zero at x.

THE 7-TANGENT QUADRICS OF THE SAME SYSTEM OF THE C_7^5

By WILLIAM P. MILNE.

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1. *Introduction.*

The Abelian canonical curve $N_8^{(4)}$ in space of four dimensions is the curve of intersection of three hyperquadrics. If a point on $N_8^{(4)}$ be taken and the curve projected on to a space of three dimensions, we get a curve of degree 7 and genus 5, which we shall denote by C_7^5. The curve C_7^5 can be regarded as the residual intersection of two cubic surfaces intersecting each other in a conic. Its equation will be of the form,

$$\frac{x}{P} = \frac{y}{Q} = \frac{z}{R}, \tag{1}$$

where P, Q, R are quadrics, and the point of intersection of the planes x, y, z is called the "pole" of the curve.

The properties of the C_7^5 have received considerable attention at the hands of Sturm* and Montesano†. The C_7^5 also presents itself in the study of other subjects, such as "Bilinear Conic Complexes"‡ and "On the analytical representation of congruences of conics"§, etc. So far as I know, however, the properties of the C_7^5 most nearly related to those of the Abelian canonical curve $N_8^{(4)}$ have not been intensively studied. This forms the subject of the present paper.

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* R. Sturm, *Synthetische Untersuchungen über Flächen 3 Ordnung* (Leipzig, 1867).

† D. Montesano, *Torino Atti*, 27 (1892), 660.

‡ D. Montesano, *Napoli Atti*, 2 (1913), 15.

§ C. G. F. James, *Proc. Camb. Phil. Soc*, 21 (1922), 3.

2. *The 7-tangent quadrics of the* C_7^5.

We define two 8-tangent hyperquadrics of the $N_8^{(4)}$ as belonging to the same system, if their $2\times 8 = 16$ points of contact lie on the same hyperquadric. Let H_0 be a hyperquadric touching the $N_8^{(4)}$ eight times, and let F_0 be one of the points of contact. Let the $N_8^{(4)}$ be the curve of intersection of the three hyperquadrics H_1, H_2, H_3. Then it is easy to prove that $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ can be so determined that $\lambda_0 H_0+\lambda_1 H_1+\lambda_2 H_2+\lambda_3 H_3$ shall be a hypercone having its vertex at F_0. Thus we shall have a 7-tangent hypercone whose vertex is F_0. Consider two such 7-tangent hypercones whose vertices are each at F_0 and which belong to the same system. Then, plainly, a hypercone exists having its vertex at F_0 and passing through each of the 14 points of contact. Projecting from F_0 on to a space of three dimensions, we see that *the points of contact of any two 7-tangent quadrics, of the same system, with a* C_7^5 *lie on the same quadric.*

The object of the present paper is to obtain by the methods of synthetic geometry the complete configuration of 7-tangent quadrics of the same system belonging to the C_7^5.

3. *General properties of the sextic surface* Γ.

Consider the curve C_7^5 and let us take 7 points (A) arbitrarily upon it. Let a quadric through the 7 points (A) cut the C_7^5 again in the 7 points (B). We say that (A) and (B) form two "residual" sets of 7 points. Two sets (A) and (A') will be called "co-residual", if each is residual to the same set (B).

Let us now consider three co-residual groups (A), (A'), (A''), and let (P) be a variable group residual with all three. Defining an "associated system of 8 points" as being the 8 points common to three quadrics, we wish to find the locus of that point P which, with the 7 points (P), form an associated system. Plainly a (1, 1)-correspondence exists between the quadrics $(A)(P)$, $(A')(P)$, $(A'')(P)$, which may be represented in the form

$$\left.\begin{aligned} \xi U +\eta V +\zeta W &= 0 \\ \xi U'+\eta V'+\zeta W' &= 0 \\ \xi U''+\eta V''+\zeta W'' &= 0 \end{aligned}\right\} \qquad (2)$$

Since the 8 points of intersection of the above three quadrics consist of the 7 points (P) and the eighth associated point P, we see that the locus

of P is the sextic surface

$$\Gamma \equiv \begin{vmatrix} U & V & W \\ U' & V' & W' \\ U'' & V'' & W'' \end{vmatrix} = 0. \tag{3}$$

We may view this result in a somewhat different way. Let (ξ, η, ζ) be regarded as the coordinates of a point on the plane π. Then the condition that the quadric $\xi U+\eta V+\zeta W = 0$ shall pass through an assigned point (x, y, z, t) is linear in (ξ, η, ζ), and therefore a line l in π corresponds to the point (x, y, z, t). Similarly, we obtain other two lines l' l'' in π if we use the sets of points (A'), (A'') respectively. If (x, y, z, t) be chosen at random, the lines l, l', l'' in π will not in general be concurrent, the condition for concurrency being that (x, y, z, t) shall lie on Γ. Hence, leaving on one side the 7 points (P), we see that a (1, 1)-correspondence exists between the points (x, y, z, t) of Γ and (ξ, η, ζ) of π. Thus, since Γ can be mapped, point for point, on a plane, we see that

Γ is a rational surface.

It follows that (x, y, z, t) can each be expressed rationally in terms of homogeneous polynomials in ξ, η, ζ of degree n, where we ascertain the value of n by finding the degree of the rational curves on Γ that correspond to straight lines in π. For simplicity, consider the line $\zeta = 0$. We have then to obtain the locus of P as defined by

$$\xi U + \eta V = 0,$$

$$\xi U' + \eta V' = 0,$$

$$\xi U'' + \eta V'' = 0.$$

This is plainly the curve of degree 12,

$$\begin{Vmatrix} U & V \\ U' & V' \\ U'' & V'' \end{Vmatrix} = 0, \tag{4}$$

since two of its constituent quartic surfaces $UV'-U'V = 0$, and $UV''-U''V = 0$ intersect in a curve of degree 16, part of which is the

quadri-quadric $U = V = 0$. It can easily be seen that this curve of degree 12 resolves itself into the C_7^5 and a further quintic curve. For the set of 7 points (P) lies on both $\xi U + \eta V = 0$ and $\xi U' + \eta V' = 0$ and hence on $UV' - U'V = 0$. Thus the C_7^5, on which lies the set of points (P), must lie *in toto* on this quartic surface. It follows that the C_7^5 lies on every quartic surface defined by the determinants of (4), and hence each of the above curves of degree 12 resolves itself into the C_7^5 and a further quintic curve. Thus straight lines on π correspond to quintic curves on Γ, and these quintic curves are, from the method of their definition, rational. Two such quintic curves will cut each other in one point, since straight lines in π intersect in one point.

If (A) be a set of 7 assigned points on the C_7^5, and if a quadri-quadric curve be given passing through these 7 points, the locus of the eighth associated point P of the set of 7 points (P) in which the C_7^5 is cut by any quadric containing the above quadri-quadric curve is a rational quintic curve. Two such quintic curves intersect in one point.

The rational surface Γ can be mapped upon a plane by expressing the coordinates of any point (x, y, z, t) upon it as homogeneous polynomials in (ξ, η, ζ) of degree 5.

It will be convenient to designate such rational quintic curves as "generating quintics".

If we take three sets of 7 points (B), (B'), (B''), residual to (A), (A'), (A''), such that $(A)(B)$ lie on U, $(A)(B')$ on V, $(A)(B'')$ on W, $(A')(B)$ on U', &c , we shall obtain the corresponding nets of quadrics,

$$\xi' U + \eta' U' + \zeta' U'' = 0,$$

$$\xi' V + \eta' V' + \zeta' V'' = 0,$$

$$\xi' W + \eta' W' + \zeta' W'' = 0,$$

and another system of generating quintic curves derived from (B), (B'), (B'') in the same way as the former system is derived from (A), (A'), (A''). These two systems of generating quintic curves we shall refer to as "conjugate". Plainly, also, the complete system of quadrics containing two sets of 7 points, one of which is residual to (A) and the

other to (B) is given by the equation :

$$\xi'(\xi U+\eta V+\zeta W)+\eta'(\xi U'+\eta V'+\zeta W')+\zeta'(\xi U''+\eta V''+\zeta W'')=0. \quad (5)$$

We next proceed to investigate the relationship existing between Γ and the C_7^5. We have already seen that the quartic surface,

$$\begin{vmatrix} U & V \\ U' & V' \end{vmatrix} = 0,$$

cuts Γ in the two curves of the twelfth degree,

$$\begin{vmatrix} U & V \\ U' & V' \end{vmatrix} = 0,$$

$$\begin{vmatrix} U & V \\ U'' & V'' \end{vmatrix}$$

and

$$\begin{Vmatrix} U & V & W \\ U' & V' & W' \end{Vmatrix} = 0,$$

each of which has been shown to degenerate into the C_7^5 and a generating quintic curve, the two quintics thus obtained being conjugate. Thus every quartic surface defined by the minors of the determinant,

$$\Gamma \equiv \begin{vmatrix} U & V & W \\ U' & V' & W' \\ U'' & V'' & W'' \end{vmatrix}$$

intersects the sextic surface Γ in the C_7^5 taken twice over and in two conjugate generating quintic curves. We deduce at once that

the C_7^5 is a double curve on Γ.

It follows, immediately, that Γ must have an equation of the form

$$\begin{aligned}\Gamma \equiv A(yR-zQ)^2+B(zP-xR)^2+C(xQ-yP)^2 \\ +2F(zP-xR)(xQ-yP)+2G(xQ-yP)(yR-zQ) \\ +2H(yR-zQ)(zP-xR)=0, \quad (6)\end{aligned}$$

since the number of conditions absorbed by a surface of degree N having to contain a C_n^p as a double-curve is $(3N-4)n-5(p-1)$, which becomes becomes in this case 78, since $N=6$, $n=7$, $p=5$. But a sextic surface can be made to satisfy $\frac{7.8.9}{6}-1 \equiv 83$ conditions, leaving the

possibility of satisfying $83-78 \equiv 5$ further conditions. Thus the form (6) is the most general form for Γ.

We therefore see that Γ can be generated geometrically in the following way. Let

$$\lambda x+\mu y+\nu z = 0 \tag{7}$$

$$\lambda P+\mu Q+\nu R = 0 \tag{8}$$

be corresponding members of the net of planes and the net of quadrics by which the C_7^5 is defined. Let λ, μ be connected by the relation

$$K \equiv A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu = 0. \tag{9}$$

Then Γ will be generated by the conics (7), (8) subject to the condition (9).

Let there be given a cone K of the second degree having the pole of C_7^5 as its vertex. The conics through the 6 points of the C_7^5 in which the curve is cut by the tangent planes to K generate a sextic surface Γ having the C_7^5 as a double curve.

We shall in future refer to K as the "Generating Cone" of the surface Γ.

The generating cone K passes through the twelve pinch-points on the double curve C_7^5 of the sextic surface Γ.

Consider a straight line p connecting the pole of the C_7^5 with any other point P on the curve. Then the two tangent planes to K containing p each cut the sextic surface Γ in a generating conic. The two tangent planes to Γ at P are therefore the two planes each containing the tangent at P to the C_7^5 and one of the tangents at P to these two conics. If P lie on K, the two tangent planes through p to K coincide, and hence P is a pinch-point.

Let us now view Γ as a conical hypersurface projected on to a space of three dimensions from the point F_0 on the $N_8^{(4)}$ in space of four dimensions, F_0 being the vertex of the above conical hypersurface. The following result is immediately obtained :—

Γ *is the enveloping-cone drawn from the point*

$$F \equiv (x, y, z, t, u) \equiv (0, 0, 0, 0, 1)$$

on the Abelian normal curve

$$xu+P = yu+Q = zu+k = 0$$

in space of four dimensions, to the hypersurface

$$a(xu+P)^2+b(yu+Q)^2+c(zu+R)^2+2f(yu+Q)(zu+R)$$
$$+2g(zu+R)(xu+P)+2h(xu+P)(yu+Q) = 0,$$

where $A \equiv bc-f^2$, *&c., this being the general form* (10) *of hypersurface of degree* 4 *containing the* $N_8^{(4)}$ *as a double curve.*

The condition that the plane (7) shall touch the quadric (8) is of the fifth degree in λ, μ, ν and hence planes through the pole of the C_7^5, intersecting the C_7^5 in 6 further points that lie on a conic degenerating into two trisecants of the curve, envelop a cone Φ of class 5. But there are 10 planes which touch both K and Φ, and hence *there are* 10 *planes through the pole of the* C_7^5 *that intersect* Γ *in* 10 *pairs of straight lines, which are each trisecants of the* C_7^5.

We shall see later that these are the only straight lines on Γ.

A sextic surface having the C_7^5 as a double curve can, without loss of generality, be expressed in the form $(xQ-yP)^2 = \rho(yR-zQ)(zP-xR)$. Hence, *any cubic surface through the* C_7^5 *intersects* Γ *in two conics each meeting the* C_7^5 *in* 6 *points. In particular, when the two conics become coincident, the cubic surface will touch* Γ *at points lying on that conic.*

4. *Mapping of the sextic surface* Γ.

We are now in a position to obtain the general equations whereby the surface Γ can be mapped on a plane π. Every plane section of Γ, being a sextic curve having 7 double points where the plane is met by the C_7^5, is of genus 3. It has already been shown that these sextic curves, which are sections of Γ, map into quintic curves, which must also be of genus 3 and must therefore possess 3 double points or their equivalent amongst the base-points of the mapping. Since Γ is of degree 6, the fundamental quintic curves of the mapping must possess either (i) 3 double and 7 single base-points, or (ii) 1 triple and 10 single base-points. If we consider (i) in detail, we find that there are only 3 straight lines on the surface, whereas we have already revealed the existence of 20 straight lines on Γ. The mapping of Γ on the plane π is therefore of the type represented by (ii).

The fundamental curves in the mapping of Γ *on* π *are therefore curves of degree* 5, *having one triple base-point* E_0 *and* 10 *single base-points* $E_r (r = 1, 2, \ldots, 10)$.

To obtain the algebraic expression for the mapping, consider the pencil of quartic curves having double-points at E_0 and passing through the 10 points E_r. Let F, G be their two further points of intersection and choose E_0FG as the triangle of reference, such that $\xi = 0$, $\eta = 0$, $\zeta = 0$ are the sides E_0F, E_0G, FG respectively. Let two quartics of the pencil be

$$u_2\xi^2 + u_3\xi + u_2'\xi\eta = 0,$$

$$v_2\xi^2 + v_3\xi + v_2'\xi\eta = 0,$$

where u_r, v_r are homogeneous and of degree r in ξ, η. The mapping can then be carried out by means of the equations :

$$\left.\begin{aligned} x &= \alpha\xi(u_2\xi^2 + u_3\xi + u_2'\xi\eta) + \{(u_2v_2' - v_2u_2')\,\xi + (u_3v_2' - v_3u_2')\} \\ y &= \beta\eta(u_2\xi^2 + u_3\xi + u_2'\xi\eta) + \{(u_2v_2' - v_2u_2')\,\xi + (u_3v_2' - v_3u_2')\} \\ z &= \gamma\xi(v_2\xi^2 + v_3\xi + v_2'\xi\eta) + \{(u_2v_2' - v_2u_2')\,\xi + (u_3v_2' - v_3u_2')\} \\ t &= \delta\eta(v_2\xi^2 + v_3\xi + v_2'\xi\eta) + \{(u_2v_2' - v_2u_2')\,\xi + (u_3v_2' - v_3u_2')\} \end{aligned}\right\}. \quad (11)^*$$

5. *Curves on the sextic surface* Γ.

Proceeding in the usual way we obtain the following systems of curves lying entirely on the surface Γ :—

I. 20 *straight lines.*

(i) 10 straight lines e_r corresponding to the base-points E_r.

(ii) 10 straight lines ϵ_r corresponding to the straight lines E_0E_r.

There are only 20 *straight lines on* Γ, *which appear as the intersecting pairs* e_r, ϵ_r.

II. *Conics.*

There are no conics on the surface other than those which correspond to straight lines in π *passing through* E_0, *and the planes of these conics must therefore be the tangent planes of the generating cone* K.

* Compare : Th. Reye, *Math. Annalen*, 48 (1897), 140 ; F. Enriques, *Math. Annalen*, 46 (1895), 179 ; G. Castelnuovo, *Rend. di Palermo*, 4 (1890), 85.

III. 512 *twisted cubics.*

Let T_0 be the twisted cubic on Γ corresponding to the triple base-point E_0. We may then tabulate the twisted cubics on Γ as follows, where i denotes the number of points of intersection of the twisted cubics of the various systems with T_0 and n denotes the number of such twisted cubics in each system.

TABLE A.

System	i	n	Corresponding mapping-curve on π
(1)	T_0	1	E_0
(2)	0	45	Straight line through 2 points E_r
(3)	1	210	Conic through E_0 and 4 points E_r
(4)	2	210	Cubic through 6 points E_r and having node at E_0
(5)	3	45	Quartic through 8 points E_r and having triple point at E_0
(6)	4	1	Quintic through 10 points E_r and having quadruple point at E_0

In this Table $r \neq 0$. We shall denote the twisted cubic of the system (6) by S_0.

IV. *Rational quintic curves.*

There are two types of rational quintic curves on Γ.

TABLE B.

TYPE I.

System	i	Corresponding mapping-curve on π
(1)	0	Straight line
(2)	1	Conic through E_0 and 2 points E_r
(3)	2	Cubic through 4 points E_r and having node at E_0
(4)	3	Quartic through 6 points E_r and having triple point at E_0
(5)	4	Quintic through 8 points E_r and having quadruple point at E_0
(6)	5	Sextic through 10 points E_r and having quintuple point at E_0

In this Table $r \neq 0$. It is evident that these systems of quintic curves contain as degenerate cases the straight lines, conics, and twisted cubics already discussed.

TYPE II.

System	i	n	Corresponding mapping-curves on π
(1)	0	252	Conic through 5 points E_s
(2)	1	1260	Cubic going once through E_0, twice through $1E_r$, once through $5E_s$
(3)	2	2520	Quartic going twice through E_0, twice through $2E_r$, once through $5E_s$
(4)	3	2520	Quintic going 3 times through E_0, twice through $3E_r$, once through $5E_s$
(5)	4	1260	Sextic going 4 times through E_0, twice through $4E_r$, once through $5E_s$
(6)	5	252	Septimic going 5 times through E_0, twice through $5E_r$, once through $5E_s$

In this Table $r \neq s \neq 0$.

These exhaust all the types of rational quintics on the surface Γ.

6. *Further properties of the sextic surface* Γ.

We now proceed to develop the propositions (i) that any member of the pencil of quadrics containing a quadri-quadric through the 7 points (A) cuts the C_7^5 in 7 points whose eighth associated point describes a rational quintic on Γ corresponding to a straight line in π; (ii) that a conjugate rational quintic generated in the same way from a residual set (B) corresponds to a sextic curve through 10 points E_r and having a quintuple point at E_0.

Consider a straight line l in π to which corresponds the rational quintic L_5^0 on Γ. To every point on l corresponds a unique point on L_5^0 and *vice versa*. If, however, l passes through E_0, then to the point E_0 corresponds the twisted cubic T_0, which must therefore be regarded as having every one of its points the eighth associated point of the 7 points (B_0), in which it intersects the C_7^5. Thus a quadric through (A) will contain T_0. Similarly a quadric through (B) will contains S_0.

Of all the quadrics of the net (A), *one, and only one, cuts the* C_7^5 *in* 7 *residual points* (B_0) *lying on a twisted cubic. Similarly, of all the quadrics of the net* (B), *one, and only one, cuts the* C_7^5 *in* 7 *residual points* (A_0) *lying on a twisted cubic.*

Thus the two residual sets of 7 points (A_0) and (B_0) lie on the twisted cubics S_0 and T_0 respectively. The following result presents itself immediately :—

The twisted cubics S_0 *and* T_0 *lie on the same quadric* Ψ *and intersect each other in* 4 *points. Hence, all the generators of one system of* Ψ *must be chords of both* S_0 *and* T_0, *and all the generators of the opposite system of* Ψ *must meet both* S_0 *and* T_0 *once.*

We shall in future use the residual sets of 7 points (A_0) and (B_0) as the basic sets of 7 points, by means of which the other rational curves on Γ are generated in the manner already described in the preceding sections. Thus a pencil of quadrics through S_0 can be regarded as the pencil of quadrics containing S_0 and one of its chords. Also to every line in π corresponds a pencil of quadrics through (A), *i.e.* in particular, a pencil of quadrics containing S_0 and one of its chords. Hence to every straight line in π corresponds a chord of S_0, and conversely.

The pencil of quadrics containing S_0 and any one of its chords cuts the C_7^5 in sets of 7 points whose eighth associated point describes a rational quintic on Γ. To every chord of S_0 corresponds a rational quintic of the system (1), *Type* I, *Table* B, *and conversely.*

Let us now consider the pencil of quartic surfaces containing the C_7^5, S_0, and any assigned chord p of S_0. We have already seen that the residual curve of intersection of this pencil of quartic surfaces is a generating rational quintic L_5^0 of Γ. Let us denote the ranks of C_7^5, S_0, p, L_5^0 by r_1, r_2, r_3, r_4 respectively, and the number of the mutual intersections of these curves by i_{12}, i_{13}, etc. Then, if we define the polar surface of a line with respect to two surfaces U_4, V_4 as being the surface generated by the curve of intersection of the first polars of points on the given line with respect to U_4 and V_4, we get the following equations in the usual way, the degree of the polar surface in this case being $4+4-2 \equiv 6$,

$$r_1+i_{12}+i_{31}+i_{14} = 7\times 6 = 42,$$

$$r_2+i_{23}+i_{12}+i_{24} = 3\times 6 = 18,$$

$$r_3+i_{31}+i_{23}+i_{34} = 1\times 6 = 6,$$

$$r_4+i_{14}+i_{24}+i_{34} = 5\times 6 = 30.$$

Now, using the known facts that $r_1 = 22$, $r_2 = 4$, $r_3 = 0$, $r_4 = 8$, $i_{12} = 7$, $i_{14} = 13$, $i_{23} = 2$, we deduce that $i_{34} = 4$, $i_{24} = 5$, $i_{31} = 0$. Thus the assigned chord of S_0 intersects the corresponding generating quintic curve in 4 points and is therefore its quadrisecant.

All quadrics containing the twisted cubic S_0 and an assigned chord p of S_0 cut the C_7^5 in a set of 7 points whose eighth associated point describes a rational quintic on the sextic surface Γ, having the assigned chord p as its quadrisecant, and intersecting S_0 5 times.

The quadrisecants of all generating quintics of the same system intersect Γ in two further points that lie on a twisted cubic.

Consider next the particular case when p is a chord of both S_0 and T_0. The quintic curve will then resolve itself into T_0 and a conic which is met

by p twice, whence the plane of this conic must contain the generator p of the quadric Ψ, and must therefore be a tangent plane of Ψ. The generating cone K of Γ, which is touched by the planes containing the conics of Γ, must therefore be the enveloping cone of the quadric Ψ whose vertex is the pole of the C_7^5.

The generating cone K of Γ is the enveloping cone drawn from the pole of the C_7^5 to the quadric Ψ containing the twisted cubics S_0 and T_0.

All quadrics through S_0 and a chord of both S_0 and T_0 cut the C_7^5 in 7 points whose eighth associated point describes the conic of Γ lying in the plane containing both this chord and the pole of the C_7^5. In particular, if the above chord of S_0 be that generator of Ψ which meets e_r, the required locus will be the straight line ϵ_r.

To the straight line joining any two of the four points of intersection of S_0 and T_0 corresponds one of two conjugate generating quintic curves according as the line is regarded as a chord of S_0 or of T_0. In either case the straight line is the quadrisecant of each quintic curve, and cuts it in the four remaining points where it meets Γ.

Consider next the line $E_r E_s$ in π and let p_{rs} be the corresponding chord of S_0. Then one quadric of the pencil $S_0 p_{rs}$ will contain e_r and another will contain e_s, since e_r, e_s correspond to the points E_r, E_s respectively. Hence e_r will either meet S_0 *twice* and will not meet p_{rs} or it will meet both p_{rs} and S_0 *once*. But we have already seen that e_r meets S_0 *once* only, and hence E_r must meet p_{rs} *once* only. Similarly e_s meets p_{rs} once only. Thus p_{rs} is the unique chord of S_0 which meets both e_r and e_s.

All quadrics containing S_0, and any one of the 45 chords of S_0 which meet two of the lines e_r, e_s of Γ, which intersect S_0 once only, cut the C_7^5 in a set of 7 points, whose eighth associated point generates a twisted cubic of the system (2) of Table A.

It follows that the twisted cubics of the system (5) of Table A can be generated in a similar way by using T_0 and the chords of T_0 that meet both ϵ_r and ϵ_s.

We proceed next to consider the method of generating geometrically the twisted cubics of the system (3) of Table A. To a conic of π corresponds a rational curve of degree 10 on Γ. Hence, if the point (ξ, η, ζ) describes a conic on π, the quadric $\xi U + \eta V + \zeta W = 0$ will intersect the C_7^5 in 7 points whose eighth associated point describes the above rational curve of degree 10 on Γ.

The contact quadrics of any quartic surface of the type

$$AU^2+BV^2+CW^2+2FVW+2GWU+2HUV = 0,$$

having nodes at the 7 points (A) and at their eighth associated point,

intersect the C_7^5 *in* 7 *points, the locus of whose eighth associated point is a rational curve on* Γ *of degree* 10.

In particular, if instead of the 7 points (A), we choose the 7 points (A_0) lying on S_0, we obtain the particular form of quartic surface which contains S_0 as a double curve, and which is, in fact, the form of quartic surface studied by Clebsch [*Math. Annalen*, 2 (1870), 445–466]. Such a surface can be regarded as being generated by those chords of S_0 that belong to a linear complex. The conics of the plane π, to which correspond twisted cubics on Γ, are those conics that pass through E_0 and 4 points $E_r (r \neq 0)$. But to the point E_r corresponds the quadric containing S_0 and e_r, and hence e_r will touch the above Clebschian quartic surface. Hence two coincident chords of S_0 will meet e_r and will belong to the required linear complex. Now we know that, in the case of any quadric containing S_0, two of its generators which are chords of S_0 belong to a given linear complex, and it is also easy to show that a unique linear complex can be found whose pairs of lines of the above type common to five such quadrics coincide in pairs (compare the theorem that a unique conic exists intersecting each of five co-planar lines in two coincident points). Hence, choosing Ψ and four quadrics each containing S_0 and one of the lines e_r as our five quadrics, we obtain the 210 linear complexes required. The 210 twisted cubics of the system (4), Table A, can be similarly obtained by means of T_0.

7. *7-tangent quadrics of the* C_7^5.

Let us consider now the particular case when the original 7 points (A) are the points of contact of a quadric with the C_7^5. The systems of points (A) and (B) may now be regarded either as residual or as co-residual. In fact, the two systems of points (A) and (B) become coincident. Consider the three sets of 7 points (A), (A'), (A''), and let the 7-tangent quadrics U_0, V_0, W_0 touch the C_7^5 at the points (A), (A'), (A'') respectively. Let U_1, V_1, W_1 be the quadrics containing both the sets of 7 points, (A') and (A''), (A'') and (A), (A) and (A') respectively. Then, if we so select our triangle of reference in π that to (ξ, η, ζ) occupying the positions (1, 0, 0), (0, 1, 0), (0, 0, 1) correspond the quadrics U_0, V_0, W_0, the corresponding quadrics (2) will become

$$\left.\begin{aligned} \xi U_0 + \eta W_1 + \zeta V_1 &= 0 \\ \xi W_1 + \eta V_0 + \zeta U_1 &= 0 \\ \xi V_1 + \eta U_1 + \zeta W_0 &= 0 \end{aligned}\right\}, \qquad (12)$$

The sextic surface Γ will then assume the modified form Γ', where

$$\Gamma' \equiv \begin{vmatrix} U_0 & W_1 & V_1 \\ W_1 & V_0 & U_1 \\ V_1 & U_1 & W_0 \end{vmatrix} = 0. \tag{13}$$

The following results are now plain :—

The general equation to the quadric containing the points of contact of any two 7-tangent quadrics of the same system of the C_7^5 is of the form

$$\xi_1\xi_2 U_0+\eta_1\eta_2 V_0+\zeta_1\zeta_2 W_0+(\eta_1\zeta_2+\eta_2\zeta_1)\,U_1+(\zeta_1\xi_2+\zeta_2\xi_1)\,V_1 \\ +(\xi_1\eta_2+\xi_2\eta_1)\,W_1 = 0.$$

In particular, the general equation to the 7-tangent quadrics of the same system is

$$\xi^2 U_0+\eta^2 V_0+\zeta^2 W_0+2\eta\zeta U_1+2\zeta\xi V_1+2\xi\eta W_1 = 0.$$

The sextic surface Γ', where

$$\Gamma' \equiv \begin{vmatrix} U_0 & W_1 & V_1 \\ W_1 & V_0 & U_1 \\ V_1 & U_1 & W_0 \end{vmatrix} = 0,$$

may therefore be regarded as the envelope of the 7-tangent quadrics of the same system, or as the locus of the eighth associated point of the 7 points of contact of the 7-tangent quadrics of the same system.

We proceed to investigate in greater detail the properties of Γ, and we shall find it convenient for this purpose to use Cayley's designation of an "octadic" surface as being a quartic surface having 8 associated points as nodes.

Consider the octadic surface represented by the minor of U_0 in the determinantal form Γ', viz. $V_0W_0-U_1^2$. It follows from § 3 that, since the systems of points (A) and (B) become coincident, the two systems of conjugate generating quintics will also become coincident, and hence that the octadic surface $V_0W_0-U_1^2=0$ will touch Γ' along a generating rational quintic curve. Hence, *the surface Γ' is touched along each of its generating quintic curves by an octadic surface containing the C_7^5 and the contact-quadrics of such octadics are the 7-tangent quadrics of the C_7^5. The 8 nodes of the octadic surface lie on Γ'.*

The following result also presents itself immediately :—

The twisted cubics S_0 and T_0 coincide into a single twisted cubic S_0' or T_0' and Γ' is touched along S_0' (or T_0') by the quadric Ψ'. S_0' (or T_0') will touch the generating cone K' of Γ' at three points, and the plane con-

taining these three points will be the polar plane of the pole of the C_7^5 with respect to Ψ'.

We have seen that one generating quintic curve of Γ' consists of S_0', e_1', ϵ_1'. Let the conjugate generating quintic that becomes coincident with it consist of S_0', ϵ_{10}', e_{10}', so that e_1', ϵ_1' shall ultimately coincide with ϵ_{10}', e_{10}' respectively. Let the point of intersection of e_1', ϵ_1' be H_1'. Then the planes $H_1'e_{10}'$ and $H_1'\epsilon_{10}'$ would each become tangent planes to Γ' at H_1' in the limit, and hence H_1' must be a node on Γ'. Reasoning in the same way with regard to the other cases of coincidence among the lines of the surface Γ', and supposing that e_1', ϵ_1'; e_2', ϵ_2'; e_3', ϵ_3'; e_4', ϵ_4'; e_5', ϵ_5' coincide with ϵ_{10}', e_{10}'; ϵ_9', e_9'; ϵ_8', e_8'; ϵ_7', e_7'; ϵ_6', e_6' respectively, we see that Γ' possesses 5 nodal points H_1', H_2', H_3', H_4', H_5'.

Γ' possesses 5 nodal points, at each of which intersect a pair of lines on the surface, and the planes containing these 5 pairs of lines each pass through the pole of the C_7^5 and touch the generating cone K'.

It may be noted that the general sextic surface Γ containing the C_7^5 as a double curve possesses 5 disposable constants, which, on *a priori* grounds, suggests the possibility of Γ being able to acquire 5 nodes.

Consider, again, the cone Φ of class 5 enveloped by those planes that pass through the pole of the C_7^5 and cut the C_7^5 in 6 further points lying on two trisecants. Plainly there are in general 10 planes through the pole of the C_7^5 which are tangent planes to each of the cones Φ and K, and cut the surface Γ in its 10 pairs of lines. But, in the case of Γ', these 10 pairs of lines present themselves by coincidence as 5 pairs, and hence the common tangent planes of Φ' and K' must consist of 5 coincident pairs of planes, *i.e.* the cone K' is a 5-tangent cone of Φ'.

The generating cone K' of Γ' is a 5-tangent cone of Φ'.

To every system of 7-tangent quadrics of a C_7^5 corresponds a sextic surface Γ', containing the C_7^5 as a double curve, and possessing, in addition, 5 nodal points. The generating cone K' of Γ' is a 5-tangent cone of the cone Φ' of class 5.

Γ' will be the enveloping-cone from a point on the Abelian normal curve $N_8^{(4)}$ in space of 4 dimensions to a 5-nodal hypersurface of degree 4 containing $N_8^{(4)}$ as a double curve. Hence, to every system of 8-tangent hyper-quadrics of the $N_8^{(4)}$ corresponds a 5-nodal hyper-surface of degree 4 containing the $N_8^{(4)}$ as a double curve.

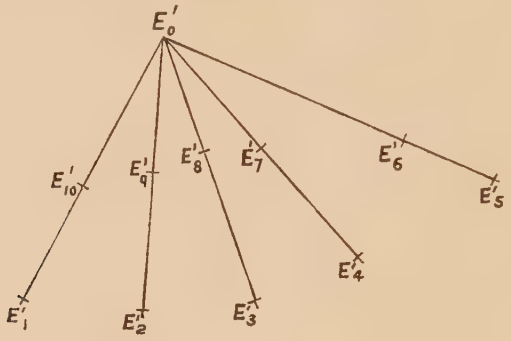
8. *The mapping of the 5-nodal sextic surface Γ'.*

We have seen that the generating rational quintics of Γ correspond to straight lines in π, and that the conjugate rational quintics correspond

to sextic curves in π, having 5-ple points at E_0, and passing once through the 10 points E_r. But in the case of Γ', the two systems of conjugate quintics must coincide, and therefore the sextic curve, having a 5-ple point at E_0 and passing through the 10 points E_r must degenerate into a straight line and a quintic curve having a 5-ple point at E_0', and passing once through the 10 points E_r. This latter quintic curve must, therefore, consist of 5 straight lines intersecting in E_0', and such that on each line lie two of the base points E_r'.

In the case of Γ', *the base points of the mapping consist of a triple point* E_0' *and* 10 *single points lying two by two on* 5 *straight lines through* E_0'.

Let the base points be distributed as in the following diagram :—



To straight lines on π' correspond rational quintic curves on Γ', and hence the straight line $E_0'E_1'E_{10}'$—to the points E_0', E_1', E_{10}' of which correspond the curves S_0', e_1', e_{10}' respectively—transforms into a curve of degree $5-(3+1+1)\equiv 0$ on Γ', *i.e.* into a point through which pass S_0', e_1', e_{10}'. This is plainly the nodal point H_1' as before defined, and similarly with regard to the other points H_2', H_3', H_4', H_5'.

The twisted cubic S_0' *passes through the* 5 *nodes of* Γ'.

Since every straight line in π' intersects all the 5 lines through E_0' and the other base points, we see that every generating rational quintic on Γ' passes through the 5 nodal points.

Every generating rational quintic on Γ' *passes through the* 5 *nodes of the surface, and any two generating quintic curves intersect in the* 5 *nodal points of* Γ' *and in one further point.*

There is no need to particularize further the properties of Γ'. The characters and relations of the various curves on Γ' proceed at once from its mapping on π'.

I take the opportunity of expressing my indebtedness to Mr. P. Fraser, of Bristol University, for his help and advice during the preparation of the paper.

PHYSIOLOGISCHE PROBLEME, WELCHE AUF "BELASTETE" RANDWERTAUFGABEN FÜHREN.

Von O. FRANK *und* A. SOMMERFELD (in München).

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Bei einer Reihe von physiologischen oder physiologisch-technischen Problemen kommen mechanische Zustände in Betracht, die schliesslich auf die Schwingungen eines einfachen Systems zurückgeführt werden können. Das System besteht aus einem elastischen Rohr, in dem sich Flüssigkeit befindet. An dem einen Ende ist das Rohr durch eine elastische Membran verschlossen. Die Aufgabe besteht darin, die Ereignisse zu beschreiben, wenn an einem Ende des Rohrs eine freie oder erzwungene Welle erregt wird. Die mechanischen Verhältnisse bleiben grundsätzlich dieselben, wenn die Rohrwand starr und die Flüssigkeit kompressibel oder wenn beide Teile elastisch deformierbar sind.

Die Lösung hat eine praktische Bedeutung nach zwei Richtungen. Es kann sich sowohl um die Berechnung der Schwingungs*frequenzen* handeln als um die Darstellung der Schwingungs*formen*. Die Berechnung der Frequenzen genügt für eine grosse Reihe von Aufgaben. Sie bietet keine besonderen Schwierigkeiten. Handelt es sich um die rechnerische Darstellung der Schwingungsformen, so treten durch die Randbedingung, die durch die Einfügung der elastischen Membran in das System gegeben sind, Schwierigkeiten auf, die durch die Fouriersche Analyse im gewöhnlichen Sinne nicht überwunden werden können. Sie erfordern ähnliche Methoden wie sie zur Lösung des Problems der belasteten Saite notwendig werden. Solche Randwertaufgaben werden als "belastet" bezeichnet.

Die Beispiele, die diesem Schema entsprechen, sind die folgenden:

(1) Der Mechanismus des äusseren und mittleren Ohres stellt ein gekoppeltes System von mehreren schwingenden Einzelsystemen dar. Für den vorliegenden Gegenstand interessiert nur die Luftsäule in dem äusseren Gehörgang in ihrer Wechselwirkung mit dem Trommelfell. Dieses idealisierte Ohr wird durch Fig. 1 dargestellt.

(2) Auf das gleiche Schema führt aber auch die Mechanik des Kreislaufs. Die Wellen, die vom Herzen in die peripheren Gebiete entsandt werden, werden hier reflektiert und gelangen wieder zum Herzen zurück. Dass sie in der Peripherie so reflektiert werden, wie wenn das Gefäss hier verschlossen wäre, geht aus der Form des peripheren Pulses bestimmt hervor. Es fragt sich nun, wie die Reflexion an der Wurzel des Arteriensystems erfolgt. An der Wurzel des Gefäss-systems wirkt die Elastizität der Herzwand, solange die Klappen nicht geschlossen sind. Sind sie geschlossen, so ist es wesentlich die Elastizität der *Klappen*, welche in Betracht zu ziehen ist.

(3) Für die Pathologie bedeutungsvoll könnte die Untersuchung der Wellenfrequenz sein, wenn eine Erweiterung der Wurzel des Gefässes bei einem Aorten-Aneurysma vorliegt, das dehnbarer ist als der übrige Teil des Gefäss-systems.

(4) Grundsätzlich ähnlich sind die Vorgänge, die *während des Schlusses der Aortenklappen* und unmittelbar nachher stattfinden. Hier handelt es sich um Erscheinungen, die in der Technik als "Widerstoss" bezeichnet werden. Bei dem Widerstoss im gewöhnlichen Sinn besteht das Ventil aus einer starren Platte, bei dem Kreislauf aus den elastischen Taschenklappen. Die genaue Darstellung der Vorgänge, die beim Schluss der Klappen stattfinden, die Feststellung des Energieverlustes, der unter physiologischen und pathologischen Umständen eintritt, die Ableitung des zweiten Herztones aus der Stosswirkung des Klappenschlusses, gehören bekanntlich zu den wichtigsten Aufgaben der Kreislauf-Physiologie und -Pathologie.

(5) Eine Reihe von physiologisch-technischen Problemen kann ebenfalls durch die mathematische Analyse des Schemas gefördert werden. Wenn die *elastischen Manometer*, die in der Kreislaufmethodik gebraucht werden, auf ihre höchste Leistungsfähigkeit gebracht werden sollen, so muss ebensowohl die Kompressibilität der Flüssigkeit als die Ausdehnungsfähigkeit der Wand berücksichtigt werden. Unmittelbar kann das Schema auf die seit *Marey* so vielfach gebrauchte *Luft-transmission* für die Registrierung von Bewegungsvorgängen angewandt werden.

(6) Schliesslich sei noch angegeben, dass die elektrischen Wellen in einem Draht, dessen Länge gegen die Wellenlänge der Schwingungen nicht vernachlässigt werden darf und an dessen Ende ein Kondensator angeschlossen ist, wieder auf dasselbe Problem führen.

Als Beispiele für die analytische Behandlung wählen wir eines aus dem Gebiete der Schallübertragung (vgl. § 1 bis 3), Gerade bei ihm wird die Eigenart der Systeme zur Geltung kommen in dem Auftreten der belasteten Orthogonalitätsbedingungen. Von den für die Analyse des Kreislaufs wichtigen Beispielen ist eines ausgewählt (vgl. § 5), bei dem nach Ermittlung der Eigenfrequenzen die Behandlung auf Grund der Fourierschen Analyse keine Schwierigkeiten bereitet.

1. *Mathematischer Ansatz für das Problem des idealisierten Ohrs.*

Im Anschlusse an Fig. 1 entspreche $x = 0$ dem Anfang des äusseren Gehörgangs (der Ohrmuschel $x = l$ dem Trommelfell, p sei der Störungsdruck, d.h. der Ueberdruck über dem Normaldruck, ρ die Dichte im ungestörten Zustand. Die Geschwindigkeit u werde als überall gleich gerichtet und innerhalb des Querschnittes gleich gross angenommen, ihr Quadrat wird vernachlässigt. Dann lautet die *Bewegungsgleichung*

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial u}{\partial t} \tag{1}$$

und die *Kontinuitätsgleichung*, in der wir die zeitliche Dichteänderung durch die damit proportionale Druckänderung ersetzen :

$$\frac{\partial p}{\partial t} = -\kappa \frac{\partial u}{\partial x}. \tag{2}$$

κ bedeutet allgemein den Volumelastizitätsmodul, dessen Grösse identisch ist mit dem Normaldruck, wenn es sich um isotherme Zustandsänderungen handelt. Die Schallgeschwindigkeit sei a ; sie wird wie bekannt

$$a = \sqrt{\left(\frac{\kappa}{\rho}\right)}.$$

Die Anfangsbedingungen für $t = 0$ sowie die Grenzbedingungen

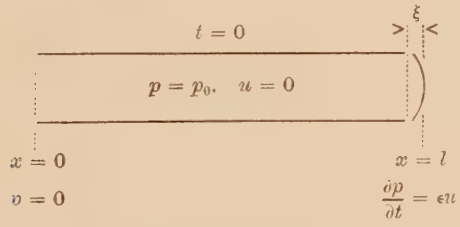

Fig. 1.

für $x = 0$ und $x = l$ sind aus Fig. 1 ersichtlich. Wir nehmen

also an, dass die Luft anfangs in Ruhe sei, aber unter einem (event. längs x variablem) Ueberdruck p_0 stehe. Nun wird an der Stelle $x = 0$ der Ueberdruck plötzlich aufgehoben, die Wellenbewegung beginnt, wobei für $t > 0$ an der Stelle $x = 0$ ständig der Normaldruck herrscht. Die *Randbedingung für* $x = l$ bedeutet, dass die elastische Membran auf eine Druckänderung dp mit einer zu dp proportionalen zusätzlichen Ausbiegung ξ reagiert, wobei $dp = \epsilon d\xi$ ist. ξ ist als Durchschnitt der Ausbiegung über den Querschnitt F zu denken, so dass $F\xi$ den Rauminhalt zwischen der ausgebogenen Membran und der Ebene $x = l$ bedeutet. Da die Luft bei $x = l$ die Ausbiegung der Membran mitmacht, wird daselbst $u = d\xi/dt$ und $\epsilon u = dp/dt$. ϵ ist ein für die Membran charakteristischer Elastizitätsmodul (von der Dimension atm/cm).

Wir wollen jetzt dasselbe Problem einmal durch die Unbekannte p ausdrücken, das andere Mal allein durch u. Indem wir u eliminieren, erhalten wir aus (1) and (2) bekanntlich :

$$\frac{\partial^2 p}{\partial t^2} = a^2 \frac{\partial^2 p}{dx^2}. \tag{3}$$

Um die Anfangsbedingung $u = 0$, welche für $t = 0$ und beliebige x zwischen 0 und l gilt, durch p auszudrücken, müssen wir sie nach x

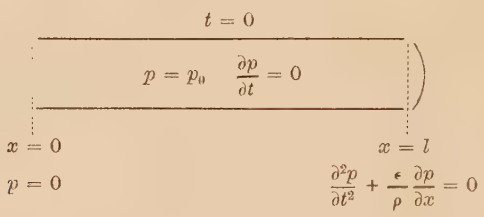

Fig. 2.

differentieren und Gl. (2) anwenden. Entsprechend müssen wir die Randbedingung für $x = l$, welche für $t > 0$ gilt, nach t differentieren und Gl. (1) anwenden. Das Ergebnis zeigt das Schema der Fig. 2.

Andererseite ergibt sich bei der Elimination von p zunächst

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}; \tag{3a}$$

sodann müssen wir die Anfangsbedingung $p = p_0$ nach x differentieren, um Gl. (1) benutzen zu können. Bei der Randbedingung für $x = l$

dagegen können wir direkt Gl. (2) benutzen ohne vorherige Differentiation. Im ganzen haben wir jetzt das Schema der Fig. 3.

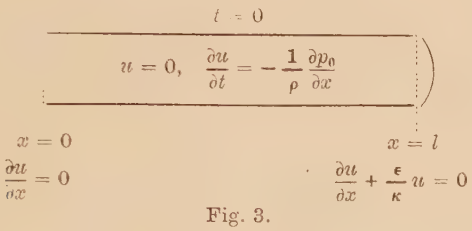

Fig. 3.

Die Randbedingung der Fig. 3 hat die z. B. in der Theorie der Warmeleitung übliche Form. Wir werden sehen, dass wir bei dieser Formulierung des Problems mit den üblichen Methoden der Fourier-Analyse auskommen. Dagegen weicht die entsprechende Bedingung der Fig. 2 davon insofern ab, als $\partial^2 p/\partial t^2$ an stelle von p steht. Wir werden zeigen, dass dies auf die oben angegebene Schwierigkeit führt und zu einer Erweiterung der Fourierschen Analyse Anlass gibt.

Uebrigens möge bemerkt werden, dass unsere Formulierung des akustischen Problems in erster Linie den Verhältnissen beim physiologischen Experiment (vgl. No. 5 der Einleitung) entspricht. In zweiter Linie dient es als Modell für die Wahrnehmung eines Knalls, also einer besonderen Art von Geräuschen, nicht eines reinen Tons.

2. *Integration des Problems für die Unbekannte p bezw. u.*

Nach Gl. (3) und den Bedingungen (Fig. 2):

$$p = 0 \text{ für } x = 0, \quad \frac{dp}{dt} = 0 \text{ für } t = 0$$

lautet der Fouriersche Ansatz für p

$$p = \sum_n A_n \sin w_n x \cos w_n a t. \tag{4}$$

Die Randbedingung für $x = l$ (aus Fig. 2) bestimmt w_n als Wurzel der Gl.

$$-w_n^2 a^2 \sin w_n l + \frac{\epsilon}{\rho} w_n \cos w_n l = 0,$$

d.h.

$$\lambda \operatorname{tg} \lambda = \mu \quad \text{mit} \quad \lambda = w_n l, \quad \mu = \frac{\epsilon l}{\rho a^2} = \frac{\epsilon l}{\kappa}. \tag{5}$$

Bekanntlich gibt es unendlich viele Wurzeln dieser Gleichung. Dementsprechend hat man unendlich viele Koeffizienten A zur Verfügung

und vermutet, dass man durch sie der Anfangsbedingung $p = p_0$ genügen könne :

$$p_0 = \Sigma A_n \sin w_n x. \tag{6}$$

Versucht man aber dies in Fourierscher Weise zu tun, so stösst man auf die Schwierigkeit, dass

$$\int_0^l \sin w_n x \sin w_m x \, dx \neq 0. \tag{7}$$

Es ist nämlich

$$\int_0^l \cos (w_m \pm w_n) x \, dx = \frac{\sin (w_m \pm w_n) l}{w_m \pm w_n} = \cos w_m l \cos w_n l \, \frac{\operatorname{tg} w_m l \pm \operatorname{tg} w_n l}{w_m \pm w_n}$$

$$= \pm \mu \, \frac{\cos w_m l \cos w_n l}{w_m w_n l} \quad [\text{wegen (5)}].$$

Durch Addition oder Subtraktion der beiden hierin enthaltenen Aussagen, erhält man für $m \neq n$

$$\int_0^l \cos w_m x \cos w_n x \, dx = 0, \tag{8 a}$$

$$\int_0^l \sin w_m x \sin w_n x \, dx = -\mu \, \frac{\cos w_m l \cos w_n l}{w_m w_n l}. \tag{8 b}$$

Gl (8b) zeigt in Uebereinstimmung mit (7), dass die übliche Methode zur Bestimmung der Koeffizienten A_n in Gl. (6) versagt, dagegen weist Gl. (8a) darauf hin, dass diese Methode bei der Berechnung von u zum Ziele führen wird. Nach Gl. (3a) und den Bedingungen (Fig. 3)

$$u = 0 \text{ für } t = 0, \qquad \frac{\partial u}{\partial x} = 0 \text{ für } x = 0$$

haben wir u in der Form anzusetzen :

$$u = \sum_n B_n \cos w_n x \sin w_n a t. \tag{9}$$

Die Randbedingung für $x = l$ aus Fig. 3 bestimmt natürlich w_n als Wurzel derselben Gl. (5) wie vorher. An Stelle von (6) haben wir nunmehr, um das Problem für u zu lösen, die Gleichung zu erfüllen :

$$f(x) = -\frac{1}{\rho a} \frac{\partial p_0}{\partial x} = \sum_n B_n w_n \cos w_n x. \tag{10}$$

Wir berechnen nach (8a)

$$\beta_n w_n B_n = \int_0^l f(x) \cos w_n x \, dx, \tag{11}$$

mit der Abkürzung

$$\beta_n = \int_0^l \cos^2 w_n x \, dx. \tag{11 a}$$

Die Berechnung von u ist also gelungen.

Wir kehren zur Berechnung von p zurück. Vermöge der Gl. (1) oder (2) und der Ansätze (4) und (9) besteht zwischen A_n und B_n die Beziehung

$$A_n = -\rho a B_n. \tag{12}$$

Nach Gl. (11) muss also sein:

$$\beta_n A_n = -\frac{\rho a}{w_n} \int_0^l f(x) \cos w_n x \, dx,$$

also nach der Bedeutung von $f(x)$ in (10)

$$\beta_n A_n = \frac{1}{w_n} \int_0^l \frac{dp_0}{dx} \cos w_n x \, dx.$$

Führen wir rechts eine partielle Integration aus, so entsteht:

$$\beta_n A_n = p_0(l) \frac{\cos w_n l}{w_n} + \int_0^l p_0(x) \sin w_n x \, dx. \tag{13}$$

Hierbei wurde angenommen, dass $p_0(0) = 0$ ist, dass also die für $t = 0$ vorgeschriebene Funktion $p = p_0$ der für $t > 0$ und $x = 0$ gegebenen Bedingung $p = 0$ nicht widersprechen soll. Dies ist besonders in dem praktisch allein realisierbaren Fall zu beachten, wo p_0 konstant gesetzt wird. Man hat dann den für $x > 0$ konstanten Ueberdruck p_0 durch eine an der Stelle $x = 0$ plötzlich abfallende Kurve in den Wert $p_0 = 0$ stetig überzuführen; andernfalls würde sich nach Gl. (10), (11), und (12) das widersinnige Resultat ergeben $A_n = 0$. Dagegen folgt aus (13) in diesem Falle:

$$\beta_n A_n = p_0 \frac{\cos w_n l}{w_n} + p_0 \int_0^l \sin w_n x \, dx = \frac{p_0}{w_n}.$$

Aus (13) und (6) schliessen wir: Es ist sehr wohl möglich, eine willkürliche Funktion in eine Reihe zu entwickeln, die nach $\sin w_n x$ fortschreitet, nur sind die Koeffizienten nicht nach der gewöhnlichen Fourierschen Methode zu bestimmen, sondern nach Formel (13).

Wir können letztere Formel der gewöhnlichen Bestimmungsweise annähern, wenn wir das Integrationssymbol einführen* :

$$(14) \qquad \bar{\int} f(x) \sin w_n x \, dx = \int_0^l f(x) \sin w_n x \, dx + \frac{l}{\mu} f(l) \sin w_n l.$$

Dann wird die rechte Seite von (13) bei Berücksichtigung von (5) direkt identisch mit

$$\bar{\int} p_0(x) \sin w_n x \, dx.$$

Setzen wir ferner in (14) $\sin w_n x$ für $f(x)$, so entsteht, bei abermaliger Berücksichtigung von (5),

$$(15) \qquad \bar{\int} \sin^2 w_n x \, dx = \int_0^l \sin^2 w_n x \, dx + \frac{l}{\mu} \sin^2 w_n l$$

$$= \int_0^l \sin^2 w_n x \, dx + \frac{\sin w_n l \cos w_n l}{w_n}.$$

Man verifiziert aber leicht

$$\int_0^l \cos^2 w_n x \, dx = \frac{l}{2} + \frac{\sin w_n l \cos w_n l}{2 w_n},$$

$$\int_0^l \sin^2 w_n x \, dx = \frac{l}{2} - \frac{\sin w_n l \cos w_n l}{2 w_n}.$$

Gl. (15) besagt also [vgl. (11a)].

$$\bar{\int} \sin^2 w_n x \, dx = \int_0^l \cos^2 w_n x \, dx = \beta_n$$

und Gl. (13) lässt sich endgültig schreiben

$$(16) \qquad A_n \bar{\int} \sin^2 w_n x \, dx = \bar{\int} p_0(x) \sin w_n \, dx.$$

Damit ist der Anschluss an die Fouriersche Koeffizientenberechnung gewonnen. Indem man schliesslich in (14) $\sin w_m x$ statt $f(x)$ einsetzt, erhält man

$$\bar{\int} \sin w_m x \sin w_n x \, dx = \int_0^l \sin w_m x \sin w_n x + \frac{l}{\mu} \sin w_m l \sin w_n l,$$

wo die rechte Seite wegen (8b) und (6) verschwindet.

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* A. Kneser, *Integralgleichungen*, (2 Aufl. Braunschweig, 1922), 117.

Wir können daher sagen: *Die Eigenfunktionen* $\sin w_n x$, $\sin w_m x$ *genügen der modifizierten Orthogonalitätsbedingung:*

$$\int \sin w_n x \sin w_m x \, dx = 0, \tag{17}$$

durch welche Gl. 16) als unmittelbare Folge von (6) verständlich wird.

3. *Terminologisches, die belastete Saite.*

Gl. (17) bezeichnen wir mit Kneser l.c. als "belastete Orthogonalität": die ganze Problemstellung heisst "belastete Randwertaufgabe". Ihr besonderer Charakter rührt, wie im Anschluss an Fig. 2 bereits angedeutet wurde, her von dem Vorkommen des zweiten Differentialquotienten $\partial^2 p/\partial t^2$ in der Grenzbedingung für $x = l$.

Die Bezeichnung ist entstanden aus dem Problem der inhomogenen schwingenden Saite* [Dichte $\rho(x)$]. Ihre Differentialgleichung lautet

$$\rho(x) \frac{\partial^2 p}{\partial t^2} = \kappa \frac{\partial^2 p}{\partial x^2}. \tag{18}$$

Macht man den Ansatz

$$p = e^{i w_n \sqrt{(\kappa)} t} \phi_n(x)$$

so bestimmen sich die Eigenfrequenzen w_n und die Eigenfunktionen ϕ_n aus der Gleichung:

$$\frac{d^2 \phi_n}{dx^2} + \rho(x) w_n^2 \phi_n = 0 \tag{18 a}$$

zusammen mit den Grenzbedingungen, z.B.

$$\phi_n = 0 \quad \text{für} \quad x = 0 \quad \text{und} \quad x = l. \tag{18 b}$$

Aus (18a) und (18b) folgt die Orthogonalitätsbedingung nach der Schlussweise des Greenschen Satzes

$$\int_0^l \phi_n(x) \phi_m(x) \rho(x) \, dx = 0 \quad \text{für} \quad m \neq n, \tag{19}$$

* Vgl. Lord Rayleigh, *Theory of Sound*, II (2 Aufl. London, 1894), 201. Den im Folgenden dargestellten Zusammenhang unseres Problems mit der inhomogenen Saite verdanken wir einem freundlichen Brief von Herrn R. Courant.

und von da aus die Entwicklung einer willkürlichen Funktion $f(x)$, etwa des Anfangszustandes von p, nach den Eigenfunktionen ϕ_n:

$$(20) \qquad f(x) = \Sigma A_n \phi_n(x), \quad A_n \int_0^l \phi_n^2(x)\,\rho(x)\,dx = \int_0^l f(x)\,\phi_n(x)\,\rho(x)\,dx.$$

Man gehe nun über zum Falle einer *konzentrierten Belastung*, indem man die Funktion $\rho(x)$ zwischen $x_0-\epsilon < x < x_0+\epsilon$ anwachsen lässt wie μ/ϵ, und im übrigen Gebiet etwa $\rho = 1$ wählt. Dann geht (19) im Limes $\epsilon = 0$ über in:

$$(19\,a) \qquad \int_0^l \phi_n(x)\,\phi_m(x)\,dx + \mu\phi_n(x_0)\,\phi_m(x_0) = \bar{\int} \phi_n \phi_m\,dx = 0.$$

Die hier eingeführte Bezeichnung $\bar{\int}$ entspricht durchaus derjenigen des vorigen §; der Unterschied, dass dort die Stelle $x = x_0$ mit dem Endpunkt $x = l$ zusammenfiel, ist offenbar unwesentlich. Indem wir die analoge Bezeichnung auf die in (20) vorkommenden Integrale anwenden, schreiben wir:

$$(20\,a) \qquad A_n \bar{\int} \phi_n^2\,dx = \bar{\int} f\phi_n\,dx,$$

$$\bar{\int} f\phi_n\,dx = \int_0^l f(x)\,\phi_n(x)\,dx + \mu f(x_0)\,\phi_n(x_0),$$

$$\bar{\int} \phi_n^2\,dx = \int_0^l \phi_n^2(x)\,dx + \mu\phi_n^2(x_0).$$

Wir wollen uns aber weiterhin überzeugen, dass bei der Saite ebenso wie bei unserem ursprünglichen Problem mit konzentrierter Belastung eine Grenzbedingung (bezw. "Uebergangsbedingung") auftritt, welche durch das Vorkommen von $\partial^2 p/\partial t^2$ ausgezeichnet ist. Die Differentialgleichung (18) mit $\rho = \text{const.}$ gilt jetzt natürlich nur für $x \leqq x_0$ und verlangt eine Zusatzbedingung für $x = x_0$. Bedeutet p die transversale Verrückung, κ die Spannung in der Saite, so ist $\kappa\,\partial p/\partial x$ die transversale Komponente der Spannung. Die Bewegung der Zusatzmasse μ regelt sich dann durch:

$$\mu \frac{\partial^2 p}{\partial t^2} = \kappa \left(\frac{\partial p}{\partial x}\right)_{x=x_0+\epsilon} - \kappa \left(\frac{\partial p}{\partial x}\right)_{x=x_0-\epsilon}.$$

Legen wir insbesondere die Masse μ an das Ende der Saite ($x_0 = l$), so geht die Zusatzbedingung über in

$$\frac{\partial^2 p}{\partial t^2} + \frac{\kappa}{\mu} \frac{\partial p}{\partial x} = 0 \quad \text{für} \quad x = l$$

und dies stimmt (bis auf die Bezeichnung der Konstanten) genau mit der Grenzbedingung für $x = l$ in Fig. 2 überein.

4. *Das Problem des Kabels.*

Das in No. 6 der Einleitung genannte Kabelproblem entspricht bei Vernachlässigung des Ohmschen Widerstandes genau dem akustischen Problem nach § 1 und 2. Sei e die Spannung, i der Strom, λ und κ Selbstinduktion und Kapazität für die Kabellänge 1, so lauten die Differentialgleichungen :

$$\frac{\partial e}{\partial x} = -\lambda \frac{\partial i}{\partial t} \quad \text{(Induktionsgesetz)}, \tag{21}$$

$$\frac{\partial e}{\partial t} = -\frac{1}{\kappa} \frac{\partial i}{\partial x} \quad \text{(Kontinuitätsgleichung der Ladung)}. \tag{22}$$

Anfangs- und Grenzbedingungen mögen nach dem Schema der Fig. 4 festgelegt werden : Das Kabel sei anfangs stromfrei und auf das Potentiel

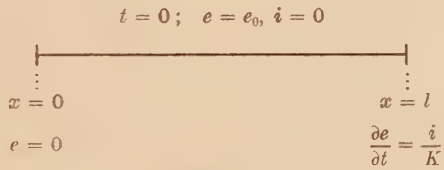

Fig. 4.

e_0 statisch geladen. Für $t = 0$ werde es am einen Ende ($x = 0$) geerdet ; am anderen Ende ($x = l$) sei der Kondensator von der Kapazität K angeschlossen ; K sei der Grössenordnung nach vergleichbar mit der Gesamtkapazität κl des Kabels.

Diese Bedingungen sowie die Differentialgl. (21), (22), sind dieselben wie in unserem akustischen Problem Fig. 1, wenn wir e mit p, i mit u identifizieren. Wir schliessen daraus :

Das elektrische Problem führt auf eine *Fourier-Entwicklung im gewöhnlichen Sinne*, wenn wir i als Unbekannte verwenden. Es führt dagegen auf eine *belastete Randwertaufgabe*, wenn wir e verwenden. Die Belastung besteht in dem angehängten Kondensator. Die Konstante μ der transcendenten Gleichung (5) wird gleich dem Verhältnis der Kapazität κl der ganzen Leitung zur Kapazität K des angehängten Kondensators.

5. *Ein Problem des Kreislaufs.*

Hier ist zunächst zu beachten, dass im Gegensatz zu §§ 1 und 2 die Flüssigkeit (Blut) als inkompressibel, die Röhre (Arterienwand) als dehnbar angenommen werden muss. Die Konstante κ in Gl. (2) ist also jetzt der Elastizitätsmodul der Arterie für transversale Dehnung. Um ein einfaches Beispiel zu definieren, legen wir die Nebenbedingungen durch das Schema 6 fest: $x = 0$ entspreche dem peripheren Teil des Gefäss-systems. Diesen behandeln wir im Sinne von No 2 der Einleitung als geschlossenes Ende und setzen daher hier $u = 0$. $x = l$ entspreche dem centralen Teil des Gefäss-systems, dem Herzen, dessen Deformierbarkeit ähnlich wirken soll wie eine elastische Membran. Der in Betracht kommende Elastizitätsmodul werde wie früher ϵ genannt.

Wir idealisieren das Problem dahin, dass wir dauernd mit demselben ϵ operieren, also von der Veränderung des Systems durch den Schluss der Klappen ganz absehen. Die Herzmuskulatur erzeugt durch ihre Kontraktion einen auf die Herzfüllung wirkenden Druck P, der als gegebene Funktion von t angesehen wird. Der ungefähre Verlauf von P

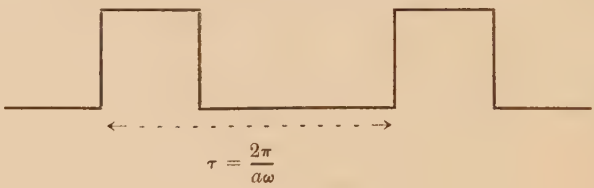

Fig. 5.

ist in Fig. 5 skizziert. Die kürzere Erhebung entspricht der Systole die längere der annähernden Ruhe während der Diastole des Herzens. Durch die Hinzufügung von P in der Gleichung für $x = l$ (vgl. Fig. 6) wird die frühere "homogene" zu einer "inhomogenen Randbedingung".

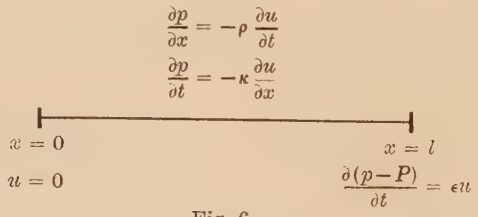
$\frac{\partial p}{\partial x} = -\rho \frac{\partial u}{\partial t}$
$\frac{\partial p}{\partial t} = -\kappa \frac{\partial u}{\partial x}$
$x = 0$
$u = 0$
$x = l$
$\frac{\partial (p - P)}{\partial t} = \epsilon u$

Fig. 6.

Wir suchen eine rein periodische Lösung des folgenden hydrodynamischen Problems (vgl. Fig. 6); von der (tatsächlich wesentlichen) Wirkung der Viskosität des Blutes ist dabei abgesehen. Die Periodenzahl

der vorgegebenen Druckkurve P haben wir im Interesse der einfacheren Schreibweise in den folgenden Formeln nicht w sondern aw genannt. Wir denken uns P in eine Fourierreihe der Periode $\tau = (2\pi)/(aw)$ entwickelt; es sei $dP/dt = \sum_n c_n \sin nawt$.

Wir setzen u und p in der folgenden Form an:

$$(23) \qquad \begin{cases} u = \Sigma A_n \sin nwx \sin nawt, \\ p - p_0 = \rho a \Sigma A_n \cos nwx \cos nawt, \end{cases}$$

Hierdurch genügen wir den Differentialgleichungen des Problems und der Grenzbedingung $u = 0$ für $x = 0$. Es bleibt noch die Grenzbedingung für $x = l$ zu erfüllen. Diese liefert:

$$(24) \qquad -A_n(\rho a^2 nw \cos nwl + \epsilon \sin nwl) = c_n.$$

Dadurch ist die Lösung (23) bestimmt (bis auf die belanglose Konstante p_0).

Der Faktor links in (24) weist darauf hin, dass die von der Herzmuskulatur erzwungene Schwingung im Falle der Resonanz durch die Eigenfrequenzen der Röhre verstärkt wird. In der Tat gibt dieser Faktor $= 0$ gesetzt, die Eigenschwingung der Röhre unter den jetzigen Grenzbedingungen bei fehlender Anregung P. Bei Berücksichtigung der Dämpfung wird diese Verstärkung herabgedrückt.

Bei der Einfachheit der gegenwärtigen Problemstellung kommt die Besonderheit der belasteten Orthogonalitätsbedingung nicht zur Geltung.

Wenn auch die Idealisierung des Systems ziemlich weitgehend erscheint, so ist zu bedenken, dass die reine Empirie auf diesem Gebiet vollständig versagt. Und auf der anderen Seite, dass die Analyse von mathematischen Modellen, die man sich von noch so verwickelten Verhältnissen bildet, stets förderlich wirkt. In diesem Sinne glauben wir, dass das hier gegebene mathematische Schema für die beabsichtigte genauere experimentelle Untersuchung des Kreislaufs nicht fruchtlos sein wird.

SOME FURTHER CONSIDERATIONS ON THE PROBLEM OF TWO OBSTACLES IN AN INFINITE STREAM

By E. G. C. Poole.

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REFERENCES:

Thiry (Thèse), *Annales Sci. de l'Ecole Normale* (3), 38 (1921), 229-329.
Cisotti, *Idromeccanica Piana*, 2 (Milano, 1922), 364-368.
Weinstein, "Ein hydrodynamischer Unitätssatz", *Math. Zeitschrift*, 19 (1923), 265-275.
Poole, *Proc. London Math. Soc.* (2), 22 (1924), 425-453.
,, ,, ,, ,, Records, May, 1924.

I.

The extensive literature of steady two-dimensional fluid motions with discontinuities, where the assigned parts of the boundary are straight, might suggest that few points of principle remain unelucidated in such problems. But this view is contradicted by a configuration as simple as that of two straight obstacles in an infinite stream. Technically, it requires the conformal representation of overlapping polygons drawn on a Riemann surface, for which the analysis lies ready to hand in Schläfli's extension of the Schwarz-Christoffel formula, and also of doubly-connected polygonal regions, for which the formula was found by Thiry. But there are other difficulties of *analysis situs* on which the theory of conformal representation sheds no light, because the regions to be represented are at first unknown. If these can be guessed correctly, then we know how to set about putting the solution into analytical form. But if every reasonable surmise turns out on verification to be false, then we have no general principle to guide us.

In this problem, a survey of the most obvious possibilities suggested that, in the general case, the current must occupy a doubly-connected region. But Thiry, who surmounted all the analytical difficulties and minutely studied this case, found it necessary to assume that the velocity-potential was many-valued, with a cyclic constant corresponding to a circulation about one obstacle and its wake. Such a solution raises a difficulty as serious as that which it removes. For in a *continuous* fluid motion, where the complete boundary is assigned, we know from existence theorems on conformal representation that a single-valued solution exists; and most mathematicians would be very reluctant to abandon the hypothesis that there is a similar solution in a problem of *discontinuous* fluid motion, where part of the boundary consists of free stream-lines.

Although the method of trial and error has enabled me to suggest two possible solutions with uniform velocity-potentials, the necessity for an examination of the question of principle is thereby merely postponed. The difficulty has so far escaped attention, because in practice the solution has usually been so obvious in the configurations hitherto examined, that the existence of a problem was hardly realized.

Our problem admits five essential degrees of freedom, corresponding to the relative size and position of the obstacles and their inclination to the current at infinity. Seven possible steady motions are shown in Fig. I, of which A and B were examined by Thiry, and are merely shown for reference. These two have four degrees of freedom without any circulation and five with a circulation, and thus correspond to the obstacles in general position. The next two configurations are generalizations of the configuration with two infinite wakes and of the doubly-connected configuration respectively, and are found to have six degrees of freedom, without assuming any circulation. The problem is thus shown to be indeterminate, like a number of problems studied by Villat and Thiry. The modification consists of the introduction of triangular spits of dead water in front of the obstacles at the branch-points of the current, a possibility which was mentioned in my earlier paper. Only one of these cases is examined in detail, as the reader can easily reconstruct the analysis of the other. The last three configurations are modifications of A and B, having four, three, and one degrees of freedom respectively. Of these G is selected for detailed examination to exhibit a curious anomaly which it presents, namely that the case of general position (if it exists at all) has fewer degrees of freedom than the symmetrical case. The method employed is suggested by an interesting paper of Weinstein's.

We ought to mention that in all these problems there are impracticable quadratures; it is logically possible, though hardly likely, that,

if these were completed, the necessary conditions which we have to impose might not all be independent, and that the number of degrees

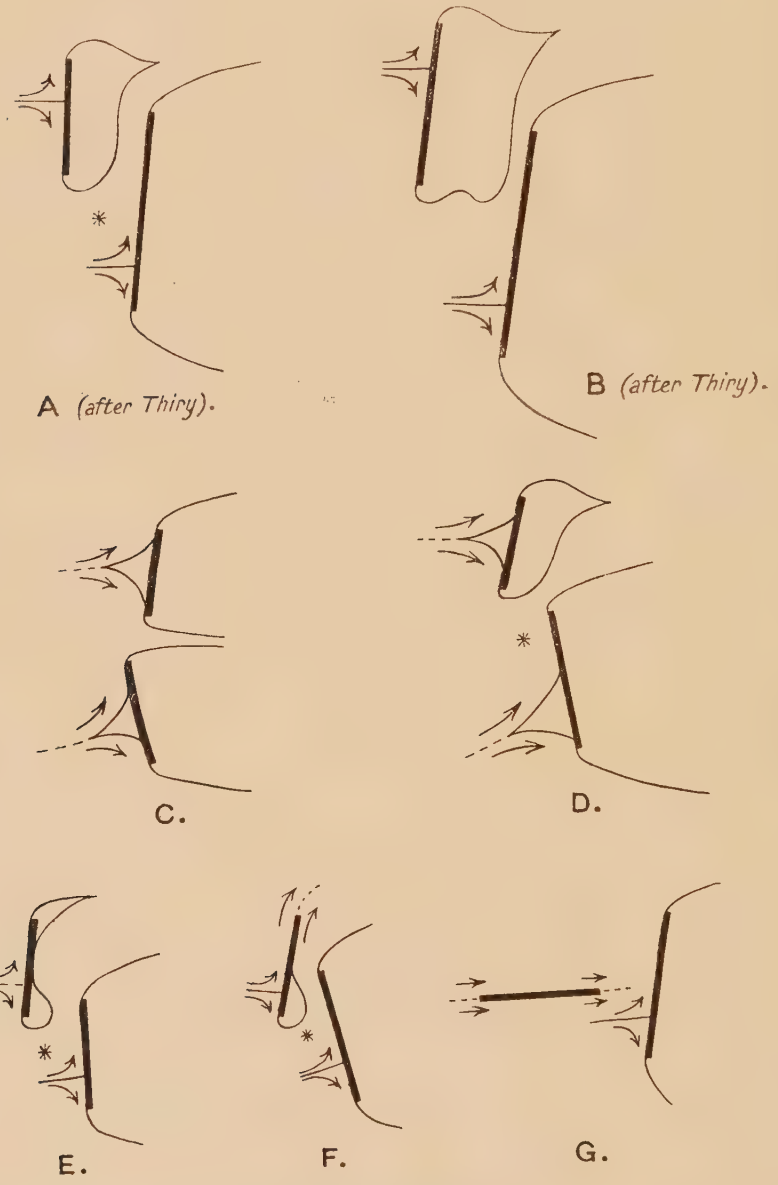

FIG. I. Synopsis

of freedom would thus be increased. Certain solutions, hitherto rejected as insufficiently general, might then be found to cover all possible data.

II.

Configuration *D*, with the usual auxiliary diagrams, is shown in Fig. II. The doubly-connected region occupied by the stream in the

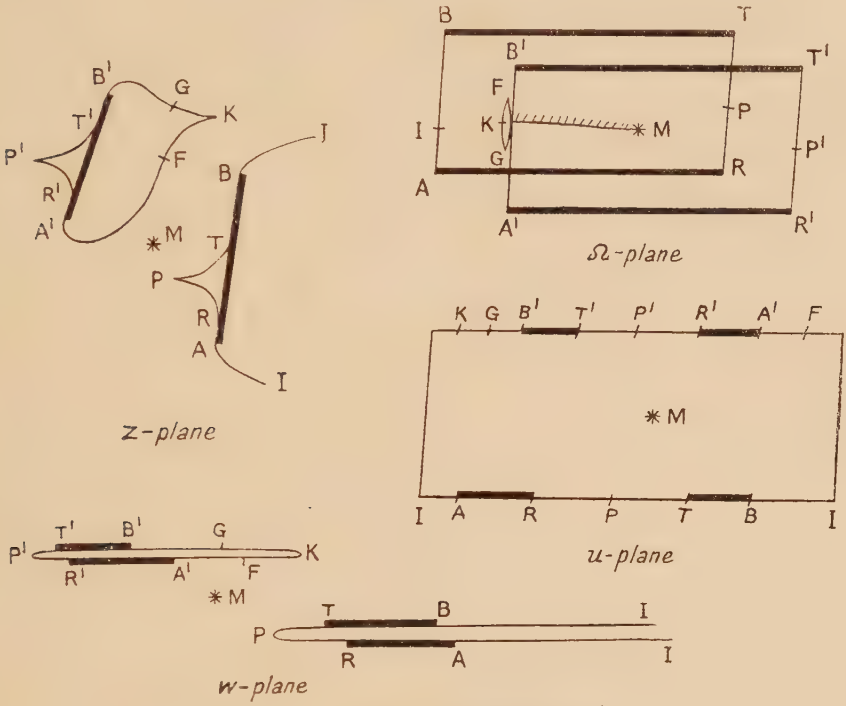

FIG. II. Details of Configuration D.

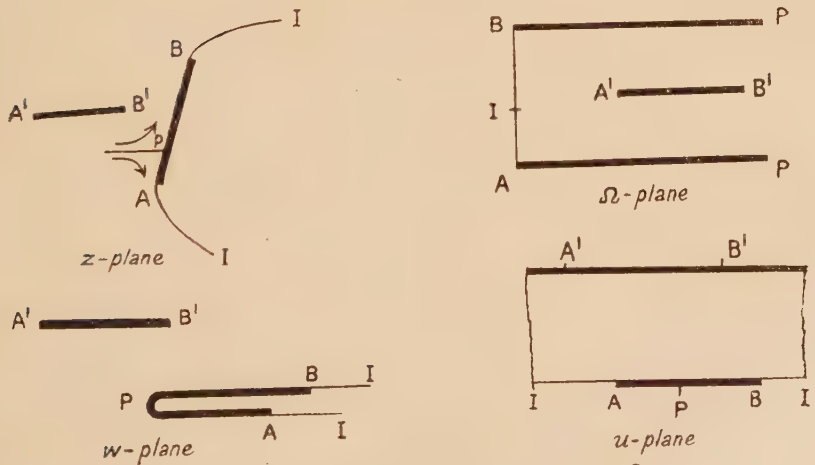

FIG. III. Details of Configuration G.

z-plane is represented on a strip parallel to the real axis of the u-plane, a length $2\omega_1$ of the strip corresponding to a complete circuit about the obstacle with a finite wake. The lower border corresponds to the obstacle with an infinite wake and to the boundary of its wake; the upper border, $I(u-\omega_3)=0$, to the other obstacle and wake. The analysis is arranged as much like Thiry's solution as possible. The difference introduced by the two spits of dead water in front of the obstacles is that the Ω-diagram does not extend to infinity, but is cut off in both sheets. The Ω-region is a Riemann surface of two rectangular sheets, with a winding point M, corresponding to a point of stationary velocity in the liquid. There is a slit FG in one sheet, which is interpolated in the boundary of the other, as in Thiry's figure.

The correspondence of points is then

$$I,\ A,\ B,\ A',\ B',\ P,\ R,\ T,\ P',\ R',\ T',\ F,\ G,\ K,\ M,$$

$$u=0,\ \alpha,\ \beta,\ \alpha',\ \beta',\ \varpi,\ \rho,\ \tau,\ \varpi',\ \rho',\ \tau',\ \phi,\ \chi,\ \kappa,\ \mu+i\nu.$$

The w-region is bounded by a semi-infinite slit and a finite slit parallel to it, and the formula for w is the same as Thiry's. We see that dw/du is uniform and real along both boundaries, so that it admits the period $2\omega_3$ by Schwarz's principle of symmetry. It admits also the period $2\omega_1$ corresponding to a complete circuit, so that it is an elliptic function. The only singularity is a pole of the third order at I, $u=0$, so that it is linear in $\wp(u)$ and $\wp'(u)$. Integrating, w will be linear in $\wp(u)$, $\zeta(u)$, and u; but since w admits the period $2\omega_1$, because the circulation is assumed to be null, we find that $\zeta(u)$ and u must occur in the combination $[\omega_1\zeta(u)-\eta_1 u]$. Hence

$$Aw+A'=\wp(u)+\epsilon\left(\zeta(u)-\frac{\eta_1}{\omega_1}\,u\right), \tag{1}$$

where A, A' are trivial constants affecting only the scale and choice of origin, and A, ϵ are real.

The parameters ϖ, ϖ', κ are given as functions of ϵ, being the zeros of dw/du.

By similar reasoning, $(d\Omega/du)^2$ is an elliptic function, with eight simple poles at the junctions of the fixed and free parts of the boundary, two double zeros at the inflexions of the free stream lines, and two other double zeros at the point of stationary velocity and its symmetric image. Thus, by adding a complete period, if necessary, to one of the parameters, we

have the form

$$\left(\frac{d\Omega}{du}\right)^2 = B\,\frac{\sigma^2(u-\phi)\,\sigma^2(u-\chi)\,\sigma^2(u-\mu-i\nu)\,\sigma^2(u-\mu+i\nu)}{\sigma(u-\alpha)\,\sigma(u-\beta)\,\sigma(u-\alpha')\,\sigma(u-\beta')} \times\sigma(u-\rho)\,\sigma(u-\tau)\,\sigma(u-\rho')\,\sigma(u-\tau'). \quad (2)$$

We have fifteen undetermined parameters, contained in the following expressions, some of which are complex,

$$\alpha,\quad \beta,\quad \alpha',\quad \beta',\quad \rho,\quad \tau,\quad \rho',\quad \tau',\quad \phi,\quad \chi,\quad \mu,\quad \nu,\quad \epsilon,\quad (\omega_3/\omega_1),\quad B.$$

The conditions imposed on these by the nature of the Ω-diagram are as follows :—

(i) Since the expression (2) is doubly-periodic,

$$\alpha+\beta+\alpha'+\beta'+\rho+\tau+\rho'+\tau' = 2\phi+2\chi+4\mu\,; \quad (3)$$

(ii) the boundaries must be closed, but it is sufficient to express the closure conditions for one boundary, since the other will then be closed automatically by Cauchy's theorem. Hence

$$I\{\Omega(B)-\Omega(A)\} = \pi, \quad (4)$$

$$R\{\Omega(B)-\Omega(A)\} = 0, \quad (5)$$

$$I\{\Omega(T)-\Omega(R)\} = \pi\,; \quad (6)$$

(iii)
$$I\{\Omega(B')-\Omega(A')\} = \pi, \quad (7)$$

because the second obstacle is straight.

These five conditions leave ten available parameters. The additive constants of integration in Ω affect only the scale of measurement of velocities and the choice of the initial line, and are therefore trivial. Two other trivial constants appear, which only affect the choice of origin in the z-plane, in the formula

$$z = \int e^{\Omega}\,dw. \quad (8)$$

The conditions imposed in the z-plane are :—

(iv) the closure conditions

$$R\{z(u+2\omega_1)-z(u)\} = 0, \quad (9)$$

$$I\{z(u+2\omega_1)-z(u)\} = 0, \quad (10)$$

(v) and the conditions that $ARTB$, $A'R'T'B'$ should be collinear :

$$I\left\{\frac{z(T)-z(A)}{z(R)-z(A)}\right\}=0, \tag{11}$$

$$I\left\{\frac{z(T')-z(A')}{z(R')-z(A')}\right\}=0. \tag{12}$$

These four additional conditions leave six degrees of freedom. To obtain Thiry's solution, we must unite the triads of point PRT, $P'R'T'$, losing four parameters, and we drop the conditions (11), (12), so that we have finally four degrees of freedom. If we now drop the condition that the circulation is null and so assume a more general form for w, in which $\zeta(u)$ and u need not appear only in a certain combination, we get back to Thiry's solution, with its five degrees of freedom.

III.

The details of configuration G are shown in Fig. III. The w-region and the formula (1) are unchanged. The Ω-region is an open rectangle bounded internally by a slit corresponding to the second obstacle, about which the fluid motion is continuous. The ends of this slit correspond to the ends of the obstacle, because the velocity is finite and continuous and Ω is analytic in z at these points. Since the angle in the Ω-plane is a multiple of that in the z-plane, and the latter is double the angle in the u-plane, we find that these points are, at least, simple zeros of $d\Omega/du$. Now we find, as before, that $(d\Omega/du)^2$ is an elliptic function; its singularities are a double pole at the branch-point P and simple poles at the junctions A, B, so that it has four zeros, which must be the double zeros at A', B'. Taking the correspondence of points

$$I, \quad A, \quad B, \quad A', \quad B', \quad P,$$

$$u=0, \quad \alpha, \quad \beta, \quad \alpha', \quad \beta', \quad \varpi,$$

we have

$$\left(\frac{d\Omega}{du}\right)^2 = B\,\frac{\sigma^2(u-\alpha')\,\sigma^2(u-\beta')}{\sigma(u-\alpha)\,\sigma(u-\beta)\,\sigma^2(u-\varpi)}. \tag{13}$$

The parameters are subject to several conditions.

(i) Since A', B', P are the zeros of dw/du, they will depend on the parameter ϵ only, being the roots of

$$\wp'(u)-\epsilon\left(\wp(u)+\frac{\eta_1}{\omega_1}\right)=0, \tag{14}$$

and, in particular, $\alpha'+\beta'+\varpi$ is congruent to zero.

(ii) The condition for an elliptic function gives

$$\alpha+\beta+2\varpi\equiv 2\alpha'+2\beta' \pmod{2\omega_1, 2\omega_3}. \tag{15}$$

(iii) The closure condition for $A'B'A'$, along which the imaginary part of Ω is constant, gives for the Ω-diagram one condition only,

$$\Omega(u+2\omega_1)=\Omega(u), \tag{16}$$

the other boundary being closed automatically by Cauchy's theorem.

(iv) The closure condition for $A'B'A'$, along which the argument of dz/dw is constant, gives for the z-plane one condition only,

$$z(u+2\omega_1)=z(u), \tag{17}$$

the other boundary being again closed automatically.

(v) At P,

$$\Omega\sim-\log(u-\varpi), \quad \left(\frac{d\Omega}{du}\right)^2\sim\frac{1}{(u-\varpi)^2},$$

and hence

$$B=\frac{\sigma(\varpi-\alpha)\,\sigma(\varpi-\beta)}{\sigma^2(\varpi-\alpha')\,\sigma^2(\varpi-\beta')}. \tag{18}$$

The five essential parameters α, β, ϵ, B, (ω_3/ω_1) are thus connected by four relations, leaving one degree of freedom.

The case of symmetry arises when one obstacle is perpendicular to the stream and the other parallel to the stream, forming merely a geometrical segment of the axis of symmetry, which offers no resistance to the current. The case is covered by the classical analysis of the one-obstacle problem, but we require to see how it arises by specializing the constants in the above formulas. The correspondence of points is :

$$\begin{array}{cccccccc} I, & A, & B, & & A', & B', & & P, \\ u=0, & \alpha, & 2\omega_1-\alpha, & \omega_3, & \omega_1+\omega_3, & & \omega_1. \end{array}$$

Since B is given by (18), we have only two parameters, a and (ω_3/ω_1), but the conditions are identically satisfied. For we have the reduced forms :—

$$w_0 = \wp(u), \tag{19}$$

$$\frac{d\Omega_0}{du} = \tfrac{1}{2}\frac{\wp'(u)\sqrt{[e_1-\wp(a)]}}{[\wp(u)-e_1]\sqrt{[\wp(u)-\wp(a)]}}, \tag{20}$$

$$\frac{dz_0}{dw_0} = e^{\Omega_0} = \frac{\sqrt{[\wp(u)-\wp(a)]}+\sqrt{[e_1-\wp(a)]}}{\sqrt{[\wp(u)-e_1]}}. \tag{21}$$

The last formula shows that condition (16) holds, and (17) gives merely

$$\int_{e_2}^{e_3}+\int_{e_3}^{e_2}\frac{\sqrt{[x-\wp(a)]}+\sqrt{[e_1-\wp(a)]}}{\sqrt{(x-e_1)}}\,dx = 0,$$

which is true.

Thus the case of symmetry has two degrees of freedom, corresponding to the arbitrary choice of two points on the axis of symmetry in the one-obstacle problem. We observe, particularly, that ϵ is zero in this case.

Let us now follow Weinstein's method and imagine a *small* deviation from symmetry. We shall suppose ϵ to be so small that its square is negligible, and we shall expand in powers of ϵ along $A'B'$, where there are no singularities of w, Ω, or z. Keeping only the first power of ϵ along this segment, we shall write $w_0+\epsilon w_1$ for w, and so on.

The approximate values of the roots of (14), when ϵ is small, are of the form $\omega+x$, where ω is any half-period and x is $O(\epsilon)$.

We find $$x\wp''(\omega) \sim \epsilon\left(\wp(\omega)+\frac{\eta_1}{\omega_1}\right),$$

and, on reduction and neglecting ϵ^2, we have

$$\varpi = \omega_1+\frac{\epsilon(e_1\omega_1+\eta_1)}{2\omega_1(e_1-e_2)(e_1-e_3)}, \tag{22}$$

$$a' = \omega_3+\frac{\epsilon(e_3\omega_1+\eta_1)}{2\omega_1(e_3-e_1)(e_3-e_2)}, \tag{23}$$

$$\beta' = \omega_1+\omega_3+\frac{\epsilon(e_2\omega_1+\eta_1)}{2\omega_1(e_2-e_3)(e_2-e_1)}. \tag{24}$$

The condition (15) is satisfied by putting

$$\alpha = \chi - \frac{\epsilon(e_1\omega_1+\eta_1)}{\omega_1(e_1-e_2)(e_1-e_3)}, \tag{25}$$

$$\beta = 2\omega_1 - \chi - \frac{\epsilon(e_1\omega_1+\eta_1)}{\omega_1(e_1-e_2)(e_1-e_3)}. \tag{26}$$

Putting $\Omega = \Omega_0+\epsilon\Omega_1+O(\epsilon^2)$, we find, by logarithmic differentiation,

$$\frac{d\Omega_1}{du} = \frac{d\Omega_0}{du}\left\{\frac{(e_1\omega_1+\eta_1)[e_1-\wp(\chi)]\wp'(u)}{2(e_1-e_2)(e_1-e_3)[\wp(u)-e_1][\wp(u)-\wp(\chi)]} - \frac{\omega_1\wp(u)+\eta_1}{\omega_1\wp'(u)}\right\}. \tag{27}$$

Similarly, putting $z = z_0+\epsilon z_1+O(\epsilon^2)$,

$$z = \int e^{\Omega}dw = \int e^{\Omega_0}\left(dw_0+\epsilon(dw_1+\Omega_1 dw_0)\right)+O(\epsilon^2),$$

$$z_1 = \int e^{\Omega_0}dw_1+\int \Omega_1 dz_0. \tag{28}$$

We have now three parameters, namely, ϵ, χ, and $(\omega_3:\omega_1)$ subject to two closure conditions, (16) and (17), which reduce to

$$\epsilon\{\Omega_1(\omega_3+2\omega_1)-\Omega_1(\omega_3)\} = O(\epsilon^2), \tag{29}$$

$$\epsilon\{z_1(\omega_3+2\omega_1)-z_1(\omega_3)\} = O(\epsilon^2), \tag{30}$$

Both conditions are satisfied by putting $\epsilon = 0$, and we have the symmetrical case with two degrees of freedom. But if we exclude this solution, but still keep ϵ small enough for the approximations used above to be valid, then the two other parameters, χ and $(\omega_3:\omega_1)$, must satisfy approximately

$$\Omega_1(\omega_3+2\omega_1) = \Omega_1(\omega_3), \tag{31}$$

$$z_1(\omega_3+2\omega_1) = z_1(\omega_3). \tag{32}$$

If the small displacement is supposed to be made from the general symmetrical configuration, the two parameters must be arbitrary. Unless, therefore, the integrations implied in (31) and (32) could be performed, and these conditions were found to be reducible to only one or no new condition, we conclude that there is no series of unsymmetrical solutions tending to the symmetrical solution as a limit.

[Note added 17 January, 1927. The corresponding problem with n plane obstacles can be solved by a configuration of type C, each obstacle having a spit of dead-water in front and an infinite wake behind.

(i) The fluid region is simply-connected and can be represented on a half-plane of t; ignoring trivial constants, which only affect scale, orientation, and choice of origin, the problem has $4n-3$ degrees of freedom.

(ii) The Ω-region is a Riemann surface of n sheets connected in pairs by $n-1$ branch-cuts, ending at $n-1$ branch-points; each sheet is a rectangular strip of breadth π, with one side abutting on the imaginary axis; this figure has also $4n-3$ degrees of freedom.

(iii) The w-region is a plane with n parallel semi-infinite slits; the n values of t which make w infinite are given by a single equation of the form $\Omega = i\gamma$; the n values corresponding to the ends of the slits are at our disposal, giving altogether $n+1$ parameters in addition to those in (ii).

(iv) The z-region gives n conditions of collinearity, one for each obstacle, so that the $5n-2$ parameters are connected by n equations, leaving $4n-2$ degrees of freedom in the solution, or one more than the minimum required; the solution is accordingly indeterminate.]

DIVERGENT CONTINUED FRACTIONS AND POLAR SINGULARITIES

By R. WILSON.

[Received 19 December, 1925.—Read 14 January, 1926.]

1. *Introduction.* A general method of developing an analytic function, given by a Taylor series, in a continued fraction has been created by Padé*. He has formed a table of double-entry of rational fractions approximating formally to the Taylor series. These rational fractions form the successive approximants to the continued fractions obtained by taking given paths in the table.

Montessus de Ballore† has shown that the continued fractions corresponding to the rows converge (except at enclosed poles) to the function within circles in which the function is meromorphic. The continued fraction for the first row converges within the circle, centre the origin, passing through the nearest pole; that for the second row converges within the circle through the second pole; and so on. He has obtained a similar result for the continued fractions arising from the columns, in which the zeros play a part similar to the poles in determining the circles of convergence.

The partial sums $s_n = c_0 + c_1 z + \ldots + c_n z^n$ of the Taylor series form the first row of the Padé table. The n-th element of the $(\nu+1)$-th row of the table can be written in the form

$$R_{n,\nu}(z) = \frac{b_{n,\nu,0}\, s_n(z) + b_{n,\nu,1}\, z\, s_{n-1}(z) + \ldots + b_{n,\nu,\nu} z^{\nu} s_{n-\nu}(z)}{b_{n,\nu,0} + b_{n,\nu,1} z + \ldots + b_{n,\nu,\nu} z^{\nu}},$$

* H. Padé, "Sur la représentation approchées d'une fonction par des fonctions rationnelles", *Annales Sci. de l'Ecole Normale* (3), 9 (1892), 1–93, Supplément.

† R. Montessus de Ballore, "Sur les fractions continues algébriques", *Rend. di Palermo*, 19 (1905), 1–73.

where the coefficients $b_{n,\nu,k}$ are the determinants of the matrix

$$\begin{vmatrix} c_{n+1}, & \dots, & c_{n-\nu+1} \\ c_{n+\nu}, & \dots, & c_n \end{vmatrix}.$$

This fraction has the formal appearance of a Borel* sum. Notice, however, that these sums differ fundamentally from those introduced by Borel*, in that the weights vary with the functions and the points considered.

Dr. Dienes† has shown that there is a relation between the character of the divergence of the Taylor series at a singular point on the circle of convergence and the nature of the singularity in question.

At his suggestion I propose to examine the corresponding problem for these continued fractions at the enclosed and boundary poles and at points on the boundary.

Let $1/a_1, 1/a_2, \dots, 1/a_{\nu+1}, \dots$ be the index-points of the poles of the given function, arranged in order of increasing moduli. The function $R_{n,\nu}(z)$ for $n \to \infty$, ν fixed, converges *within* the circle $|za_{\nu+1}| = 1$, except at the poles enclosed.

Hadamard‡ has shown that the denominator§

$$\left(1+\frac{b_{n,\nu,1}}{b_{n,\nu,0}}z+\dots+\frac{b_{n,\nu,\nu}}{b_{n,\nu,0}}z^{\nu}\right)$$

of $R_{n,\nu}(z)$ takes the form $(1-a_1z)(1-a_2z)\dots(1-a_nz)$, as $n \to \infty$.

Hence we shall discuss the character of the divergence of the sequence of fractions $R_{1,\nu}(z), \dots, R_{n,\nu}(\nu), \dots$ at the points $1/a_1, 1/a_2, \dots, 1/a_\nu, 1/a_{\nu+1}$, and at all points on $|za_{\nu+1}| = 1$ in the case for which there is one simple boundary pole at $1/a_{\nu+1}$.

2. *Method.* Let the Taylor expansion of the function, analytic in the neighbourhood of the origin, be

$$f(z) \equiv c_0+c_1z+c_2z^2+\dots+c_mz^m+\dots. \qquad (1)$$

* E. Borel, *Sur les séries divergentes.* Collection Borel.

† P. Dienes, *Sur les singularités des fonctions analytiques.* Collection Borel.

‡ J. Hadamard, "Essai sur l'étude des fonctions . . .", *Journal de Math.* (4), 8 (1892), 101–286, Deuxième Partie.

§ It is easy to show that the Padé table of a meromorphic function is *normal* along the ν-th row (ν finite) for sufficiently large values of n. Hence the *abnormal case* does not arise in the discussion.

The algebraic rational function $P_{n,\nu}(z)/Q_{n,\nu}(z)$, of degrees n in the numerator and ν in the denominator, approximating to the expansion to the first $n+\nu+1$ terms, is readily obtained in the form

$$P_{n,\nu}(z) \equiv \begin{vmatrix} \psi_n(z), & z\psi_{n-1}(z), & \ldots, & z^\nu\psi_{n-\nu}(z) \\ c_{n+1}, & c_n, & \ldots, & c_{n-\nu+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+\nu}, & c_{n+\nu-1}, & \ldots, & c_n \end{vmatrix}, \quad Q_{n,\nu}(z) \equiv \begin{vmatrix} 1, & z, & \ldots, & z^\nu \\ c_{n+1}, & c_n, & \ldots, & c_{n-\nu+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+\nu}, & c_{n+\nu-1}, & \ldots, & c_n \end{vmatrix}, \tag{2}$$

where $\psi_{n-t} = c_0+c_1z+\ldots+c_{n-t}z^{n-t}$ and $c_{-m} = 0$.

The first case is that in which all the enclosed singularities are *simple* poles.

If we take the index-points of the poles to be (as in § 1)

$$\frac{1}{a_1}, \quad \frac{1}{a_2}, \quad \ldots, \quad \frac{1}{a_\nu}, \quad \frac{1}{a_{\nu+1}}, \quad \ldots,$$

where $|a_1| \geqslant |a_2| \ldots \geqslant |a_\nu| > |a_{\nu+1}| > \rho$,

then
$$f(z) \equiv \sum_{r=1}^{\nu+1} \frac{A_r}{1-a_r z} + \phi(z), \tag{3}$$

where $\phi(z)$ is holomorphic within a circle of radius $1/\rho$.

From (1),
$$c_m \equiv \sum_{r=1}^{\nu+1} A_r a_r^m + M_m \rho^m, \tag{4}$$

where $|M_m| < M$, a fixed constant.

The continued fraction of the row $\nu =$ constant converges if

$$\lim_{n\to\infty} \frac{P_{n,\nu}(z)}{Q_{n,\nu}(z)} \equiv \lim_{n\to\infty} R_{n,\nu}(z)$$

exists and is finite. From § 1, this occurs at all points within the circle $|za_{\nu+1}| = 1$ except at the enclosed poles. We shall discuss the form of $P_{n,\nu}(z)/Q_{n,\nu}(z)$ at

(*a*) the points $z = 1/a_r$ $(r < \nu+1)$;

(*b*) the point $z = 1/a_{\nu+1}$;

(*c*) any other point on the boundary $|za_{\nu+1}| = 1$.

3. *Evaluation of* $Q_{n,\nu}(1/a_r)$. If we apply (4) to formula (3),

$$Q_{n,\nu}\left(\frac{1}{a_r}\right) = \frac{1}{a_r^\nu}\begin{vmatrix} a_r^\nu, & a_r^{\nu-1}, & \dots, & 1 \\ \sum_1^{\nu+1} A_r a_r^{n+1}+M_{n+1}\rho^{n+1}, & \dots, & \sum_1^{\nu+1} A_r a_r^{n-\nu+1}+M_{n-\nu+1}\rho^{n-\nu+1} \\ \dots & \dots & \dots & \dots \\ \sum_1^{\nu+1} A_r a_r^{n+\nu}+M_{n+\nu}\rho^{n+\nu}, & \dots, & \sum_1^{\nu+1} A_r a_r^{n}+M_{n}\rho^{n} \end{vmatrix}$$

$$= \frac{1}{a_r^\nu}\begin{vmatrix} a_r^\nu, & a_r^{\nu-1}, & \dots, & 1 \\ \sum_1^{\nu+1} A_r a_r^{n+1}, & \sum_1^{\nu+1} A_r a_r^{n}, & \dots, & \sum_1^{\nu+1} A_r a_r^{n-\nu+1} \\ \dots & \dots & \dots & \dots \\ \sum_1^{\nu+1} A_r a_r^{n+\nu}, & \sum_1^{\nu+1} A_r a_r^{n+\nu-1}, & \dots, & \sum_1^{\nu+1} A_r a_r^{n} \end{vmatrix}$$

$$\times\left[1+K_n\left(\frac{\rho}{a_{\nu+1}}\right)^n\right],$$

where $|K_n|<K$, a finite constant.

The value of the determinant is

$$\frac{1}{A_r}\begin{vmatrix} 0, & 0, & 0, & 1, & \dots, & 0 \\ a_1^{n-\nu+1}, & a_2^{n-\nu+1}, & \dots, & a_r^{n-\nu+1}, & \dots, & a_{\nu+1}^{n-\nu+1} \\ \dots, & \dots & \dots & \dots & \dots & \dots \\ a_1^n, & a_2^n, & \dots, & a_r^n, & \dots, & a_{\nu+1}^n \end{vmatrix}$$

$$\times\begin{vmatrix} A_1, & A_2, & \dots, & A_r, & \dots, & A_{\nu+1} \\ A_1 a_1, & A_2 a_2, & \dots, & A_r a_r, & \dots, & A_{\nu+1} a_{\nu+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_1 a_1^\nu, & A_2 a_2^\nu, & \dots, & A_r a_r^\nu, & \dots, & A_{\nu+1} a_{\nu+1}^\nu \end{vmatrix}$$

$$= \frac{1}{A_r a_r^{n-\nu+1}} \prod_{s=1}^{\nu+1} A_s \prod_{s=1}^{\nu+1} a_s^{n-\nu+1} \prod^{\nu+1} (a_h-a_k) \prod^{\nu+1\prime} (a_h-a_k),$$

where the index r is suppressed in Π', and the double product is taken over $2 \leqslant h \leqslant \nu+1$, $1 \leqslant k < h$.

(This notation is used throughout the paper.)

Hence

$$Q_{n,\nu}\left(\frac{1}{a_r}\right) = \frac{1}{A_r a_r^{n+1}} \prod_{s=1}^{\nu+1} A_s \prod_{s=1}^{\nu+1} a_s^{n-\nu+1} \prod^{\nu+1} (a_h - a_k) \prod^{\nu+1\prime} (a_h - a_k) \left[1 + K_n \left(\frac{\rho}{a_{\nu+1}}\right)^n\right].$$

4. *Evaluation of* $P_{n,\nu}(1/a_r)$. First we evaluate

$$\begin{vmatrix} c_{n+1}, & \cdots & \boxed{c_{n-t}} & \cdots, & c_{n-\nu+1} \\ c_{n+\nu}, & \cdots & \boxed{c_{n+\nu-t-1}} & \cdots, & c_n \end{vmatrix}$$

where the outlined column is omitted.

This minor of $P_{n,\nu}(1/a_r)$ can be written, from (4),

$$\begin{vmatrix} \sum_1^{\nu+1} A_r a_r^{n+1} + M_{n+1}\rho^{n+1}, & \cdots & \boxed{\sum_1^{\nu+1} A_r a_r^{n-t} + M_{n-t}\rho^{n-t}} & \cdots, & \sum_1^{\nu+1} A_r a_r^{n-\nu+1} + M_{n-\nu+1}\rho^{n-\nu+1} \\ \sum_1^{\nu+1} A_r a_r^{n+\nu} + M_{n+\nu}\rho^{n+\nu}, & \cdots & \boxed{\sum_1^{\nu+1} A_r a_r^{n+\nu-t-1} + M_{n+\nu-t-1}\rho^{n+\nu-t-1}} & \cdots, & \sum_1^{\nu+1} A_r a_r^{n} + M_{n}\rho^{n} \end{vmatrix}$$

and is the sum of products of corresponding determinants formed from the matrices

$$\begin{vmatrix} a_1^{n-\nu+1}, & \cdots, & a_\nu^{n-\nu+1}, & a_{\nu+1}^{n-\nu+1}, & \rho^{n-\nu+1}, & 0, & \cdots, & 0, & 0 \\ a_1^{n-\nu+2}, & \cdots, & a_\nu^{n-\nu+2}, & a_{\nu+1}^{n-\nu+2}, & 0, & \rho^{n-\nu+2}, & \cdots, & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \boxed{a_1^{n-t}, \quad \cdots, \quad a_\nu^{n-t}, \quad a_{\nu+1}^{n-t}, \quad 0, \quad 0, \quad \cdots, \quad 0, \quad 0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1^{n}, & \cdots, & a_\nu^{n}, & a_{\nu+1}^{n}, & 0, & 0, & \cdots, & \rho^{n}, & 0 \\ a_1^{n+1}, & \cdots, & a_\nu^{n+1}, & a_{\nu+1}^{n+1}, & 0, & 0, & \cdots, & 0, & \rho^{n+1} \end{vmatrix}$$

$$\begin{vmatrix} A_1, & \dots, & A_\nu, & A_{\nu+1}, & M_{n-\nu+1}, & \dots, & M_{n+1} \\ A_1\alpha_1, & \dots, & A_\nu\alpha_\nu, & A_{\nu+1}\alpha_{\nu+1}, & M_{n-\nu+2}, & \dots, & M_{n+2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_1\alpha_1^{\nu-t-2}, & \dots, & A_\nu\alpha_\nu^{\nu-t-2}, & A_{\nu+1}\alpha_{\nu+1}^{\nu-t-2}, & M_{n-t}, & \dots, & M_{n+\nu-t} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_1\alpha_1^{\nu-2}, & \dots, & A_\nu\alpha_\nu^{\nu-2}, & A_{\nu+1}\alpha_{\nu+1}^{\nu-2}, & M_{n-1}, & \dots, & M_{n+\nu-1} \\ A_1\alpha_1^{\nu-1}, & \dots, & A_\nu\alpha_\nu^{\nu-1}, & A_{\nu+1}\alpha_{\nu+1}^{\nu-1}, & M_n, & \dots, & M_{n+\nu} \end{vmatrix} \tag{5}$$

Consider

$$\frac{1}{\alpha_r^t}\psi_{n-t}\left(\frac{1}{\alpha_r}\right) = \frac{c_{n-t}}{\alpha_r^r} + \dots + \frac{c_1}{\alpha_r^{t+1}} + \frac{c_0}{\alpha_r^t}$$

$$= \frac{1}{\alpha_r^t}\sum_{s=0}^{n-t}\frac{1}{\alpha_r^t}(A_1\alpha_1^s + \dots + A_\nu\alpha_\nu^s + A_{\nu+1}\alpha_{\nu+1}^s + M_s\rho^s).$$

The term containing A_k is

$$\frac{A_k}{\alpha_r^t}\sum_{s=0}^{n-t}\left(\frac{\alpha_k}{\alpha_r}\right)^s = \frac{A_k}{\alpha_r^n}\,\frac{\alpha_k^{n-t+1} - \alpha_r^{n-t+1}}{\alpha_k - \alpha_r}.$$

Forming the contribution to $P_{n,\nu}(1/\alpha_r)$, [see (2)], of terms in the top row containing A_k, and using (5), we get

$$\frac{H_n}{\alpha_r^{n+1}}\begin{vmatrix} \alpha_k^{n-\nu+1} - \alpha_r^{n-\nu+1}, & \alpha_1^{n-\nu+1}, & \dots, & \alpha_k^{n-\nu+1}, & & & & & & \\ & & \dots, & \alpha_r^{n-\nu+1}, & \dots, & \alpha_\nu^{n-\nu+1}, & \rho^{n-\nu+1}, & 0, & \dots, & 0 \\ \alpha_k^{n-\nu+2} - \alpha_r^{n-\nu+2}, & \alpha_1^{n-\nu+2}, & \dots, & \alpha_k^{n-\nu+2}, & & & & & & \\ & & \dots, & \alpha_r^{n-\nu+2}, & \dots, & \alpha_\nu^{n-\nu+2}, & 0, & \rho^{n-\nu+2}, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_k^{n+1} - \alpha_r^{n+1}, & \alpha_1^{n+1}, & \dots, & \alpha_k^{n+1}, & & & & & & \\ & & \dots, & \alpha_r^{n+1}, & \dots, & \alpha_\nu^{n+1}, & \dots, & 0, & 0, & \dots, \rho^{n+1} \end{vmatrix}$$

where $|H_n| < H$, a finite constant, and $k \neq r$.

The contribution of terms whose top row constituent is $\frac{1}{a_r^t}\sum_{s=0}^{n-t}\frac{M_s\rho^s}{a_r^s}$ gives

$$\frac{1}{a_r^\nu}\begin{vmatrix} \sum_0^{n-\nu}\frac{M_s\rho^s}{a_r^s}, & a_1^{n-\nu+1}, \ldots, a_r^{n-\nu+1}, \ldots. a_\nu^{n-\nu+1}, a_{\nu+1}^{n-\nu+1}, \rho^{n-\nu+1}, 0, \ldots, 0 \\ a_r\sum_0^{n-\nu+1}\frac{M_s\rho^s}{a_r^s}, & a_1^{n-\nu+2}, \ldots, a_r^{n-\nu+2}, \ldots, a_\nu^{n-\nu+2}, a_{\nu+1}^{n-\nu+2}, 0, \rho^{n-\nu+2}, \ldots, 0 \\ \\ a_r^\nu\sum_0^n\frac{M_s\rho^s}{a_r^s}, & a_1^{n+1}, \ldots, a_r^{n+1}, \ldots, a_\nu^{n+1}, a_{\nu+1}^{n+1}, 0, 0, \ldots, \rho^{n+1} \end{vmatrix}$$

$$= \prod_{s=1}^{\nu} a_s^{n-\nu+1} I_n\left(\frac{\rho}{a_r}\right)^n,$$

where $|I_n| < I$, a finite constant.
The value of this finite sum of determinants is

$$\prod_{s=1}^{\nu} a_s^{n-\nu+1}\left[I_n\left(\frac{\rho}{a_r}\right)^n + J_n\left(\frac{a_{\nu+1}}{a_r}\right)^n\right], \tag{6}$$

where $|J_n| < J$, a finite constant.

The term in $(1/a_r^t)\psi_{n-t}(1/a_r^t)$ containing A_r is $(n-t+1)A_r/a_r^t$ and the contribution from such terms to $P_{n,\nu}(1/a_r)$ is, from (5),

$$\frac{A_r}{a_r^n}\prod_{s=1}^{\nu} A_s \prod_{s=1}^{\nu} a_s^{n-\nu+1}\prod^{\nu}(a_h-a_k)\begin{vmatrix} (n-\nu+1), & 1, & 1, & \ldots, 1 \\ (n-\nu+2)\,a_r, & a_1, & a_2, & \ldots, a_\nu \\ \ldots & \ldots & \ldots & \ldots \\ (n+1)\,a_r^\nu, & a_1^\nu, & a_2^\nu, & \ldots, a_\nu^\nu \end{vmatrix}\left[1+G_n\left(\frac{a_{\nu+1}}{a_\nu}\right)^n\right] \tag{7}$$

where $|G_n| < G$, a finite constant.

The value of the determinant is (save possibly for sign)

$$a_r\prod^{\nu}(a_h-a_k)\prod_{s=1}^{\nu}(a_r-a_s).$$

This gives, when we add contributions (6) and (7),

$$P_{n,\nu}\left(\frac{1}{a_r}\right) = \frac{A_r}{a_r^{\nu-1}}\prod_{s=1}^{\nu} A_s \prod_{s=1}^{\nu} a_s^{n-\nu+1}\prod^{\nu}(a_h-a_k)^2\prod_{s=1}^{\nu}(a_r-a_s)$$
$$\times\left[1+F_n\left(\frac{a_{\nu+1}}{a_r}\right)^n + G_n\left(\frac{a_{\nu+1}}{a_\nu}\right)^n\right],$$

where $|F_n| < F$, a finite constant.

5. *Order of Divergence at $z = 1/a_r$.* If we use the results of §§ 3 and 4,

$$\frac{P_{n,\nu}(1/a_r)}{Q_{n,\nu}(1/a_r)} = \frac{A_r^2}{A_{\nu+1}}\left(\frac{a_r}{a_{\nu+1}}\right)^{n-\nu+1} \frac{a_r \prod_{s=1}^{\nu}(a_r-a_s)^2}{\prod_{h=1}^{\nu}(a_{\nu+1}-a_h)\ \prod_{k=1}^{\nu}{}'(a_{\nu+1}-a_k)}$$

$$\times\left[1+F_n\left(\frac{a_{\nu+1}}{a_r}\right)^n+G_n\left(\frac{a_{\nu+1}}{a_\nu}\right)^n-K_n\left(\frac{\rho}{a_{\nu+1}}\right)^n\right].$$

Hence

$$\lim_{n\to\infty}\frac{P_{n,\nu}(1/a_r)/Q_{n,\nu}(1/a_r)}{(a_r/a_{\nu+1})^n} = \frac{A_r^2}{A_{\nu+1}}\left(\frac{a_{\nu+1}}{a_r}\right)^{\nu-1}\frac{a_r\prod_{s=1}^{\nu}(a_r-a_s)^2}{\prod_{h=1}^{\nu}(a_{\nu+1}-a_h)\ \prod_{k=1}^{\nu}{}'(a_{\nu+1}-a_k)}, \quad (8)$$

which is finite.

6. *Enclosed Multiple Poles.* Consider a multiple pole of order m at $z = 1/\beta$.

It will add to the form (3) of $f(z)$ an expression

$$\frac{B_1}{(1-\beta z)}+\frac{B_2}{(1-\beta z)^2}+\ldots+\frac{B_m}{(1-\beta z)^m}$$

$$=\frac{B_1(1-\beta z)^{m-1}+\ldots+B_{m-1}(1-\beta z)+B_m}{(1-\beta z)^m}=\frac{\theta(z)}{(1-\beta z)^m}.$$

We shall consider this multiple pole as m simple poles at

$$\frac{1}{\beta_1},\quad \frac{1}{\beta_2},\quad \ldots,\quad \frac{1}{\beta_m}$$

united, in the following manner:

$$\frac{\theta(z)}{\prod_{k=1}^{m}(1-\beta_k z)} \equiv \sum_{s=1}^{m} -\frac{\beta_s^{m-1}\cdot\theta(1/\beta_s)}{(1-\beta_s z)\prod_{k=1}^{m}(\beta_s-\beta_k)},$$

giving residues

$$A_s = -\frac{\beta_s^{m-1}\cdot\theta(1/\beta_s)}{\prod_{k=1}^{m}(\beta_s-\beta_k)} \quad (s=1, 2, \ldots, m). \quad (9)$$

If the multiple pole is neither at $1/a_r$ nor at $1/a_{\nu+1}$, we can put

$$\beta_1=\beta_2=\ldots=\beta_m=\beta$$

in (8) without affecting the nature of the result.

When we evaluate $P_{n,\nu}(1/\beta)/Q_{n,\nu}(1/\beta)$, by substituting from (9) in (8), all factors of the form $(\beta_h-\beta_k)$ cancel.

Hence, making $\beta_1 = \beta_2 = \ldots = \beta_m = \beta$, we find that

$$\lim_{n\to\infty} \frac{P_{n,\nu}(1/\beta)/Q_{n,\nu}(1/\beta)}{(\beta/a_{\nu+1})^n} = \frac{B_m^2}{A_{\nu+1}} \left(\frac{a_{\nu+1}}{\beta}\right)^{\nu-1} \frac{\beta^{2m-1} \Pi (\beta - a_s)^2}{(a_{\nu+1}-\beta)^{2m-1} \bar{\Pi} (a_{\nu+1}-a_h)^2}, \quad (10)$$

where $\bar{\Pi}$ contains no β factors.

7. *Enclosed Poles with Index-Points of Equal Moduli.* If these are simple poles, their contributions to $P_{n,\nu}(1/a_r)$ are of a lower order than that required, as in (6).

Repeating the arguments of the previous paragraph for each pole separately, it follows that this is also true of multiple poles with index-points of equal moduli.

8. *Value at the Boundary Pole.* In this case the determinant in (7) has the form

$$\begin{vmatrix} (n-\nu+1), & 1, & 1, & \ldots, & 1 \\ (n-\nu+2)\, a_{\nu+1}, & a_1, & a_2, & \ldots, & a_\nu \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (n+1)\, a_{\nu+1}^\nu, & a_1^\nu, & a_2^\nu, & \ldots, & a_\nu^\nu \end{vmatrix},$$

giving the result

$$\lim_{n\to\infty} \frac{P_{n,\nu}(1/a_{\nu+1})/Q_{n,\nu}(1/a_{\nu+1})}{n} = A_{\nu+1}.$$

9. *Value on the Boundary Circle.* It is required to evaluate

$$P_{n,\nu}(z)/Q_{n,\nu}(z),$$

where $|za_{\nu+1}| = 1$ and $z \neq 1/a_{\nu+1}$.

Methods similar to those of §§ 3 and 4 give

$$Q_{n,\nu}(z) = \prod_{s=1}^{\nu} A_s \prod_{s=1}^{\nu} a_s^{n-\nu+1} \Pi (a_h - a_k)^2 \prod_{h=1}^{\nu} (1 - a_h z) \cdot \left[1 + E_n \left(\frac{a_{\nu+1}}{a_\nu}\right)^n\right]$$

and

$$P_{n,\nu}(z) = z^{n+1} \prod_{s=1}^{\nu+1} A_s \prod_{s=1}^{\nu+1} a_s^{n-\nu+1} \left\{ \sum_{r=1}^{\nu} \Pi' (a_h - a_k)^2 \prod_{l=1}^{\nu+1} (a_r - a_l)/a_r(1 - a_r z) \right\}$$

$$\times \left[1 + D_n \left(\frac{\rho}{a_{\nu+1}}\right)^n\right]$$

$$+ Q_{n,\nu}(z) \left\{ \sum_{r=1}^{\nu+1} \frac{A_r}{1 - a_r z} + \phi_{n-\nu}(z) \right\} \left[1 + E_n \left(\frac{a_{\nu+1}}{a_\nu}\right)^n\right],$$

where $|E_n| < E$, $|D_n| < D$, two finite constants, and $\phi_{n-\nu}(z)$ represents the first $n-\nu+1$ terms in the expansion of $\phi(z)$. [See (3).]

The ratio $P_{n,\nu}(z)/Q_{n,\nu}(z)$ is of the form $C_n z^n a_{\nu+1}^n + C_n'$, where C_n, C_n' are functions of z with limiting values C, $C' \equiv f(z)$.

Hence, as $n \to \infty$, $P_{n,\nu}(z)/Q_{n,\nu}(z)$ remains finite, but, since $C \neq 0$, it does not tend to a limit at any point on the boundary except at the point $z = 1/a_{\nu+1}$, discussed in § 8.

The form of this result is unaffected by the modification, as in § 6, for enclosed multiple poles, or by that of § 7, for enclosed poles with index-points of equal moduli.

10. *Summary of Results.* The region of representation of the continued fraction is given by the theorems of Montessus de Ballore. Outside the circle mentioned in §§ 1 and 2 the function is not defined by this representation since the continued fraction diverges everywhere. Within the circle, however, the function is analytic, except at the enclosed poles.

The order of divergence at enclosed poles is independent of their multiplicity or of the equality in moduli of the index-points of adjacent poles, and is given by

$$\lim_{n\to\infty} \frac{P_{n,\nu}(1/a_r)/Q_{n,\nu}(1/a_r)}{(a_r/a_{\nu+1})^n} = \text{constant},$$

as in (8) and (10).

The order of divergence at the boundary pole is given by

$$\lim_{n\to\infty} \frac{P_{n,\nu}(1/a_{\nu+1})/Q_{n,\nu}(1/a_{\nu+1})}{n} = \text{constant},$$

The value at any other point on the boundary is indeterminate but finite.

The last two results are in conformity with the known results in the case of the Taylor expansion and are given by the first row of the Padé table. For in this case we have to consider the behaviour of the Taylor expansion of a function for which the only singularity on the circle of convergence is a simple pole.

The first result, however, is strikingly different in its order of divergence from those arising in the case of the Taylor expansion of a function.

Of chief interest is the dominating effect of the boundary pole on the order of divergence. The index points $1/a_r$, $1/a_{\nu+1}$, and the residues A_r, $A_{\nu+1}$, also play a peculiar part in the details of the results.

THEOREMS CONCERNING AN ANALYTIC FUNCTION WHICH IS BOUNDED UPON A SIMPLE CURVE PASSING THROUGH AN ISOLATED ESSENTIAL SINGULARITY

By E. F. COLLINGWOOD *and* G. VALIRON.

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1. *Introduction.* Let the function $f(z)$ be regular in a simply connected domain D, whose frontier E is a simple, closed, continuous curve passing through an isolated essential singularity of $f(z)$, which we may assume to be situated at the origin 0. We suppose that $f(z)$ is bounded on E and continuous on E except, perhaps, at 0. Let $m(r)$ be the maximum of $|f(z)|$ on E for $|z| \leqslant r$. $m(r)$ is bounded, and so it does not increase without limit as $r \to 0$.

The circumference $\gamma_r(|z| = r)$ may be supposed to cut E at more than one point, and so it divides D into a finite or infinite number of distinct domains, of which one, denoted by D_r, has 0 as a frontier point. We denote by $M_D(r)$ the maximum of $|f(z)|$ upon those arcs of γ_r intercepted by D. Suppose that $M_D(r)$ is not bounded. It then follows from the maximum modulus principle that, for all r less than a certain R_0, $M_D(r)$ is an increasing function of $1/r$ and is greater than $m(r)$. Let T_r be the portion of γ_r belonging to the frontier of D_r. It then follows from the same principle that $|f(z)| = M_D(r)$ at some point of T_r.

Here we consider the order of magnitude of $M_D(r)$ as $r \to 0$. The paper falls into three sections. In Section I we give a detailed proof, outlined by one of us in a recent note*, of the following result :

THEOREM I. *Given a positive number $\delta < 1$, there is a constant $k = k(\delta) < 1$, depending upon δ such that, for $r < R_0$,*

$$M_D(kr)^{1+\delta}\, m(r)^{1-\delta} > M_D(r)^2. \tag{1}$$

* E. F. Collingwood, "Sur un théorème de M. Lindelöf", *Comptes rendus*, 181 (1925), 844.

Our proof of this result rests upon considerations of a very general kind concerning analytic functions, and it is for this reason that we have included it. The dependence of the geometrical configuration, which determines k, upon the exponent δ is an essential feature of this method, of which we give a preliminary sketch in § 2.

It happens, however, that the problem with which we are here concerned does not really call for such a general method, but may be attacked by one of more restricted application due to Lindelöf. This is done in Section II. The use of two transformations of the simplest kind enables us to avoid the general considerations which dominate the first method and, with them, to eliminate the exponent δ. This leads us to our main result, which is

THEOREM II. *There is an absolute constant* $k<1$ *such that, for* $r<R_0$,

$$M_D(kr)\, m(r) > M_D(r)^2. \tag{2}$$

By a repeated application of this inequality we show that there is an absolute constant σ such that

$$\lim_{r\to 0} \frac{\log\log M_D(r)}{-\log r} \geqslant \sigma > 0. \tag{3}$$

A result of this type has already been established by Carleman*. We have included it here as an immediate consequence of our main inequality.

Let C_1 and C_2 be the two arcs of E defined by 0 and some other point of E, and suppose that $f(z)\to 0$ as $z\to 0$ along both C_1 and C_2. $m(r)$ then tends to zero as $r\to 0$ and our argument applies if $M_D(r)$ is bounded, and leads to the conclusion, due to Lindelöf†, that in this case $f(z)$ tends uniformly to zero as z approaches 0 along all paths in D. The inequalities (2) and (3) enable us to complete this result as follows :

THEOREM III. *If* $m(r)\to 0$ *as* $r\to 0$, *then either* $f(z)\to 0$ *uniformly as* $z\to 0$ *along all paths in* D, *or there are positive constants* σ, η, *and*

* T. Carleman, "Fonctions inverses des fonctions entières d'ordre fini", *Arkiv för Math.*, 15 (1920), No. 10.

† E. Lindelöf, "Sur un principe général de l'analyse et ses applications à la théorie de la représentation conforme", *Acta Soc. Sci. Fenn.*, 46, No. 4 (1915), 9.

k (<1) *such that, for* $r<R_0$,

$$M_D(kr)\, m(r)\, e^{-\eta r^{-\sigma}} > 1.$$

It is the presence of the factor $m(r)$ which constitutes the novel feature of this last inequality.

In Section III we give certain applications of our inequalities to the theory of integral functions.

I. *First method. Proof of Theorem* I.

2. We shall assume that T_r is wholly interior to a connected arc S_r on γ_r of length less than $2\pi r/3$. For if not it would only be necessary to make the transformation (z, z^3). We take $r<R_0$ so that $M_D(r)$ is an increasing function of $1/r$ and $m(r)<M_D(r)$.

Next put $r'=kr$, where the constant $k<1$ is to be fixed later, and denote by $D_{rr'}$ that portion of D_r which lies in the annulus $r'<|z|<r$.

Now let z_0 be a point inside $D_{rr'}$ and $\bar{z}_0$ its image with respect to the circumference γ_r and make the transformation

$$\frac{\xi-z_0}{\xi-\bar{z}_0} = -\frac{z-z_0}{z-\bar{z}_0}, \tag{4}$$

which leaves z_0 invariant and transforms γ_r into itself. Now as z_0 tends to zero the transformation (4) tends to a simple rotation through the angle π about the origin. It follows that there is a positive number μ (<1) such that, if $|z_0|/r<\mu$, the arc S_r will be transformed by (4) into another arc S_r' of γ_r having no point in common with S_r. We shall assume throughout that

$$\lambda = \frac{|z_0|}{r} = \frac{\mu}{\mu+1} < \tfrac{1}{2}; \tag{5}$$

and $k=r'/r<\lambda$.

$D_{rr'}$ is transformed by (4) into a region $D'_{rr'}$ interior to γ_r and containing z_0, which is left invariant by (4). There is therefore a region Γ common to $D_{rr'}$ and $D'_{rr'}$. The frontier of Γ contains no point of γ_r, but consists wholly of arcs of C_1, C_2, and $\gamma_{r'}(|z|=r')$ and their transforms C_1', C_2', and $\gamma_{r'}'$ by (4). It is well known that $\gamma_{r'}'$ is a circle interior to γ_r and its properties are easy to determine.

The function $f_1(z)=f(\xi)$ is regular in $D'_{rr'}$ and

$$f_1(z_0)=f(z_0).$$

The function $F(z) = f(z)f_1(z)$ is therefore regular in Γ, continuous on the frontier of Γ, and

$$F(z_0) = f(z_0)^2.$$

Now the frontier of Γ consists of two parts A and B. A consists wholly of arcs of C_1, C_1', C_2, and C_2', while B consists of arcs of $\gamma_{r'}$ and $\gamma_{r'}'$. On A we have $|F(z)| < m(r)\,M_D(r')$, since on C_1 and C_2 $|f(z)| < m(r)$ and in $D_{rr'}|f(z)| < M_D(r')$, while on B we have $|F(z)| < [M_D(r')]^2$.

We are now in a position to apply an inequality of R. and F. Nevanlinna to $F(z)$ at an interior point of Γ, in this case the point z_0. Let $g(z, z_0)$ be the Green's Function for the region Γ with pole z_0, and let $-h(z, z_0)$ be the conjugate function. The Nevanlinna* inequality may now be written

$$\log|F(z_0)| < \frac{1}{2\pi}\left\{\int_A \log|F(z)|\,dh(z, z_0) + \int_B \log|F(z)|\,dh(z, z_0)\right\}$$

$$< \frac{1}{2\pi}\left\{\log[M_D(r').m(r)]\int_A dh(z, z_0) + 2\log M_D(r')\int_B dh(z, z_0)\right\}$$

$$= \log[M_D(r').m(r)] + Q(z_0)\log\frac{M_D(r')}{m(r)},$$

where $$Q(z_0) = \frac{1}{2\pi}\int_B dh(z, z_0),$$

since $$\int_A dh(z, z_0) = 2\pi - \int_B dh(z, z_0).$$

We shall show that, given any positive $\delta < 1$, we can find $k = r'/r$ such that

(6) $$Q(z_0) < \delta.$$

This leads to the inequality

$$|f(z_0)|^2 < M_D(r')^{1+\delta}.m(r)^{1-\delta},$$

and, since we can choose z_0 in $D_{rr'}$ such that $|f(z_0)| = M_D(r_0) > M_D(r)$, the inequality (1) is established.

3. Let B consist of the arcs $\beta_1, \beta_2, \ldots \beta_\nu, \ldots$ on $\gamma_{r'}$ and their transforms $\beta_1', \beta_2', \ldots \beta_\nu', \ldots$ on $\gamma_{r'}'$, and let $\bar{\Gamma}$ be the region derived from Γ by

* R. and F. Nevanlinna, "Über die Eigenschaften analytischer Funktionen in der Umgebung einer singulären Stelle oder Linie", *Acta Soc. Sci. Fenn.*, 50, No. 5 (1922), 8.

adding to it those adjacent regions which are bounded either by an arc of Δ (the frontier of Γ) and an arc of $\gamma_{r'}$ or by an arc of Δ and an arc of $\gamma'_{r'}$. The effect of this is to replace the arcs $\beta_1, \beta_2, \ldots \beta_\nu, \ldots$ by a single arc β on $\gamma_{r'}$ containing all of them, and the arcs $\beta_1, \beta'_2, \ldots \beta'_\nu, \ldots$ by a single arc β' on $\gamma'_{r'}$. Let $\bar{g}(z, z_0)$ be the Green's function for $\bar{\Gamma}$ and $-\bar{h}(z, z_0)$ its conjugate. $\bar{\Gamma}$ contains Γ and we have, by a well known theorem, for all z in Γ,

$$g(z, z_0) < \bar{g}(z, z_0).$$

Hence, on B, $$\frac{\partial}{\partial n} g(z, z_0) \leqslant \frac{\partial}{\partial n} \bar{g}(z, z_0),$$

$\partial/\partial n$ denoting differentiation with respect to the interior normal, and so

$$Q(z_0) = \frac{1}{2\pi} \int_B \frac{\partial}{\partial n} g(z, z_0)\, ds \leqslant \frac{1}{2\pi} \int_B \frac{\partial}{\partial n} \bar{g}(z, z_0)\, ds = \frac{1}{2\pi} \int_B d\bar{h}(z, z_0)$$

$$< \frac{1}{2\pi} \left\{ \int_\beta d\bar{h}(z, z_0) + \int_{\beta'} d\bar{h}(z, z_0) \right\} = \bar{Q}(z_0).$$

To prove (6) it will be sufficient to prove that, given $\delta < 1$, we can find k such that $\bar{Q}(z_0) < \delta$. To do this we make use of an inequality of Lindelöf.

Suppose that a is any point of the frontier of $\bar{\Gamma}$, q the magnitude of the vector az_0, K any number not less than the greatest chord of Γ, and l a simple curve issuing from a interior to the circle $|z-a| < \eta$ and, except for its end points, to $\bar{\Gamma}$, then the oscillation of $\bar{h}(z, z_0)$ on l is less than

$$4 \arctan \sqrt{\left(\frac{2 \log(K/q)}{\log(K/\eta)}\right)}, \tag{7}$$

provided that

$$\eta < q \left(\frac{q}{K}\right)^{5+4\sqrt{2}}. \tag{8}$$

This is Lindelöf's* inequality.

Thus, if a is an end point of β, we can join it to the other end point by a simple curve l lying in $\bar{\Gamma}$ and in the circle $|z-a| < 2r'$. Plainly

* Lindelöf, *loc. cit.*, 16–18.

we may take $K = 2r$, and so, since $q \geqslant r_0 - r'$ where $z_0 = r_0 e^{i\theta_0}$, we have, by (7),

$$(9a) \qquad \int_\beta d\bar{h}(z, z_0) = \int_l d\bar{h}(z, z_0)$$

$$\leqslant \underset{l}{\text{oscillation}}\ \bar{h}(z, z_0) < 4 \arctan \sqrt{\left(\frac{2 \log\{2r/(r_0 - r')\}}{\log(r/r')}\right)},$$

provided that

$$(10a) \qquad 2r' < \frac{(r_0 - r')^{2p}}{(2r)^{2p-1}}, \qquad \text{where } p = 3 + 2\sqrt{2}.$$

Similarly, if s' is the radius of $\gamma'_{r'}$ and $\xi_1 = \xi(z_1)$, the transform by (4) of the point $z_1 = r' e^{i\theta_0}$, we have

$$(9b) \qquad \int_{\beta'} d\bar{h}(z, z_0) < 4 \arctan \sqrt{\left(\frac{2 \log\{2r/|\xi_1 - z_0|\}}{\log(r/s')}\right)}$$

provided that

$$(10b) \qquad 2s' < \frac{|\xi_1 - z_0|^{2p}}{(2r)^{2p-1}}.$$

We proceed to show that it is possible to adjust the ratio $h = r'/r_0$ so that the conditions (10) will be satisfied and the expressions on the right-hand side of the inequalities (9) will be arbitrarily small. Since λ is fixed this will give us our value of $k = h\lambda$. This determination of h presents certain inevitable minor complications.

4. The relation (4) may be written

$$\xi(z) = \frac{az + b}{cz + d},$$

where

$$(11) \qquad \begin{cases} a = -d = z_0 + \bar{z}_0; \quad c = 2; \quad b = -2z_0\bar{z}_0 \\ ad - bc = -(\bar{z}_0 - z_0)^2 \neq 0. \end{cases}$$

The centre of $\gamma'_{r'}$ lies on the line Oz_0 produced, and midway between the points $\xi(z_1)$ and $\xi(-z_1)$. Hence

$$s' = \left| \frac{(ad - bc) z_1}{c^2 z_1^2 - d^2} \right|,$$

and substituting the values (11) we find

$$\frac{r'}{s'} = r' \left| \frac{(\bar{z}_0+z_0)^2-4z_1^2}{(\bar{z}_0-z_0)^2 z_1} \right| = \frac{\{(r^2/r_0)+r_0\}^2-4r'^2}{\{(r^2/r_0)-r_0\}^2} < \left(\frac{\lambda^{-2}+1}{\lambda^{-2}-1}\right)^2 < 3,$$

since $\lambda < \frac{1}{2}$.

Also
$$\frac{\{(r^2/r_0)+r_0\}^2-4r'^2}{\{(r^2/r_0)-r_0\}^2} > 1,$$

and so

(12)
$$1 < \frac{r'}{s'} < 3.$$

Further

$$\frac{s'}{|\xi_1-z_0|} = \left| \frac{(ad-bc)\,z_1}{c^2 z_1^2-d^2} \cdot \frac{d+cz_1}{(a-cz_0)\,z_1+b-dz_0} \right|$$

$$= \left| \frac{z_1(\bar{z}_0-z_0)^2(\bar{z}_0+z_0-2z_1)}{(\bar{z}_0-z_0)(z_0-z_1)\{(\bar{z}_0+z_0)^2-4z_1^2\}} \right| = \left| \frac{z_1(\bar{z}_0-z_0)}{(z_0-z_1)(\bar{z}_0+z_0+2z_1} \right|$$

$$= \frac{r'\{(r^2/r_0)-r_0\}}{(r_0-r')\{(r^2/r_0)+r_0+2r'\}}.$$

Hence
$$\frac{r_0-r'}{r'} \cdot \frac{s'}{|\xi_1-z_0|} < 1.$$

Therefore

(13)
$$\frac{r_0-r'}{|\xi_1-z_0|} < \frac{r'}{s'} < 3.$$

We observe that the Lindelöf conditions (10) will both be satisfied if

(14)
$$2r' < \frac{(r_0-r')^{2p}}{(6r)^{2p-1}}.$$

For then (10a) is satisfied and, moreover,

$$\left(\frac{2s'}{|\xi_1-z_0|}\right)^{2p} < \left(\frac{2r'}{r_0-r'}\right)^{2p} < \left(\frac{2r'}{6r}\right)^{2p-1} < \left(\frac{s'}{r}\right)^{2p-1},$$

and so
$$2s' < \frac{|\xi_1-z_0|^{2p}}{(2r)^{2p-1}},$$

and (10b) is also satisfied.

Now (14) may be written

$$h = \frac{r'}{r_0} < \tfrac{1}{2}\left(\frac{\lambda}{6}\right)^{2p-1}(1-h)^{2p},$$

and we see that this condition will be satisfied if

(15) $$h < \lambda^{8p-2}.$$

For then $$(1-h) > 1-2^{2-8p} > \tfrac{3}{4}, \qquad \text{since } \lambda < \tfrac{1}{2},$$

and it will be sufficient to have

$$h < \frac{3}{8}\left(\frac{\lambda}{8}\right)^{2p-1}.$$

But $$\frac{3}{8}\left(\frac{\lambda}{8}\right)^{2p-1} > \tfrac{1}{4}(\lambda^4)^{2p-1} > \lambda^{8p-2}, \quad \text{since } \frac{\lambda}{8} > \lambda^4 \text{ because } \lambda < \tfrac{1}{2}.$$

This proves our assertion.

Now suppose that $H \geqslant 1$ and

(16) $$2\log\frac{2r}{r_0-r'}\Big/\log\frac{r}{r'} < \frac{1}{H}$$

or $$(\lambda h)^{1/2H} < \lambda.\frac{1-h}{2},$$

$$h < \lambda^{2H-1}\left(\frac{1-h}{2}\right)^{2H}.$$

In order that this condition, and therefore (16), may be satisfied it is sufficient that

(17) $$h < \lambda^{6H-1}.$$

For then $(1-h) > 1-2^{-5} > \frac{3}{4}$, and it will be sufficient to have

$$h < (\tfrac{3}{8})^{2H}\lambda^{2H-1},$$

and, since $\lambda^2 < \frac{3}{8}$, our assertion is proved.

It is now plain that, if we take

$$h = \lambda^{6H+8p-7},$$

and

(18) $$k = h\lambda = \lambda^{6(H-1)+8p}$$

both (15) and (17) will be satisfied.

Now

$$\sqrt{\left(\frac{2\log\{2r/|\xi_1-z_0|\}}{\log(r/s')}\right)} < \sqrt{\left(\frac{2\log\{6r/(r_0-r')\}}{\log(r/r')}\right)}$$

$$< \sqrt{\left(2.\frac{2\log\{2r/(r_0-r')\}}{\log(r/r')}\right)} < 2\sqrt{\left(\frac{1}{H}\right)},$$

since

$$\frac{2r}{r_0-r'} > \frac{2}{\lambda} > 3.$$

Let δ be any positive number less than 1 and choose

$$H = \cot^2\frac{\delta\pi}{6}.$$

Then we have, by (9a) and (9b),

$$\bar{Q}(z_0) < \frac{1}{2\pi}\left\{\frac{4\delta\pi}{6}+4\arctan\left(2\tan\frac{\delta\pi}{6}\right)\right\} < \delta,$$

with $$\frac{r'}{r} = k = \exp\left\{\left(6\cot^2\frac{\delta\pi}{6}+18+16\sqrt{2}\right)\log\lambda\right\}.$$

This completes the proof of Theorem I.

II. *Second method. Proof of Theorem* II.

5. As before we suppose that $r < R_0$, $r' = kr$, where $k < 1$ is be fixed later. The frontier of $D_{rr'}$ is made up (i) of T_r, (ii) of arcs of $\gamma_{r'}$, (iii) of at least one arc $C_1(r, r')$ of C_1, one of whose extremities is on γ_r and the other on $\gamma_{r'}$, all other points of $C_1(r, r')$ being interior to the annulus $r' < |z| < r$, and (iv) of other arcs both of whose end points lie either on γ_r or on $\gamma_{r'}$.

There are two cases to be considered :

(*a*) The argument of z varies by less than 4π, as z describes $C_1(r, r')$, starting from the point on γ_r.

(*b*) This variation in argument is at least 4π.

6. *Case* (*a*). Let p be an integer, and make the transformation $z = Z^p$. $D_{rr'}$ is transformed into Δ and $f(z)$ becomes

$$F(Z) = f(Z^p),$$

which is regular in Δ and continuous on the frontier. Δ lies between the transform Γ_1 of $C_1(r, r')$, and a curve Γ_2 derived from Γ_1 by a rota-

tion through an angle $2\pi/p$ about O. On Γ_1 the argument of Z varies by less than $4\pi/p$. Hence, in Δ,

$$\phi_0 < \arg Z < \phi_0 + \frac{4\pi}{p} + \frac{2\pi}{p},$$

where ϕ_0 is the minimum value of $\arg Z$ on Γ_1. Putting $p = 36$, Δ lies between two rays, making an angle $6\pi/p = \pi/6$, and in the annulus

$$R' = (kr)^{\frac{1}{36}} < |Z| < R = r^{\frac{1}{36}}.$$

We now apply the method of Lindelöf. On the circumference

$$|Z| = (R'+R)/2$$

there is a point Z_0 in Δ at which

$$|F(Z_0)| = |f(z_0)| > M_D(r), \quad (z_0 = Z_0^p)$$

since $D_{rr'}$ contains $T_{r''}$ if $r' < r'' < r$. About this point Z_0 we turn Δ through the angle $\frac{1}{2}\pi$, which gives a domain Δ' in which the function

$$\Phi(Z) = F\{Z_0 + i(Z - Z_0)\}$$

is regular. On those portions of the contours of Δ and Δ' arising from C_1 and C_2 we have $|F(Z)| < m(r)$ and $|\Phi(Z)| < m(r)$ respectively, while at all interior points of Δ and Δ' $|F(Z)$ or $|\Phi(Z)|$ is less than $M_D(r')$. Now if k is sufficiently small the frontier of Δ'', the domain common to Δ and Δ', which contains Z_0, will consist only of arcs of the frontiers of these two domains which are not arcs of the circles $|Z| = R'$ and $|Z| = R$. A simple geometrical calculation shows that it is in fact sufficient to have

$$\frac{R'}{R} = \frac{2\sqrt{(3)}-1}{11}.$$

Then on the frontier of Δ'' $|\Phi(Z)\,F(Z)| < m(r)\,M_D(r')$ while $\Phi(Z)\,F(Z)$ is regular in Δ''. Therefore

$$M_D(kr)\,m(r) > |\Phi(Z_0)\,F(Z_0)| = |f(z_0)|^2 \tag{19}$$

$$> [M_D(r)]^2, \tag{2}$$

where

$$k = \left(\frac{2\sqrt{(3)}-1}{11}\right)^{36}. \tag{19a}$$

7. *Case* (*b*). On $C_1(r, r')$ the argument of z varies by at least 4π. Let A be a point of γ_r exterior to the arc S_r of length $2\pi r/3$ which contains T_r, and at an angular distance of, say, $\pi/12$ from one of the end points of S_r. $C_1(r, r')$ cuts OA, and there are two successive points of intersection, B and B', between which the change in argument of z is equal to 2π. For if this were not so the difference between the argument of any point of $C_1(r, r')$ and that of OA would be less than 2π, and so the difference between the arguments of any two points of $C_1(r, r')$ would be less than 4π, which is contrary to hypothesis. Let G be the arc of $C_1(r, r')$ joining B to B', and G* the simple closed curve formed by G and the segment BB'. G* is completely interior to the annulus $r' < |z| < r$, and so in the portion of $D_{rr'}$ exterior to G^* the maximum of $|f(z)|$ is greater than $M_D(r)$. This maximum is attained on the contour, and so at a point z_0 of the interval BB'.

Now let z' be the image of z with respect to OA, and introduce the function $\phi(z) = \bar{f}(z')$, where $\bar{f}$ is the conjugate of f. $\phi(z)$ exists and is regular in the domain D_1, which is the image of $D_{rr'}$ with respect to OA. At all points of OA belonging to $D_{rr'}$

$$|\phi(z)| = |f(z)|.$$

Let V be the domain containing z_0, and consisting of points common to $D_{rr'}$ and D_1. Each point of V can be joined to z_0 by a path interior to both D_1 and $D_{rr'}$ and not crossing the frontiers of either; and the frontier of V consists of portions of the frontiers of $D_{rr'}$ and D_1. It is plain that the frontier of V includes no point of either $\gamma_{r'}$ or γ_r. For G and its image G' must have at least one point of intersection. There is therefore a closed curve J, consisting of portions of G and G' passing through the point B' and containing $\gamma_{r'}$ in its interior. If a point of $\gamma_{r'}$ were a frontier point of V we could find a point P, arbitrarily near to this point, interior to V and to J. Since z_0 is exterior to J, any path joining z_0 to P must cross J, which is impossible, since J consists of portions of the frontiers of $D_{rr'}$ and D_1. Similarly there is a closed curve passing through B interior to γ_r and containing z_0 in its interior. Precisely the same argument shows that no point of γ_r can be a frontier point of V. The frontier of V, therefore, consists wholly of arcs of C_1 and C_2 and of their images C_1' and C_2'; and on all of these $|\phi(z)f(z)|$ is less than $m(r)M_D(r')$. Since the function $\phi(z)f(z)$ is regular in V we have

$$M_D(r')\,m(r) > |f(z_0)|^2 > M_D(r)^2. \tag{20}$$

Hence the inequality (2) is established in all cases with the value

$$\frac{r'}{r} = k = \left(\frac{2\sqrt{(3)}-1}{11}\right)^{36}.$$

This completes the proof of Theorem II.

8. *Proof of inequality* (3). The inequality (2) holds for all values of $r < R_0$. Hence, for $r < R_0$,

$$M_D(kr)\, m(r) > M_D(r)^2$$

$$M_D(k^2 r)\, m(kr) > M_D(kr)^2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$M_D(k^n r)\, m(k^{n-1} r) > M_D(k^{n-1} r)^2,$$

and, combining these inequalities, we have

$$M_D(k^n r)\, m(k^{n-1} r)\, m(k^{n-2} r)^2 \ldots m(r)^{2^{n-1}} > M_D(r)^{2^n}. \tag{21}$$

We may suppose, without loss of generality, that $m(r) < 1$, and we can choose $r_0 (< R_0)$ such that $M_D(r_0) > 1$. It then follows from (21) that

$$M_D(k^n r_0) > M_D(r_0)^{2^n},$$

and, writing $r = k^n r_0$,

$$\frac{\log \log M_D(r)}{-\log r} > \sigma \left(1 + \frac{\log r_0}{-\log r}\right) + \frac{\log \log M_D(r_0)}{-\log r}, \tag{22}$$

where

$$\sigma = -\frac{\log 2}{\log k} > 0.$$

Hence

$$\varliminf_{r \to 0} \frac{\log \log M_D(r)}{-\log r} \geqslant \sigma, \tag{3}$$

which is the inequality (3).

9. *Proof of Theorem* III. Put $\eta = 2 r_0^{\sigma} \log M_D(r_0) > 0$, since $M_D(r_0) > 1$. It then follows, from (2) and (22), that

$$M_D(kr)\, m(r)\, e^{-\eta r^{-\sigma}} > 1. \tag{23}$$

Suppose now that $\lim_{r \to 0} m(r) = 0$; that is to say, $f(z) \to 0$ as $z \to 0$ along both C_1 and C_2. It is then plain that our arguments apply, without

modification, if $f(z)$ is bounded in D, say $|f(z)|<K$, provided that we substitute K for $M_D(kr)$ throughout. Inequalities (19) and (20) then give

$$M_D(r_0)<\sqrt{\{K.m(r)\}}\to 0 \quad \text{as} \quad r\to 0,$$

where $kr<r_0<r$. This is Lindelöf's result. Combining it with (23) we obtain Theorem III. Our argument is quite independent of the Phragmén-Lindelöf principle, which was an essential part of Lindelöf's proof of his theorem. In our first method, on the other hand, we rely upon a result of R. and F. Nevanlinna, which may be regarded as a generalization of this principle.

III. *Applications and concluding remarks.*

9. Theorems I and II carry us further than Carleman's result in cases where $m(r)$ decreases sufficiently rapidly; to be precise, where

$$\overline{\lim_{r\to 0}} \frac{\log\log\{1/m(r)\}}{-\log r} > \frac{2}{\pi^2}. \tag{24}$$

The relation existing between $M_D(r)$ and $m(r)$ is analogous to that existing between the maximum and minimum modulus of a function $F(z)$ regular in the neighbourhood of an isolated essential singularity. Supposing this singularity to be at the origin and the order to be finite, it has been shown by one of us* that the inequality

$$M(kr)^H m_1(r) > 1, \tag{25}$$

$M(r)$ being the maximum and $m_1(r)$ the minimum of $|F(z)|$ on $|z|=r$, is satisfied for $k<1$, for a value $H=H(k)>0$, and for a sequence of values of r such that every interval R, $kR(R<R_0)$ contains one of them. There is a corollary to this proposition which it is interesting to compare with Theorem I. Suppose that on every circle $|z|=r(r<R_0)$ there is a point at which

$$|F(z)|\leqslant m_1(r), \tag{26}$$

$m_1(r)$ being a decreasing function. Then it follows from the three-circles theorem that inequality (25) is satisfied for all sufficiently small values of r. If we transfer our attention from the values of the exponents H and δ to the fact that the set of points in which (26) is satisfied may be

* G. Valiron, "Recherches sur le théorème de M. Picard dans la théorie des fonctions entières", *Annales Sci. de l'Ecole Normale* (3), 39 (1922), 317, or *General theory of integral functions* (1923), 89.

discontinuous, it appears that this result is more complete than that given by inequality (1). This leads us to ask the following question: Are there functions $F(z)$ such that on every circle $|z| = r (r < R_0)$ there is a point at which

$$|F(z)| < M(kr)^{-H}, \quad k \leqslant 1, \quad H = H(k) > 0,$$

but no continuous curve on which this inequality is satisfied (with appropriate values of H and k)?

If the answer is in the negative it will follow that every value a which is exceptional, in the sense of R. Nevanlinna, as regards the distribution of the moduli of the zeros of $F(z)-a$ must be an asymptotic value.

It is scarcely necessary to observe, however, that for the functions $F(z)$ here considered our theorems give results which cannot be derived from the theorem concerning the minimum modulus if there are several curves on which $m(r)$ satisfies inequality (24) and between which $F(z)$ is unbounded.

10. Our second method leads to other new results if, simply assuming that $m(r)$ is bounded, we adopt a hypothesis about the form of the domain D_r. These hypotheses correspond to the cases (a) and (b), and lead to results which may be compared with those of Denjoy*. We restrict ourselves to the case in which the function considered is unbounded in D_r.

A. Suppose that, for a sequence of values of R tending to 0, D_r belongs to the domain

$$R^\gamma < |z| < R, \quad \gamma > 1,$$

$$\phi_0 < \arg z < \phi_0 + \epsilon(R),$$

where $\epsilon(R)$ is non-increasing. Then

$$\overline{\lim_{r \to 0}} \frac{\log \log M_D(r)}{-\log r} \geqslant \left(1 - \frac{1}{\gamma}\right) \frac{\log 2}{\mu(\epsilon)}, \tag{27}$$

$\mu(\epsilon)$ being a function of $\epsilon = \lim_{r \to 0} \epsilon(R)$ such that $\mu(0) = 0$.

To see this we apply the method of case (a), followed by the repetition argument of § 8. It follows that if D satisfies these conditions with $\epsilon = 0$, and if $M_D(r)$ is of finite order, then $f(z)$ must be bounded in D.

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* A. Denjoy, "Sur les fonctions entières de genre fini", *Comptes rendus*, 145 (1908), 106–108.

Applying this last result to the integral functions of the form

$$F(z) = e^{-z} \sum_0^\infty \frac{k_n}{n!} z^n, \quad \tfrac{1}{2} < k_n < 1,$$

introduced by one of us* to show that a function may be of regular growth, while the distribution of the zeros of $F(z)-a$ may be irregular for a non-enumerable set of values of a, we see that these functions have no asymptotic values.

B. Suppose that, for a sequence of values of R tending to zero, the portion of C_1 belonging to the annulus

$$R^\gamma < |z| < R, \quad \gamma > 1,$$

has a polar equation of the form

$$r = \theta(\phi), \tag{28}$$

$\theta(\phi)$ being monotonic and such that

$$|\theta(\phi+2\pi)-\theta(\phi)| \leqslant \epsilon(R)\,\theta(\phi), \tag{29}$$

where $\epsilon(R)$ is once more non-increasing. Then inequality (27) is again satisfied. This follows from the proof of Theorem II in the case (b).

In particular, it is not possible for an integral function of finite order to be bounded upon a spiral of the form (28) when $\theta(\phi)$ satisfies (29) with $\lim_{r\to 0} \epsilon(R) = 0$†. To emphasize the interest of this proposition we recall the fact that there are functions of finite order which are bounded upon logarithmic spirals‡.

11. In the case where $f(z)$ is regular in the neighbourhood of an isolated essential singularity we may replace the hypothesis of the existence of a complete curve C on which $f(z)$ is bounded by the hypothesis that there exists a discontinuous sequence of curves which cross a sequence of annuli and on which $f(z)$ is bounded. Certain consequences of this hypothesis have been discussed by one of us in a recent note§, and will be

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* G. Valiron, "Sur les zéros des fonctions entières d'ordre fini", *Rend. di Palermo*, 42 (1918).

† This result was conjectured by one of us in 1917. G. Valiron, "Sur les chemins de détermination des fonctions entières", *Bull. Soc. Math. de France*, 45 (1917), 153.

‡ We are unable to give a reference, but such functions are easy to construct.

§ E. F. Collingwood, "Sur un théorème de M. Valiron", *Comptes rendus*, 182 (1926), 40. The writer of this note takes this opportunity of saying that the argument is in part fallacious; for the assertion that the set of partially asymptotic values is enumerable is false. The argument can be amended so as to show that the relevant set is of linear measure zero. The same error occurs in a later note in *Comptes rendus*, 183 (1926), 847, where Theorem II is false and should be deleted.

more fully developed elsewhere. We restrict ourselves here to pointing out a case where the inequality given by Theorem III may replace the more precise inequality given by the method of Carleman. The method of § 8 leads at once to this result : *If $F(z)$ is regular in the neighbourhood of infinity and the numbers $K>0$ and $\gamma>1$ are given, and if there is a number $R=R(K,F)>R_0$ such that $|F(z)|<K$ on a continuous curve C_R, crossing the annulus*

$$R<|z|<R^{\gamma},$$

then

$$\log M(R^{\gamma}) \geqslant R^{(\gamma-1)\log 2/[1+\delta(R)]\log k}, \tag{30}$$

where

$$\delta(R)\to 0 \quad as \quad R\to\infty .$$

As an immediate corollary we have the following proposition, which generalizes Carleman's theorem in a new way :

THEOREM IV. *If $F(z)$ is regular in the neighbourhood of infinity, of order ρ, and such that, for a fixed number $\gamma>1$, there is an infinite sequence of annuli (R_1, R_1^{γ}), (R_2, R_2^{γ}), ... (R_n, R_n^{γ}), ..., where $\lim_{n\to\infty} R_n=\infty$, each annulus (R_n, R_n^{γ}) being crossed by a continuous curve C_n on which $|f(z)|<K$, then*

$$\rho \geqslant \left(1-\frac{1}{\gamma}\right)\frac{\log 2}{\log k}. \tag{31}$$

It follows that if $F(z)$ is of zero order there cannot be a number $\gamma>1$ and an indefinitely increasing sequence of numbers (R_n) with the property indicated.

This leads to a proposition which had already been deduced by one of us* from the theorem of Carleman : *If $F(z)$ is of zero order, and $\Delta_1, \Delta_2, \ldots, \Delta_n, \ldots$, are the domains in which $|F(z)|<K$, each of which contains at least one zero of all the functions $F(z)-x$ provided $|x|<K$, then the ratio of the logarithms of the maximum and minimum values of $|z|$ in Δ_n tends to unity as n tends to infinity.* In other words, the branches of the inverse function $\phi(u)$ of $F(z)$ have the following property : *For $|u|<K$ all but a finite number of the functions $\log\phi(u)$ have the same relative continuity.*

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* G. Valiron, "Remarque sur un théorème de M. Carleman", *Comptes rendus*, 176 (1923), 943, 944.

ON INTEGRAL FUNCTIONS WITH REAL NEGATIVE ZEROS

By E. C. Titchmarsh.

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1. The familiar fact, that the density of the zeros of an integral function is related to the rate of increase of the modulus of the function, takes a very precise form in the case of functions of special kinds. The functions to be considered here are of the form

$$f(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{a_n}\right),$$

where the numbers $a_1, a_2, \ldots$ are all real and positive, and the product is absolutely convergent; and the asymptotic behaviour assumed is what Valiron has called perfectly regular growth*. It may be supposed that $a_{n+1} \geqslant a_n$ for all values of n. Let $n(r)$ be the number of numbers a_n for which $a_n \leqslant r$. Then we have

Theorem I. *If, as* $r \to \infty$,

$$n(r) \sim \lambda r^{\rho} \quad (0 < \rho < 1,\ \lambda > 0),$$

then, as $x \to \infty$ *(*x *real),*

$$\log f(x) \sim \pi\lambda \operatorname{cosec} \pi\rho \,.\, x^{\rho}.$$

For
$$\begin{aligned}
\log f(x) &= \sum_{n=1}^{\infty} \log\left(1+\frac{x}{a_n}\right) \\
&= \sum_{n=1}^{\infty} n \left\{\log\left(1+\frac{x}{a_n}\right) - \log\left(1+\frac{x}{a_{n+1}}\right)\right\} \\
&= \sum_{n=1}^{\infty} n \int_{a_n}^{a_{n+1}} \frac{x\,dr}{r(x+r)} \\
&= x\int_0^{\infty} \frac{n(r)\,dr}{r(x+r)}.
\end{aligned}$$

* G. Valiron, *Integral functions* (Toulouse, 1923), 44–45.

Hence
$$\log f(x) \sim x \int_0^\infty \frac{\lambda r^\rho \, dr}{r(x+r)}$$
$$= \lambda x^\rho \int_0^\infty \frac{t^{\rho-1}\, dt}{1+t}$$
$$= \pi\lambda \operatorname{cosec} \pi\rho \, . \, x^\rho.$$

2. The converse of Theorem I is true also.

THEOREM II. *If, as* $x \to \infty$,
$$\log f(x) \sim \pi\lambda \operatorname{cosec} \pi\rho \, . \, x^\rho \quad (0 < \rho < 1, \ \lambda > 0),$$
then
$$n(r) \sim \lambda r^\rho.$$

This is, in the case $\rho = \frac{1}{2}$, equivalent to a theorem which I have proved elsewhere*. It is of the same nature as the "positive Tauberian" theorems of Hardy and Littlewood†, and, in fact, it is deduced from a form of the Hardy-Littlewood theorem, and I am unable to prove it in any other way. On the other hand, from Theorem II it is possible to deduce the Hardy-Littlewood theorem, so that the two things are really equivalent. It is remarkable that no complex analysis is used in the proof. The theorem of Hardy and Littlewood to be used is :

If $F(x)$ *is a positive integrable function such that, when* $t \to 0$,
$$\int_0^\infty F(x)\, e^{-xt}\, dx \sim t^{-\beta} \quad (\beta > 0),$$
then, when $x \to \infty$,
$$\int_0^x F(u)\, du \sim \frac{x^\beta}{\Gamma(\beta+1)}.$$

We also require the following lemma, proved in my paper already referred to :

LEMMA. *If* $\phi(x)$ *and* $\phi'(x)$ *are integrable over any finite interval, and, when* $x \to \infty$,
$$\phi(x) \sim x^{-\beta}, \quad \phi'(x) = O(x^{-\beta-1}) \quad (0 < \beta < 1),$$

* E. C. Titchmarsh, *London Math. Soc. Journal*, 1 (1926), 35–37.

† G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 13 (1913), 174–191.

then, when $t \to 0$,

$$\int_0^\infty \phi(x) \begin{matrix}\cos\\ \sin\end{matrix} xt\, dx \sim \begin{matrix}\sin\\ \cos\end{matrix} \tfrac{1}{2}\pi\beta\,\Gamma(1-\beta)\, t^{\beta-1}.$$

Proof of Theorem II. We have

$$\log f(x) \geqslant x \int_1^\infty \frac{n(r)\,dr}{r(x+r)} > xn(x) \int_x^\infty \frac{dr}{r(x+r)} = n(x)\log 2.$$

Hence $$n(x) = O(x^\rho).$$

Also $$\frac{f'(x)}{f(x)} = \int_0^\infty \frac{n(r)\,dr}{(x+r)^2} = O\left\{\int_0^\infty \frac{r^\rho\,dr}{(x+r)^2}\right\} = O(x^{\rho-1}).$$

Now $$\frac{\log f(x^2)}{x^2} = 2\int_0^\infty \frac{n(r^2)\,dr}{r(x^2+r^2)}.$$

Multiply each side by

$$x^\alpha \cos(xt - \tfrac{1}{2}\alpha\pi) \quad (t>0,\ 1-2\rho < \alpha < 2-2\rho)$$

and integrate from 0 to ∞. We may, as in my previous paper, invert the order of integration on the right, and we obtain

$$\int_0^\infty \log f(x^2)\cos(xt-\tfrac{1}{2}\alpha\pi)\, x^{\alpha-2}\,dx = \pi\int_0^\infty n(r^2)\, r^{\alpha-2} e^{-rt}\,dr.$$

In the lemma, put

$$\phi(x) = \frac{\sin\pi\rho}{\pi\lambda}\, x^{\alpha-2}\log f(x^2), \quad \beta = 2-2\rho-\alpha.$$

Then

$$\int_0^\infty \log f(x^2)\cos(xt-\tfrac{1}{2}\alpha\pi)\, x^{\alpha-2}\,dx$$

$$= \frac{\pi\lambda}{\sin\pi\rho}\left\{\cos\tfrac{1}{2}\alpha\pi\int_0^\infty \phi(x)\cos xt\,dx + \sin\tfrac{1}{2}\alpha\pi\int_0^\infty \phi(x)\sin xt\,dx\right\}$$

$$\sim \frac{\pi\lambda}{\sin\pi\rho}\left\{\cos\tfrac{1}{2}\alpha\pi\sin(1-\rho-\tfrac{1}{2}\alpha)\pi + \sin\tfrac{1}{2}\alpha\pi\cos(1-\rho-\tfrac{1}{2}\alpha)\pi\right\} \times\Gamma(2\rho+\alpha-1)\, t^{1-2\rho-\alpha}$$

$$= \pi\lambda\,\Gamma(2\rho+\alpha-1)\, t^{1-2\rho-\alpha}.$$

Hence $$\int_0^\infty n(r^2)\, r^{\alpha-2} e^{-t}\,dr \sim \lambda\Gamma(2\rho+\alpha-1)\, t^{1-2\rho-\alpha}.$$

Hence, by the theorem of Hardy and Littlewood,

$$\int_0^x n(r^2)\, r^{\alpha-2}\, dr \sim \frac{\lambda}{2\rho+\alpha-1}\, x^{2\rho+\alpha-1}.$$

Thus*, for $x > x_0(\epsilon)$

$$\frac{\lambda(1-\epsilon)}{2\rho+\alpha-1}\, x^{2\rho+\alpha-1} < \int_0^x n(r^2)\, r^{\alpha-2}\, dr < \frac{\lambda(1+\epsilon)}{2\rho+\alpha-1}\, x^{2\rho+\alpha-1}.$$

Hence

$$\int_x^{x+x\delta} n(r^2)\, r^{\alpha-2}\, dr < \frac{\lambda(1+\epsilon)(1+\delta)^{2\rho+\alpha-1}-\lambda(1-\epsilon)}{2\rho+\alpha-1}\, x^{2\rho+\alpha-1}$$

$$= \frac{\lambda}{2\rho+\alpha-1}\, \{(2\rho+\alpha-1)\,\delta+O(\delta^2)+O(\epsilon)\}\, x^{2\rho+\alpha-1}.$$

On the other hand,

$$\int_x^{x+x\delta} n(r^2)\, r^{\alpha-2}\, dr \geqslant n(x^2) \int_x^{x+x\delta} r^{\alpha-2}\, dr > n(x^2)\, \frac{x\delta}{x^{2-\alpha}(1+\delta)^{2-\alpha}}.$$

Hence
$$n(x^2) < \lambda(1+\delta)^{2-\alpha} \left\{1+O(\delta)+O\left(\frac{\epsilon}{\delta}\right)\right\} x^{2\rho},$$

and the required upper bound is obtained on taking, *e.g.*, $\delta = \sqrt{\epsilon}$. The lower bound may be obtained in a similar way. This proves Theorem II.

2.1. In order to deduce the theorem of Hardy and Littlewood from Theorem II, we require another lemma similar to that already used.

LEMMA. *If $\phi(x)$ and $\phi'(x)$ are integrable over any finite interval not ending at $x = 0$, and if*

$$\phi(x) \sim x^{-\beta} \quad (x \to 0), \qquad |\phi'(x)| < Kx^{-\beta-1} \quad (\textit{all } x),$$

where $0 < \beta < 1$, then, when $t \to \infty$,

$$\int_0^\infty \phi(x) \begin{matrix}\cos\\ \sin\end{matrix} xt\, dx \sim \begin{matrix}\sin\\ \cos\end{matrix} \tfrac{1}{2}\pi\beta\, \Gamma(1-\beta)\, t^{\beta-1}.$$

Consider the sine case, which is the one to be actually used. Let

$$\int_0^\infty \phi(x) \sin xt\, dx = \int_0^{\delta/t} + \int_{\delta/t}^{\Delta/t} + \int_{\Delta/t}^\infty = I_1+I_2+I_3.$$

* The final argument is substantially that of E. Landau, *Rend. di Palermo*, 26 (1908), 218.

Then $$I_1 = O\Big(\int_0^{\delta/t} x^{-\beta}\,dx\Big) = O(\delta^{1-\beta}\,t^{\beta-1}),$$

$$I_3 = -\phi\Big(\frac{\Delta}{t}\Big)\,\frac{1-\cos\Delta}{t} - \frac{1}{t}\int_{\Delta/t}^{\infty} \phi'(x)(1-\cos xt)\,dx$$

$$= O(\Delta^{-\beta}\,t^{\beta-1}) + \frac{1}{t}\int_{\Delta/t}^{\infty} O(x^{-\beta-1})\,dx$$

$$= O(\Delta^{-\beta}\,t^{\beta-1}).$$

Let $m(\xi) = \underset{x\leqslant\xi}{\text{Max}}\,|\,x^{\beta}\,\phi(x)-1\,|$, so that $m(\xi)\to 0$ as $\xi\to 0$. Then

$$I_2 = \int_{\delta/t}^{\Delta/t} x^{-\beta}\sin xt\,dx + \int_{\delta/t}^{\Delta/t} \{x^{\beta}\,\phi(x)-1\}\,x^{-\beta}\sin xt\,dx$$

$$= t^{\beta-1}\int_{\delta}^{\Delta} u^{-\beta}\sin u\,du + O\Big\{m\Big(\frac{\Delta}{t}\Big)\int_{\delta/t}^{\Delta/t} x^{-\beta}\,dx\Big\}$$

$$= t^{\beta-1}\Big[\int_0^{\infty} u^{-\beta}\sin u\,du + O(\delta^{1-\beta}) + O(\Delta^{-\beta}) + O\Big\{m\Big(\frac{\Delta}{t}\Big)\,\Delta^{1-\beta}\Big\}\Big].$$

Hence

$$\int_0^{\infty} \phi(x)\sin xt\,dx$$

$$= t^{\beta-1}\Big[\int_0^{\infty} u^{-\beta}\sin u\,du + O(\delta^{1-\beta}) + O(\Delta^{-\beta}) + O\Big\{m\Big(\frac{\Delta}{t}\Big)\,\Delta^{1-\beta}\Big\}\Big],$$

and the result follows on choosing δ small enough, Δ large enough, and then t large enough.

2.2. *The theorem of Hardy and Littlewood follows from Theorem* II. Suppose that

$$F(x)\geqslant 0, \quad \int_1^{\infty} F(x)\,e^{-xt}\,dx \sim t^{-\beta} \quad (0<\beta<1).$$

Let $$F_1(x) = \int_1^x F(y)\,dy, \quad F_2(x) = [F_1(x)] = F_1(x)+O(1).$$

Then $$\int_1^{\infty} F_1(x)\,e^{-xt}\,dx = \frac{1}{t}\int_1^{\infty} F(x)\,e^{-xt}\,dx \sim t^{-\beta-1},$$

and so

$$\int_1^{\infty} F_2(x)\,e^{-xt}\,dx = \int_1^{\infty} F_1(x)\,e^{-xt}\,dx + O\Big(\frac{1}{t}\log\frac{1}{t}\Big) \sim t^{-\beta-1}.$$

Integrating each side, we obtain

$$\int_1^\infty \frac{F_2(x)}{x}\, e^{-xt}\, dx \sim \frac{1}{\beta}\, t^{-\beta}.$$

Let $$\phi(t) = \beta \int_1^\infty \frac{F_2(x)}{x}\, e^{-xt}\, dx.$$

Then $\phi(t)$ satisfies the conditions of the lemma, so that

$$\int_0^\infty \phi(t) \sin xt\, dt \sim \cos \tfrac{1}{2}\pi\beta\, \Gamma(1-\beta)\, x^{\beta-1}.$$

Now $$\int_0^\infty \sin xt\, dt \int_1^\infty \frac{F_2(r)}{r}\, e^{-rt}\, dr = x \int_1^\infty \frac{F_2(r)}{r(x^2+r^2)}\, dr,$$

if we may invert the order of integration. Now

$$F_2(r) = \int_1^r F(x)\, dx + O(1) \leqslant e \int_1^\infty F(x)\, e^{-(x/r)}\, dx + O(1) = O(r^\beta).$$

Hence the associated double integral is absolutely convergent, by comparison with the integral

$$\int_1^\infty \int_0^\infty r^{\beta-1}\, e^{-rt}\, dr\, dt.$$

This justifies the inversion. We have thus proved that

$$\int_1^\infty \frac{F_2(r)\, dr}{r(x^2+r^2)} \sim \frac{1}{\beta} \cos \tfrac{1}{2}\pi\beta\, \Gamma(1-\beta)\, x^{\beta-2}.$$

This is a case of Theorem II, and it follows that

$$F_2(r) \sim \frac{r^\beta}{\Gamma(\beta+1)}, \quad F_1(r) \sim \frac{r^\beta}{\Gamma(\beta+1)}.$$

3. In the preceding sections it has been shown that functions of the type considered are characterized by their asymptotic behaviour on the positive real axis. It will now be shown that they are also, to a certain extent, characterized by their asymptotic behaviour on the negative real axis.

As before, we prove a direct theorem, from $n(x)$ to $\log|f(-x)|$, and a converse. The direct theorem is rather more complicated than the previous one, owing to the exceptional behaviour of the function in

the immediate neighbourhood of its zeros. The converse theorem is of an entirely different character from Theorem II. It has nothing to do with the theorem of Hardy and Littlewood, but is deduced from the well known theorems of Phragmén and Lindelöf. This is true, at any rate, in the case $\rho < \frac{1}{2}$. I have not been able to reach any satisfactory conclusion in the case $\frac{1}{2} < \rho < 1$.

The direct theorem for $f(-x)$ is as follows:

THEOREM III. *Let*

$$n(x) \sim \lambda x^\rho \quad (0 < \rho < 1).$$

Then, if ϵ is any given positive number,

$$\log|f(-x)| > (\pi\lambda \cot \pi\rho + \epsilon)\, x^\rho \quad [(x > x_0(\epsilon)],$$

while, if η is any other positive number,

$$\log|f(-x)| > (\pi\lambda \cot \pi\rho - \epsilon)\, x^\rho,$$

for $0 < x < X$, $\{X > X_0(\epsilon, \eta)\}$, except in a set of measure less than ηX.

We may express the result by saying that

$$\log|f(-x)| \sim \pi\lambda \cot \pi\rho \,.\, x^\rho$$

in a set of density 1.

The proof is in two parts.

(i) We have $$a_n \sim (n/\lambda)^{1/\rho}.$$

Hence, given $\delta > 0$, we can determine L, so that

$$\left\{\frac{n(1-\delta)}{\lambda}\right\}^{1/\rho} < a_n < \left\{\frac{n(1+\delta)}{\lambda}\right\}^{1/\rho} \quad (n > L).$$

Let $$M = \left[\frac{\lambda x^\rho}{1+\delta}\right], \quad N = \left[\frac{\lambda x^\rho}{1-\delta}\right].$$

Then

$$\log|f(-x)| \leqslant \sum_{n=1}^{L} \log\left(\frac{x}{a_n} - 1\right) + \sum_{L+1}^{M} \log\left(\frac{x}{a_n} - 1\right) + \sum_{N+1}^{\infty} \log\left(1 - \frac{x}{a_n}\right),$$

since $$\log\left|1 + \frac{x}{a_n}\right| < 0 \quad (M+1 \leqslant n \leqslant N),$$

if δ is small enough. Now

$$\sum_{n=1}^{L} \log\left(\frac{x}{a_n}-1\right) = O(\log x),$$

$$\sum_{L+1}^{M} \log\left(\frac{x}{a_n}-1\right) < \sum_{L+1}^{M} \log\left\{\left(\frac{\lambda}{(1-\delta)\,n}\right)^{1/\rho} x-1\right\}$$

$$< \int_0^{M+1} \log\left\{\left(\frac{\lambda}{(1-\delta)\,t}\right)^{1/\rho} x-1\right\} dt$$

$$= \frac{\lambda x^\rho}{1-\delta} \int_0^{(1-\delta)(M+1)/\lambda x^\rho} \log\left(u^{-(1/\rho)}-1\right) du$$

$$= \lambda x^\rho \left\{\int_0^1 \log\left(u^{-(1/\rho)}-1\right) du + O\left(\delta \log \frac{1}{\delta}\right)\right\}.$$

Similarly

$$\sum_{N+1}^{\infty} \log\left(1-\frac{x}{a_n}\right) = \lambda x^\rho \left\{\int_1^\infty \log\left(1-u^{-(1/\rho)}\right) du + O\left(\delta \log \frac{1}{\delta}\right)\right\}.$$

Hence

$$\log|f(-x)| < \lambda x^\rho \left\{\int_0^\infty \log|1-u^{-(1/\rho)}|\, du + O\left(\delta \log \frac{1}{\delta}\right)\right\} + O(\log x)$$

$$< (\pi\lambda \cot \pi\rho + \epsilon)\, x^\rho \quad [x > x_0(\epsilon)].$$

(ii) Let $k > 1$, and let p be an integer, depending on x, such that

$$k^p \leqslant x < k^{p+1}.$$

Let $$M = \left[\frac{\lambda k^{p\rho}}{1+\delta}\right], \quad N = \left[\frac{\lambda k^{(p+1)\rho}}{1-\delta}\right],$$

and choose L as before. Then

$$\log|f(-x)| = \sum_{n=1}^{L} \log\left(\frac{x}{a_n}-1\right) + \sum_{L+1}^{M} \log\left(\frac{x}{a_n}-1\right)$$

$$+ \sum_{M+1}^{N} \log\left|1-\frac{x}{a_n}\right| + \sum_{N+1}^{\infty} \log\left(1-\frac{x}{a_n}\right).$$

Here $$\sum_{n=1}^{L} \log\left(\frac{x}{a_n}-1\right) > 0,$$

$$\begin{aligned}\sum_{L+1}^{M} \log\left(\frac{x}{a_n}-1\right) &> \sum_{L+1}^{M} \log\left\{\left(\frac{\lambda}{(1+\delta)\,n}\right)^{1/\rho} x-1\right\} \\ &> \int_L^M \log\left\{\left(\frac{\lambda}{(1+\delta)\,t}\right)^{1/\rho} x-1\right\} dt \\ &> \frac{\lambda x^\rho}{1+\delta} \int_{[(1+\delta)L]/\lambda x^\rho}^{1} \log\,(u^{-(1/\rho)}-1)\,du \\ &= \lambda x^\rho \left\{\int_0^1 \log\,(u^{-(1/\rho)}-1)\,du + O(\delta)\right\} + O(\log x).\end{aligned}$$

Similarly

$$\sum_{N+1}^{\infty} \log\left(1-\frac{x}{a_n}\right) > \lambda x^\rho \left\{\int_1^\infty \log\,(1-u^{-(1/\rho)})\,du + O(\delta)\right\}.$$

Hence

$$\log|f(-x)| > \left(\pi\lambda \cot \pi\rho + O(\delta)\right) x^\rho + O(\log x) + \sum_{M+1}^{N} \log\left|1-\frac{x}{a_n}\right|.$$

It remains to consider the last term on the right. Putting

$$x = a_M + (a_N - a_M)\,\xi,$$

we have

$$\prod_{M+1}^{N} (x-a_n) = (a_N-a_M)^{N-M} \prod_{M+1}^{N} \left(\xi - \frac{a_n-a_M}{a_N-a_M}\right).$$

The product involving ξ is a polynomial of degree $N-M$, whose zeros lie between 0 and 1. Hence*

$$\left|\prod_{M+1}^{N} \left(\xi - \frac{a_n-a_M}{a_N-a_M}\right)\right| > (2e)^{-H(N-M)} \quad (0<\xi<1),$$

except in a set of measure $1/H$. Hence

$$\left|\prod_{M+1}^{N} (x-a_n)\right| > \left(\frac{a_N-a_M}{2^H e^H}\right)^{N-M}$$

for $a_M < x < a_N$, and so *a fortiori* for $k^p \leqslant x < k^{p+1}$, except in a set of measure $(a_N-a_M)/H$. Hence, except in this set,

$$\left|\prod_{M+1}^{N} \left(1-\frac{x}{a_n}\right)\right| > \frac{1}{a_{M+1} \ldots a_N} \left(\frac{a_N-a_M}{2^H e^H}\right)^{N-M}.$$

* See G. Valiron, *loc. cit.*, 79.

Also $$a_{M+1}\dots a_N < \prod_{M+1}^{N}\left\{\frac{n(1+\delta)}{\lambda}\right\}^{1/\rho} < \left\{\frac{N(1+\delta)}{\lambda}\right\}^{(N-M)/\rho}$$

$$< k^{(\rho+1)(N-M)}\left(\frac{1+\delta}{1-\delta}\right)^{(N-M)/\rho},$$

and $$a_N - a_M > k^{\rho-1} - k^{\rho}.$$

Hence, outside the exceptional set,

$$\left|\prod_{M+1}^{N}\left(1-\frac{x}{a_n}\right)\right| > \left(\frac{k-1}{2^H e^H k}\right)^{N-M}\left(\frac{1-\delta}{1+\delta}\right)^{(N-M)/\rho}.$$

Now $$N-M = \lambda\left(\frac{k^{(\rho+1)\rho}}{1-\delta} - \frac{k^{\rho\rho}}{1+\delta}\right) + O(1)$$

$$= \frac{\lambda k^{\rho\rho}(k^{\rho}-1)}{1-\delta} + \frac{2\delta\lambda k^{\rho\rho}}{1-\delta^2} + O(1)$$

$$< Ax^{\rho}(k^{\rho}-1) + A\delta x^{\rho} + O(1).$$

Hence

$$\log\left|\prod_{M+1}^{N}\left(1-\frac{x}{a_n}\right)\right|$$

$$> \{Ax^{\rho}(k^{\rho}-1) + A\delta x^{\rho} + A\}\left\{\log\left(1-\frac{1}{k}\right) - H\log 2e - A\delta\right\}$$

$$> -\epsilon x^{\rho} \quad [x > x_0(\epsilon, H)],$$

the result being obtained by first fixing H, then choosing k sufficiently near to 1, then δ sufficiently small, then x_0 sufficiently large.

The measure of the sub-set of the interval $(k^{\rho}, k^{\rho+1})$ which is exceptional, is less than

$$\frac{a_N - a_M}{H} < \frac{1}{H}\left[\left\{\frac{N(1+\delta)}{\lambda}\right\}^{\rho} - \left\{\frac{M(1-\delta)}{\lambda}\right\}^{\rho}\right]$$

$$< \frac{1}{H}\left\{\left(\frac{1+\delta}{1-\delta}\right)^{\rho} k^{\rho+1} - \left(\frac{1-\delta}{1+\delta}\right)^{\rho} k^{\rho}\right\}$$

$$< \frac{2}{H}(k^{\rho+1} - k^{\rho}) + O\left(\frac{\delta k^{\rho}}{H}\right)$$

$$< \frac{3}{H}(k^{\rho+1} - k^{\rho}),$$

if $\delta/(k-1)$ be small enough. Hence the part of the interval $(0, k^P)$ which is exceptional is of measure less than $3k^P/H$. This proves the theorem.

[*Added January* 30, 1927.—An immediate consequence of this theorem, in the case $\rho = \frac{1}{2}$, together with the results of my paper in *Proc. London Math. Soc.* (2), 25 (1926), 283–302, is that, if $f(t)$ is integrable over (a, b), and for $x > 0$,

$$\left| \int_a^b f(t) \cos xt \, dt \right| < e^{-\epsilon x} \quad (\epsilon > 0)$$

in a set of positive upper density, then $f(t) = 0$ almost everywhere.]

4. We have now to consider the converse of Theorem III. We shall suppose throughout this section that $0 < \rho < \frac{1}{2}$. The converse is not true in this case without some further condition. For suppose that $f(x)$ satisfies the conditions of Theorem III. Let

$$f_1(x) = \frac{\sinh \pi\sqrt{x}}{\pi\sqrt{x}} f(x) = \prod_1^\infty \left(1 + \frac{x}{n^2}\right) \prod_1^\infty \left(1 + \frac{x}{a_n}\right).$$

Then it is easily seen that $f_1(-x)$ satisfies the same asymptotic relations as $f(-x)$, but the corresponding $n_1(x) \sim \sqrt{x}$. If we make an assumption which just excludes this possibility, then the converse is true.

THEOREM IV. *If*

$$\log |f(-x)| < (\pi\lambda \cot \pi\rho + \epsilon) x^\rho \quad (x > x_0)$$

and

$$\log |f(-x)| > (\pi\lambda \cot \pi\rho - \epsilon) x^\rho$$

in a set of density 1, *and if also* $n(x) = o(\sqrt{x})$, *then*

$$n(x) \sim \lambda x^\rho.$$

This theorem will be deduced from the following theorem of Phragmén and Lindelöf* :

Let $F(z)$ *be an analytic function, regular for* $\theta_0 \leqslant \theta \leqslant \theta_1$ $(z = re^{i\theta})$, *and let*

$$|F(z)| = O\{e^{(A \cos \theta\rho + B \sin \theta\rho) r^\rho}\} \tag{4.1}$$

for $\theta = \theta_0$, $\theta = \theta_1$, *where* $\theta_1 - \theta_0 < \pi/\rho$. *If, further*

$$|F(z)| = O(e^{r^\alpha}) \quad \left(\rho < \alpha < \frac{\pi}{\theta_1 - \theta_0}\right),$$

for $\theta_0 \leqslant \theta \leqslant \theta_1$, *then* (4.1) *actually holds uniformly for* $\theta_0 \leqslant \theta \leqslant \theta_1$.

* E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 381–406.

The conclusion still holds if $\alpha = \pi/(\theta_1 - \theta_0)$, *provided that*

$$|F(z)| = O(e^{\epsilon r^\alpha})$$

in the angle, for every positive value of ϵ.

Since $n(x) = o(\sqrt{x})$,

$$\log|f(re^{i\theta})| \leqslant \sum_{n=1}^{\infty} \log\left(1+\frac{r}{a_n}\right) = r\int_0^\infty \frac{n(t)\,dt}{t(r+t)}$$

$$= o\left\{r\int_0^\infty \frac{\sqrt{t}\,dt}{t(r+t)}\right\} = o(\sqrt{r}).$$

We may therefore use the last part of the theorem quoted, with

$$\theta_0 = -\pi, \quad \theta_1 = \pi, \quad \alpha = \tfrac{1}{2}.$$

We have, ϵ being any positive number,

$$\log|f(re^{i\theta})| < \pi(\lambda+\epsilon)\cot\pi\rho \,.\, r^\rho$$

for $\theta = -\pi$, $\theta = \pi$. Hence

$$\log|f(re^{i\theta})| < \pi(\lambda+\epsilon)\frac{\cos\theta\rho}{\sin\pi\rho}\, r^\rho \quad (r > r_0),$$

uniformly for $-\pi \leqslant \theta \leqslant \pi$. Hence, for $r > r_0$,

$$N(r) = \int_0^r \frac{n(x)}{x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,d\theta$$

$$< \tfrac{1}{2}(\lambda+\epsilon)\frac{r^\rho}{\sin\pi\rho}\int_{-\pi}^{\pi}\cos\theta\rho\,d\theta$$

$$= \frac{\lambda+\epsilon}{\rho}\, r^\rho.$$

To obtain a corresponding inequality on the other side, we have to apply the theorem of Phragmén and Lindelöf to the function $F(z) = 1/f(z)$, which is regular, except on the line $\theta = \pi$. In order to do this we require an inequality satisfied by $F(z)$ on two lines very near to, but not coincident with, $\theta = \pi$. Now

$$\left|1-\frac{re^{i\delta}}{a_n}\right| \geqslant \left|1-\frac{x}{a_n}\right|,$$

if $$a_n^2 - 2a_n r\cos\delta + r^2 \geqslant a_n^2 - 2a_n x + x^2,$$

i.e. if $$2a_n(x - r\cos\delta) \geqslant x^2 - r^2,$$

which is true for every a_n if $r\cos\delta \leqslant x \leqslant r$. Hence for such values of r and x,

$$|f(re^{\pm i(\pi-\delta)})| \geqslant |f(-x)|.$$

But, if $r > r_0(\delta)$, there are points x in the interval $(r\cos\delta, r)$ at which

$$|f(-x)| \geqslant e^{\pi(\lambda-\frac{1}{2}\epsilon)x^\rho \cot\pi\rho}.$$

Let x_1 be, for example, the greatest of the limits of the set where this is true. Then

$$\begin{aligned}|f(re^{\pm i(\pi-\delta)})| &\geqslant |f(-x_1)| \\ &\geqslant e^{\pi(\lambda-\frac{1}{2}\epsilon)r^\rho \cot\pi\rho\cos^\rho\delta} \\ &> e^{\pi(\lambda-\epsilon)r^\rho\cot\pi\rho} \quad [\delta < \delta_0(\epsilon),\ r > r_1(\epsilon,\delta)].\end{aligned}$$

Hence
$$|F(re^{i\theta})| < e^{-\pi(\lambda-\epsilon)r^\rho\cot\pi\rho}$$

for $\theta = -\pi+\delta$, $\theta = \pi-\delta$, and, indeed, also throughout the angle between these lines, since $|F(re^{i\theta})|$ is greatest on the bounding lines. Hence

$$|F(re^{i\theta})| = O(e^{-\pi(\lambda-\epsilon)r^\rho\cot\pi\rho\sec(\pi-\delta)\rho\cos\theta\rho})$$

uniformly for $-\pi+\delta \leqslant \theta \leqslant \pi-\delta$. Thus

$$\log|f(z)| > \pi(\lambda-\epsilon)r^\rho\cot\pi\rho\sec(\pi-\delta)\rho\cos\theta\rho$$

in this angle. Hence

$$\frac{1}{2\pi}\int_{-\pi+\delta}^{\pi-\delta}\log|f(re^{i\theta})|\,d\theta > \tfrac{1}{2}(\lambda-\epsilon)\frac{\cot\pi\rho}{\cos(\pi-\delta)\rho}\,r^\rho\int_{-\pi+\delta}^{\pi-\delta}\cos\theta\rho\,d\theta$$

$$= \frac{\lambda-\epsilon}{\rho}\,\frac{\tan(\pi-\delta)\rho}{\tan\pi\rho}\,r^\rho.$$

Also, in a set of density 1,

$$\log|f(-x)| > \pi(\lambda-\epsilon)\cot\pi\rho\,x^\rho,$$

so that
$$\frac{1}{2\pi}\int_{\pi-\delta}^{\pi+\delta}\log|f(re^{i\theta})|\,d\theta > \delta(\lambda-\epsilon)\cot\pi\rho\,r^\rho.$$

Hence, for points of this set,

$$N(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|\,d\theta > \frac{\lambda-\epsilon}{\rho\tan\pi\rho}\{\tan(\pi-\delta)\rho+\delta\rho\}\,r^\rho.$$

Finally, give r any sufficiently large value. Then there is a point r_k of the above set such that

$$r \geqslant r_k > r(1-\epsilon).$$

Hence $$N(r) \geqslant N(r_k)$$

$$> \frac{\lambda-\epsilon}{\rho \tan \pi\rho} \{\tan(\pi-\delta)\rho + \delta\rho\} r_k^\rho$$

$$\geqslant \frac{\lambda-\epsilon}{\rho \tan \pi\rho} \{\tan(\pi-\delta)\rho + \delta\rho\} (1-\epsilon)^\rho r^\rho$$

$$> \frac{\lambda-\eta}{\rho} r^\rho \quad [r > r_0(\eta)].$$

Combining this with the previous result, we have

$$N(r) \sim \frac{\lambda}{\rho} r^\rho.$$

Hence, by the theorem of Landau already referred to,

$$n(r) \sim \lambda r^\rho.$$

5. *The relations between the above theorems.* The theorems which have been proved may be expressed as relations between the eight inequalities :

(A_1) $n(x) < (\lambda+\epsilon) x^\rho$, (A_2) $n(x) > (\lambda-\epsilon) x^\rho$,

(B_1) $N(x) < \frac{\lambda+\epsilon}{\rho} x^\rho$, (B_2) $N(x) > \frac{\lambda-\epsilon}{\rho} x^\rho$,

(C_1) $\log|f(x)| < (\pi\lambda \operatorname{cosec} \pi\rho+\epsilon) x^\rho$, (C_2) $\log|f(x)| > (\pi\lambda \operatorname{cosec} \pi\rho-\epsilon) x^\rho$,

(D_1) $\log|f(-x)| < (\pi\lambda \cot \pi\rho+\epsilon) x^\rho$, (D_2) $\log|f(-x)| > (\pi\lambda \cot \pi\rho-\epsilon) x^\rho$.

Each of these is for $x > x_0(\epsilon)$, and D_2 is in a set of density 1, and D_1 has to be combined with $n(x) = o(\sqrt{x})$. With this restriction, the connexions between the inequalities are as follows :

$$A_1 \to B_1, \qquad A_2 \to B_2,$$

$$B_1+B_2 \to A_1, \quad B_1+B_2 \to A_2,$$

$$A_1 \to C_1, \qquad A_2 \to C_2,$$

$$C_1+C_2 \to A_1, \quad C_1+C_2 \to A_2,$$

$$A_1+A_2 \to D_1, \quad A_1+A_2 \to D_2,$$

$$D_1 \to B_1, \qquad D_2 \to B_2.$$

It also follows from our analysis, not merely that the asymptotic behaviour of any one of $n(x)$, $f(x)$, or $f(-x)$, determines that of the other two, but that any one determines the asymptotic behaviour of $f(re^{i\theta})$ for any value of θ. For example, if $\log|f(x)| \sim \pi\lambda x^\rho \operatorname{cosec}\pi\rho$ then, by Theorem II, $n(x) \sim \lambda x^\rho$. Hence, by Theorem III,

$$\log|f(-x)| \sim \pi\lambda x^\rho \cot\pi\rho$$

in a set of density 1; and hence, applying the theorem of Phragmén and Lindelöf as in the proof of Theorem IV, we obtain

$$\log|f(re^{i\theta})| \sim \pi\lambda r^\rho \operatorname{cosec}\pi\rho \cos\theta\rho$$

for every value of θ, and, indeed, uniformly in any angle not containing $\theta = \pi$.

6. *The case* $\frac{1}{2} \leqslant \rho < 1$. In the case $\rho = \frac{1}{2}$ the converse of Theorem III is not true. For $\cot\pi\rho$ is then zero, so that λ does not appear in D_1 or D_2, so that it is obviously impossible to deduce A_1 or A_2 from D_1 and D_2.

In the case $\frac{1}{2} < \rho < 1$ it remains uncertain whether the converse of Theorem III is true or not. We have, however, the following result:

THEOREM V. *If D_1 and D_2 hold for $\frac{1}{2} < \rho < 1$, then*

$$\underline{\lim}\,\frac{n(x)}{x^\rho} \leqslant \lambda, \quad \overline{\lim}\,\frac{n(x)}{x^\rho} \geqslant \lambda.$$

To prove this we require another of the theorems of Phragmén and Lindelöf*. *Suppose that $F(z)$ is regular, and that*

$$h(\theta) = \overline{\lim}\,\frac{\log|F(re^{i\theta})|}{r^\rho}$$

is finite, in any angle concerned. Then, if $h(\theta)$ has a maximum or minimum for $\theta = \theta'$, it follows that

$$h(\theta) \geqslant h(\theta')\cos\rho(\theta-\theta') \quad \left(\theta' - \frac{\pi}{\rho} < \theta < \theta' + \frac{\pi}{\rho}\right).$$

Suppose that $n(r) < \mu r^\rho$ $(r > r_0)$, where $\mu < \lambda$. Then

$$\log|f(x)| = x\int_0^\infty \frac{n(r)\,dr}{r(x+r)} < \pi(\mu+\epsilon)\operatorname{cosec}\pi\rho \,.\, x^\rho \quad (x > x_0).$$

* *Loc. cit.*, 401–402.

Take the $F(z)$ of the theorem quoted to be $f(z)$. Then

$$h(0) \leqslant \pi\mu \operatorname{cosec} \pi\rho.$$

Also $h(\theta)$ has a maximum at $\theta = 0$. Hence

$$h(\theta) \geqslant h(0) \cos\theta\rho \quad \left(-\frac{\pi}{\rho} < \theta < \frac{\pi}{\rho}\right).$$

In particular, $h(\pi) \geqslant h(0) \cos\pi\rho.$

But, by hypothesis, $h(\pi) = \pi\lambda \cot\pi\rho.$

Hence $h(0) \cos\pi\rho \leqslant \pi\lambda \cot\pi\rho.$

Since $\cos\pi\rho < 0$, this gives

$$h(0) \geqslant \pi\lambda \operatorname{cosec} \pi\rho,$$

which involves a contradiction.

Suppose, on the other hand, that $n(r) > \nu r^\rho$ $(r \geq r_0)$, where $\nu > \lambda$. Then

$$\log f(x) > \pi(\nu - \epsilon) \operatorname{cosec} \pi\rho x^\rho \quad (x > x_0).$$

Take $F(z)$ to be $1/f(z)$. Then $h(\theta)$ has a minimum at $\theta = 0$. Hence, as before,

$$h(\pi - \delta) \geqslant h(0) \cos(\pi - \delta)\rho \quad (\delta > 0).$$

But, as in the proof of Theorem IV, we have

$$h(\pi - \delta) < -\pi(\lambda + \epsilon) \cot\pi\rho$$

if δ be small enough. Hence

$$h(0) \cos(\pi - \delta)\rho < -\pi(\lambda + \epsilon) \cot\pi\rho.$$

Since δ and ϵ are arbitrary, it follows that

$$h(0) \cos\pi\rho \leqslant -\pi\lambda \cot\pi\rho,$$

i.e. $$h(0) \geqslant -\pi\lambda \operatorname{cosec} \pi\rho,$$

i.e. $$\overline{\lim} \frac{-\log f(x)}{x^\rho} \geqslant -\pi\lambda \operatorname{cosec} \pi\rho,$$

i.e. $$\underline{\lim} \frac{\log f(x)}{x^\rho} \leqslant \pi\lambda \operatorname{cosec} \pi\rho.$$

This again gives a contradiction.

THE EXPRESSION OF HYPERGEOMETRIC SERIES IN TERMS OF SIMILAR SERIES

By C. Fox.

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1. Adopting the notation of Barnes*, let us write

$$ {}_pf_q(\alpha_1 \dots \alpha_p;\ \rho_1 \dots \rho_q;\ x) = \sum_{r=0}^{\infty} \frac{\Gamma(\alpha_1+r) \dots \Gamma(\alpha_p+r)}{\Gamma(\rho_1+r) \dots \Gamma(\rho_q+r)} \cdot \frac{x^r}{r!} \qquad (1.1) $$

$$ = \frac{\Gamma(\alpha_1) \dots \Gamma(\alpha_p)}{\Gamma(\rho_1) \dots \Gamma(\rho_q)} \, {}_pF_q(\alpha_1 \dots \alpha_p;\ \rho_1 \dots \rho_q;\ x). \qquad (1.2) $$

For brevity, these functions will sometimes be denoted by

$$ {}_pf_q(\alpha\,;\ \rho\,;\ x) $$

and

$$ {}_pF_q(\alpha\,;\ \rho\,;\ x) $$

respectively, as is done by Barnes.

If $q > p-1$, the series on the right of (1.1) represents an integral function; whilst, if $q = p-1$, the series converges only inside or (with certain exceptions) on the circle $|x| = 1$.

Now

$$ {}_pf_q(\alpha_1 \dots \alpha_p;\ \rho_1-m,\ \rho_2 \dots \rho_q;\ x) $$

$$ = \frac{1}{2\pi i} \int_C \Gamma(-s) \, \frac{\Gamma(\alpha_1+s) \dots \Gamma(\alpha_p+s)}{\Gamma(\rho_1-m+s) \dots \Gamma(\rho_q+s)} \, (-x)^s \, ds, \qquad (1.3) $$

the integral being taken along a contour C, which encloses all the poles of $\Gamma(-s)$, but none of the other poles of the integrand. For the purposes of this paper, the contour C will be taken as a rectangle, except for necessary loops, having the infinite sides parallel to the x axis, and the sides of finite length parallel to the y axis.

* "The asymptotic expansion of integral functions", *Proc. London Math. Soc.* (2), (1906), 59–116.

Now a well known theorem of Gauss* asserts that

$$\frac{\Gamma(\rho_1+s)\,\Gamma(\rho_1-m)}{\Gamma(\rho_1-m+s)\,\Gamma(\rho_1)} = \sum_{r=0}^{\infty} \frac{\Gamma(r-s)\,\Gamma(r-m)\,\Gamma(\rho_1-m)}{\Gamma(-s)\,\Gamma(-m)\,\Gamma(r+\rho_1-m)} \cdot \frac{1}{r!}, \qquad (1.4)$$

and so, combining this with (1 . 3), we get

$${}_pf_q(a_1 \dots a_p;\ \rho_1-m,\ \rho_2 \dots \rho_q;\ x)$$

$$= \frac{1}{2\pi i} \frac{\Gamma(\rho_1)}{\Gamma(\rho_1-m)} \int_C \sum_{r=0}^{\infty} \frac{\Gamma(r-m)\,\Gamma(\rho_1-m)}{\Gamma(-m)\,\Gamma(r+\rho_1-m)}$$

$$\times \frac{\Gamma(r-s)}{r!} \cdot \frac{\Gamma(a_1+s) \dots \Gamma(a_p+s)}{\Gamma(\rho_1+s) \dots \Gamma(\rho_q+s)} (-x)^s\, ds. \qquad (1.5)$$

It is easily seen that

$$\sum_{r=0}^{\infty} \frac{\Gamma(r-m)\,\Gamma(r-s)}{\Gamma(r+\rho_1-m)\,r!}$$

is uniformly convergent with regard to s provided that

$$R(s+\rho_1+1) > 0.$$

Again, from the asymptotic formula

$$\Gamma(a_1+s) \sim \exp\left\{(s+a_1-\tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log 2\pi + O\left(\frac{1}{s}\right)\right\}, \qquad (1.6)$$

when $|\arg s| < \pi$, and the fact that, as $s \to \infty$ along C, $I(s)$ remains finite, we have

$$\frac{\Gamma(a_1+s) \dots \Gamma(a_p+s)}{\Gamma(\rho_1+s) \dots \Gamma(\rho_q+s)}$$

$$\sim \exp\{(p-q-1)\, s \log s - (p-q-1)\, s + s \log|x| + O(\log s)\}. \qquad (1.7)$$

Hence, provided that either

$$p-q-1 < 0$$

or

$$p-q-1 = 0 \quad \text{and} \quad |x| < 1,$$

we may interchange the order of integration and summation in (1 . 5), even though the integration is over a range of values of s extending to $+\infty$ for the real part of s.

---

* Whittaker and Watson, *Modern analysis* (1915), 275.

Hence

$$
\begin{aligned}
&{}_pf_q(a_1 \dots a_p;\ \rho_1-m,\ \rho_2 \dots \rho_q;\ x) \\
&= \frac{1}{2\pi i}\frac{\Gamma(\rho_1)}{\Gamma(\rho_1-m)}\sum_{r=0}^{\infty}\frac{\Gamma(r-m)\,\Gamma(\rho_1-m)}{\Gamma(-m)\,\Gamma(r+\rho_1-m)}\frac{1}{r!} \\
&\qquad \times\int_C \Gamma(r-s)\frac{\Gamma(a_1+s)\dots\Gamma(a_p+s)}{\Gamma(\rho_1+s)\dots\Gamma(\rho_q+s)}(-x)^s\,ds \qquad (1.8) \\
&= \Gamma(\rho_1)\sum_{r=0}^{\infty}\frac{(-x)^r}{r!}\frac{\Gamma(r-m)}{\Gamma(-m)\,\Gamma(r+\rho_1-m)}\,{}_pf_q(a+r;\ \rho+r;\ x). \qquad (1.9)
\end{aligned}
$$

In the same way

$$
\begin{aligned}
&{}_pf_q(a_1+m,\ a_2 \dots a_p;\ \rho_1 \dots \rho_q;\ x) \\
&= \frac{1}{2\pi i}\frac{\Gamma(m+a_1)}{\Gamma(a_1)}\int_C \frac{\Gamma(a_1+m+s)\,\Gamma(a_1)}{\Gamma(a_1+s)\,\Gamma(a_1+m)} \\
&\qquad \times\Gamma(-s)\frac{\Gamma(a_1+s)\dots\Gamma(a_p+s)}{\Gamma(\rho_1+s)\dots\Gamma(\rho_q+s)}(-x)^s\,ds, \qquad (1.10)
\end{aligned}
$$

and so, using (1.4) and changing the order of integration and summation, we have

$$
\begin{aligned}
&{}_pf_q(a_1+m,\ a_2 \dots a_p;\ \rho_1 \dots \rho_q;\ x) \\
&\qquad = \sum_{r=0}^{\infty}\frac{(-x)^r}{r!}\frac{\Gamma(r-m)\,\Gamma(a_1+m)}{\Gamma(-m)\,\Gamma(r+a_1)}\,{}_pf_q(a+r;\ \rho+r;\ x). \qquad (1.11)
\end{aligned}
$$

(1.9) and (1.11) are easily proved by summing the coefficients of the powers on the right-hand sides with the help of (1.4). I do not think, however, that they are at all obvious, and so the method of arriving at them, which is extremely simple, is given; moreover, this method can be used to give many other identities of a similar nature.

2. Equations (1.9) and (1.11) contain many well known theorems as special cases.

When, in (1.9), $p=0$, $q=1$, all the functions involved become Bessel functions, and we have a theorem due to Lommel*.

When $\rho_1 = a_1$, (1.9) and (1.11) give the expansions of ${}_pf_q(a;\ \rho;\ x)$ in terms of functions of the type ${}_{p-1}f_{q-1}(a;\ \rho;\ x)$. As particular instances

* Watson, *Theory of Bessel functions* (1922), 143, § 5.23.

we have Kummer's* formula

$$ {}_1F_1(a;\ \beta;\ x) = e^x\, {}_1F_1(\beta-a;\ \beta;\ -x) \qquad (2.1) $$

and Gauss's formula

$$ {}_2F_1(a,\ \beta;\ \gamma;\ x) = \frac{1}{(1-x)^a}\, {}_2F_1\left(a,\ \gamma-\beta;\ \gamma;\ \frac{x}{x-1}\right). \qquad (2.2) $$

When m is a positive integer, the series on the right of (1.9) and (1.11) terminate, and we have

$$ {}_pf_q(a_1 \dots a_p;\ \rho_1-m,\ \rho_2 \dots \rho_q;\ x) $$

$$ = \Gamma(\rho_1) \sum_{r=0}^{m} \frac{x^r}{\Gamma(r+\rho_1-m)}\, {}_mC_r\, {}_pf_q(a+r;\ \rho+r;\ x) \qquad (2.3) $$

and

$$ {}_pf_q(a_1+m,\ a_2 \dots a_p;\ \rho_1 \dots \rho_q;\ x) $$

$$ = \sum_{r=0}^{m} \frac{\Gamma(a_1+m)}{(\Gamma a_1+r)}\, x^r\, {}_mC_r\, {}_pf_q(a+r;\ \rho+r;\ x). \qquad (2.4) $$

All the formulae so far have been proved true for all values of $|x|$ when $q > p-1$, but only for $|x|<1$ when $q = p-1$. Equations (2.3) and (2.4), however, also hold when $q = p-1$ for the case $|x| = 1$, provided that all the series involved converge. This statement is easily proved with the help of Abel's continuity theorem.

When $p = 2$, $q = 1$, the value $x = 1$ in (2.3) gives Saalschütz's† formula; whilst in (2.4) we simply revert to Gauss's formula, already used in § 1.

Again, Gauss has shown that

$$ {}_2F_1(2a,\ 2\beta;\ a+\beta+\tfrac{1}{2};\ \tfrac{1}{2}) = \frac{\Gamma(a+\beta+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})} \qquad (2.5) $$

and, combining this with (2.3), we get

$$ {}_2F_1(2a,\ 2\beta;\ a+\beta+\tfrac{1}{2}-m;\ \tfrac{1}{2}) $$

$$ = \frac{\Gamma(a+\beta+\frac{1}{2})}{\sqrt{\pi}}\, 2^{2a+2\beta-2} \sum_{r=0}^{m} 2^r\, {}_mC_r\, \frac{\Gamma(a+\beta+\frac{1}{2}-m)\,\Gamma(a+\frac{1}{2}r)\,\Gamma(\beta+\frac{1}{2}r)}{\Gamma(r+a+\beta+\frac{1}{2}-m)\,\Gamma(2a)\,\Gamma(2\beta)}, \qquad (2.6) $$

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* Cf. Whittaker and Watson, *loc. cit.*, 332 (I).

† L. Saalschütz, "Eine Summations formel", *Zeitschrift für Math.*, 35 (1890), 186–188.

whilst, combining (2 . 5) with (2 . 4), we get

$$ {}_2F_1(2\alpha+m,\ 2\beta;\ \alpha+\beta+\tfrac{1}{2};\ \tfrac{1}{2}) = \frac{\Gamma(\alpha+\beta+\frac{1}{2})}{\Gamma(2\beta)}\, 2^{2\beta-1} \sum_{r=0}^{m} {}_mC_r\, \frac{\Gamma(\beta+\frac{1}{2}r)}{\Gamma(\alpha+\frac{1}{2}+\frac{1}{2}r)} \qquad (2.7) $$

where, in both cases, m is a positive integer. On putting $\beta = \alpha+\frac{1}{2}$, in (2 . 7), we get an immediate verification.

When $p = 3$, $q = 2$, and m is a positive integer, we get, from (2 . 3),

$$ {}_3F_2(a_1,\ a_2\ a_3;\ a_1-m,\ \rho_2;\ 1) $$

$$ = \frac{\Gamma(\rho_2)\,\Gamma(\rho_2-a_2-a_3)}{\Gamma(\rho_2-a_2)\,\Gamma(\rho_2-a_3)}\, {}_3F_2\left(\begin{matrix} -m,\ a_2,\ a_3; \\ 1+a_2+a_3-\rho_2,\ a_1-m; \end{matrix}\ 1\right), \qquad (2.8) $$

provided that $R(\rho_2-a_2-a_3-m) > 0$.

The same result, with a_1 replaced by a_1+m, can be obtained from (2 . 4). Equation (2 . 8) is a special case of a formula discovered by Ramanujan* and others.

From (2 . 5) and (2 . 3), when $p = 3$, $q = 2$, $a_1 = \alpha$, $a_2 = 2\beta$, $a_3 = 2\gamma$, $\rho_1 = \alpha$, $\rho_2 = \beta+\gamma+\frac{1}{2}$, we have

$$ {}_3F_2(\alpha,\ 2\beta,\ 2\gamma;\ \alpha-m,\ \beta+\gamma+\tfrac{1}{2};\ \tfrac{1}{2}) $$

$$ = \frac{\Gamma(\alpha-m)\,\Gamma(\beta+\gamma+\frac{1}{2})}{\Gamma(2\beta)\,\Gamma(2\gamma)\,\sqrt{\pi}}\, 2^{2\beta+2\gamma-2} \sum_{r=0}^{m} {}_mC_r\, \frac{\Gamma(\beta+\frac{1}{2}r)\,\Gamma(\gamma+\frac{1}{2}r)}{\Gamma(r+\alpha-m)}\, 2^r, \qquad (2.9) $$

when m is a positive integer.

3. Since $2^{2\alpha-1}\,\Gamma(\alpha)\,\Gamma(\alpha+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2\alpha)$, we have

$$ {}_pf_q(\alpha;\ \rho;\ x) = \frac{\sqrt{\pi}}{\Gamma\{\frac{1}{2}(\rho_1+1)\}} $$

$$ \times \frac{1}{2\pi i} \int_C \left[\frac{\Gamma\{\frac{1}{2}(\rho_1+1)\}\,\Gamma(\frac{1}{2}\rho_1+s)}{\Gamma\{\frac{1}{2}(\rho_1+1)+\frac{1}{2}(s-1)\}\,\Gamma\{\frac{1}{2}(\rho_1+1)+\frac{1}{2}s\}}\, \frac{\Gamma(-s)}{2^{\rho_1+s-1}} \right. $$

$$ \left. \times \frac{\Gamma(a_1+s)\ldots\Gamma(a_p+s)}{\Gamma(\frac{1}{2}\rho_1+s)\,\Gamma(\rho_2+s)\ldots\Gamma(\rho_q+s)}\, (-x)^s \right] ds. \qquad (3.1) $$

* G. H. Hardy, "A chapter from Ramanujan's notebook", *Proc. Camb. Phil. Soc.*, 21 (1923), 498 (5 . 2).

But, from (1 . 4),

$$\frac{\Gamma\{\frac{1}{2}(\rho_1+1)\}\,\Gamma(\frac{1}{2}\rho_1+s)}{\Gamma\{\frac{1}{2}(\rho_1+1)+\frac{1}{2}(s-1)\}\,\Gamma\{\frac{1}{2}(\rho_1+1)+\frac{1}{2}s\}} = \sum_{r=0}^{\infty} \frac{\Gamma(2r-s)\,\Gamma\{\frac{1}{2}(\rho_1+1)\}}{\Gamma(-s)\,\Gamma\{r+\frac{1}{2}(\rho_1+1)\}} \frac{1}{r!}\frac{1}{2^{2r}}. \quad (3.2)$$

Substituting in (3 . 1), and changing the order of integration and summation as before, we have

$${}_pf_q(a;\ \rho;\ x) = \frac{\sqrt{\pi}}{2^{\rho_1-1}} \sum_{r=0}^{\infty} \frac{1}{\Gamma\{r+\frac{1}{2}(\rho_1+1)\}} \frac{x^{2r}}{r!}\frac{1}{2^{4r}} \times {}_pf_q(a_1+2r,\ \ldots,\ a_p+2r;\ \tfrac{1}{2}\rho_1+2r,\ \rho_2+2r,\ \ldots,\ \rho_q+2r;\ \tfrac{1}{2}x). \quad (3.3)$$

When $\rho_1 = 2a$, the functions on the right of (3 . 3) are of lower order than that on the left.

When $\rho_1 = 2a$, $p = q = 1$, (3 . 3) reduces to a formula due to Kummer*

$${}_1F_1(a;\ 2a;\ 2x) = e^x\,{}_0F_1(a+\tfrac{1}{2};\ \tfrac{1}{4}x^2), \quad (3.4)$$

and when $\rho_1 = 2a$, $p = 2$, $q = 1$, it reduces to another formula also due to Kummer†

$${}_2F_1(a,\ \beta;\ 2a;\ x) = (1-\tfrac{1}{2}x)^{-\beta}\,{}_2F_1\left(\tfrac{1}{2}\beta,\ \tfrac{1}{2}(\beta+1);\ a+\tfrac{1}{2};\ \frac{x^2}{(2-x)^2}\right). \quad (3.5)$$

Some further deductions are

$$\sum_{r=0}^{\infty} \frac{(\frac{1}{4}x)^{2r}}{r!\ \Gamma(r+a+\frac{1}{2})} J_{\nu+2r-1}(x) = \frac{(\frac{1}{2}x)^{\nu-1}}{\Gamma(a+\frac{1}{2})\,\Gamma(\nu)}\,{}_1F_2\left(a;\ 2a,\ \nu;\ -\tfrac{1}{2}x^2\right), \quad (3.6)$$

obtained from (3 . 3) by taking $p = 1$, $q = 2$, $\rho_1 = 2a_1$. Also

$$\{J_\nu(x)\}^2 = \left(\frac{x}{2\sqrt{2}}\right)^{\nu} \sum_{r=0}^{\infty} \frac{1}{\Gamma(\nu+1+r)\,r!} \left(\frac{x}{2\sqrt{2}}\right)^{2r} J_{\nu+2r}(x\sqrt{2}). \quad (3.7)$$

obtained by using the formula‡

$$\{J_\nu(x)\}^2 = \frac{1}{\{\Gamma(\nu+1)\}^2} (\tfrac{1}{2}x)^{2\nu}\,{}_1F_2(\nu+\tfrac{1}{2};\ \nu+1,\ 2\nu+1;\ -x^2), \quad (3.8)$$

and expanding the right-hand side by (3 . 3).

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* Whittaker and Watson, *loc. cit.* (1915), 332 (II).

† Kummer, *Journal für Math.*, 15 (1836), 78 (52).

‡ Watson, *loc. cit.*, 147 (5).

4. If we use Dougall's theorem* that

$$\sum_{r=0}^{\infty}\left(1+\frac{2r}{a}\right)\frac{\Gamma(a+r)}{\Gamma(a)}\frac{\Gamma(r-x)}{\Gamma(-x)}\frac{\Gamma(r-y)}{\Gamma(-y)}\frac{\Gamma(r-z)}{\Gamma(-z)}$$

$$\times\frac{\Gamma(1+x+a)}{\Gamma(r+1+x+a)}\frac{\Gamma(1+y+a)}{\Gamma(r+1+y+a)}\frac{\Gamma(1+z+a)}{\Gamma(r+1+z+a)}\frac{1}{r!}$$

$$=\frac{\Gamma(x+y+z+a+1)\,\Gamma(x+a+1)\,\Gamma(y+a+1)\,\Gamma(z+a+1)}{\Gamma(x+z+a+1)\,\Gamma(x+y+a+1)\,\Gamma(y+z+a+1)\,\Gamma(a+1)}, \qquad (4.1)$$

if the series on the left terminates, many identities can be established by the method already used. Thus

$${}_pf_q(a;\ \rho;\ x)=\Gamma(\rho_1+\rho_2-1-a)$$

$$\times\sum_{r=0}^{\infty}\left\{(a+2r)\frac{\Gamma(r+a+1-\rho_1)}{\Gamma(a+1-\rho_1)}\frac{\Gamma(r+a+1-\rho_2)}{\Gamma(a+1-\rho_2)}\frac{\Gamma(a+r)}{\Gamma(r+\rho_1)\,\Gamma(r+\rho_2)}\frac{(-x)^r}{r!}\right.$$

$$\left.\times\,{}_pf_q(a_1+r,\ldots,a_p+r;\ \rho_1+\rho_2-1-a+r,\ 2r+1+a,\ \rho_3+r,\ldots,\rho_q+r;\ x)\right\}, \qquad (4.2)$$

$${}_pf_q(a;\ \rho;\ x)=\frac{1}{2^{\rho_2-2\rho_1}}\sum_{r=0}^{\infty}\left\{(\rho_1-\tfrac{1}{2}+2r)\frac{\Gamma(\rho_1-\frac{1}{2}\rho_2+r)}{\Gamma(\rho_1-\frac{1}{2}\rho_2)}\frac{\Gamma(\rho_1-\frac{1}{2}+r)}{\Gamma\{r+\frac{1}{2}(\rho_2+1)\}}\frac{x^{2r}}{r!}\right.$$

$$\left.\times\,{}_pf_q(a_1+2r,\ldots,a_p+2r;\ 2\rho_1+4r,\ \tfrac{1}{2}\rho_2+2r,\ \rho_3+2r,\ldots,\rho_q+2r;\ x)\right\}, \qquad (4.3)$$

$${}_pf_{q+1}(a,\ a_2\ldots a_p;\ \tfrac{3}{2}a,\ \tfrac{1}{2}(3a+1),\ \rho_2\ldots\rho_q;\ x)$$

$$=3\,\frac{2^{3a-1}}{\sqrt{\pi}}\sum_{r=0}^{\infty}(a+2r)\frac{\Gamma(a+r)}{r!}\,(-\tfrac{4}{3}x)^{3r}$$

$$\times\,{}_{p-1}f_q(a_2+3r,\ldots,a_p+3r;\ 3a+1+6r,\ \rho_2+3r,\ldots,\rho_q+3r;\ \tfrac{4}{3}x). \qquad (4.4)$$

(4.2) is obtained by replacing x in (4.1) by s, and then using the method of § 1. For (4.3) x must be replaced by $\frac{1}{2}s$ and y by $\frac{1}{2}(s-1)$ in (4.1). For (4.4) x must be replaced by $\frac{1}{3}s$, y by $\frac{1}{3}(s-1)$, and z by $\frac{1}{3}(s-2)$ in (4.1).

For certain relations between the a's and the ρ's these equations simplify considerably. In the case when the right-hand sides are Bessel

* J. Dougall, "On Vandermonde's theorem, etc.", *Proc. Edin. Math. Soc.*, 25 (1907), 114–132.

functions they become

$$ {}_1F_2(\rho_1+\rho_2-1-a\,;\ \rho_1,\ \rho_2\,;\ -\tfrac{1}{4}x^2) $$

$$ = \left(\frac{2}{x}\right)^a \sum_{r=0}^{\infty} (a+2r)\,\frac{\Gamma(r+a+1-\rho_1)}{\Gamma(a+1-\rho_1)}\,\frac{\Gamma(r+a+1-\rho_2)}{\Gamma(a+1-\rho_2)} $$

$$ \times \frac{\Gamma(\rho_1)}{\Gamma(r+\rho_1)}\,\frac{\Gamma(\rho_2)}{\Gamma(r+\rho_2)}\,\frac{\Gamma(a+r)}{r!}\,J_{a+2r}(x), \quad (4.5) $$

$$ {}_1F_2(a\,;\ \rho+\tfrac{1}{2},\ 2a\,;\ -\tfrac{1}{4}x^2) = \frac{1}{\sqrt{\pi}}\left(\frac{4}{x}\right)^{2\rho}\Gamma(\rho+\tfrac{1}{2})\sum_{r=0}^{\infty}(\rho+2r)\,\frac{\Gamma(a+\tfrac{1}{2})}{\Gamma(r+a+\tfrac{1}{2})} $$

$$ \times \frac{\Gamma(\rho+\tfrac{1}{2}-a+r)}{\Gamma(\rho+\tfrac{1}{2}-a)}\,\frac{\Gamma(\rho+r)}{r!}\,J_{2\rho+4r}(x), \quad (4.6) $$

and

$$ {}_1F_2\left\{a\,;\ \tfrac{3}{2}a,\ \tfrac{1}{2}(3a+1);\ -\tfrac{3}{16}x^2\right\} $$

$$ = 3\left(\frac{2}{x}\right)^{3a}\Gamma(3a)\sum_{r=0}^{\infty}(a+2r)\,\frac{\Gamma(a+r)}{\Gamma(a)}\,\frac{1}{r!}\,J_{3a+6r}, \quad (4.7) $$

respectively.

The convergence conditions are, for (4.5), $\rho_1+\rho_2<\frac{3}{2}a+\frac{3}{4}$; for (4.6), $\rho<a+\frac{1}{4}$; and for (4.1), $2a<1$.

5. Special cases of these last three equations are interesting. The left-hand side of (4.5) breaks up into products of two Bessel functions in three cases, giving the results*

$$ \{J_\nu(\tfrac{1}{2}x)\}^2 = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sum_{r=0}^{\infty}\left\{(2\nu+\tfrac{1}{2}+2r)\,\frac{\Gamma(r+\frac{1}{2})\,\Gamma(r+\nu+\frac{1}{2})\,\Gamma(r+2\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(r+\nu+1)\,\Gamma(r+2\nu+1)}\right. $$

$$ \left.\times\frac{1}{r!}\,J_{2\nu+\frac{1}{2}+2r}(x)\right\}, \quad (5.1) $$

$$ J_\nu(\tfrac{1}{2}x)\,J_{-\nu}(\tfrac{1}{2}x) = \frac{\cos\nu\pi}{\pi}\left(\frac{2}{x}\right)^{\frac{1}{2}}\left\{\sum_{r=0}^{\infty}(\tfrac{1}{2}+2r)\,\frac{\Gamma(r+\nu+\frac{1}{2})\,\Gamma(r-\nu+\frac{1}{2})}{\Gamma(r+\nu+1)\,\Gamma(r-\nu+1)}\right. $$

$$ \left.\times\frac{\Gamma(r+\frac{1}{2})}{r!}\,J_{\frac{1}{2}+2r}(x)\right\}, \quad (5.2) $$

and

$$ J_\nu(\tfrac{1}{2}x)\,J_{\nu+1}(\tfrac{1}{2}x) = \frac{2\nu+1}{\pi}\left(\frac{1}{2x}\right)^{\frac{1}{2}}\sum_{r=0}^{\infty}\left\{(2\nu+\tfrac{3}{2}+2r)\,\frac{\Gamma(r+\nu+\frac{1}{2})\,\Gamma(r+2\nu+\frac{3}{2})}{\Gamma(r+\nu+2)\,\Gamma(r+2\nu+2)}\right. $$

$$ \left.\times\frac{\Gamma(r+\frac{1}{2})}{r!}\,J_{2\nu+\frac{3}{2}+2r}(x)\right\}. \quad (5.3) $$

* (5.2) can be obtained by combining two formulae in Nielsen, *Handbuch der Theorie der Cylinderfunktionen* (1904), namely Equation (1), 63, with (2), 275.

These expansions, which hold for all values of ν*, are not deducible from the powerful and general expansion of the product of two Bessel functions given by Bateman†.

On multiplying these expansions by Bessel functions of suitable order and integrating, with the help of the theorem

$$\int_0^\infty J_{\alpha+2r}(x)\,J_{\alpha+2s}(x)\,\frac{dx}{x} = 0 \quad (r \neq s), \quad = \frac{1}{2\alpha+4r} \quad (r = s), \qquad (5.4)$$

(provided that r and s are integers), we can express the integral of a product of three Bessel functions as a single term in many cases.

The integral

$$\int_0^\infty \left(\frac{x}{2}\right)^{2\alpha} [{}_1F_2(\rho_1+\rho_2-1-\alpha;\ \rho_1,\ \rho_2;\ -\tfrac{1}{4}x^2)]^2\,\frac{dx}{x}, \qquad (5.5)$$

with the help of (4 . 5) and (5 . 4), can obviously be expressed in terms of an infinite series. In some cases this series can be summed; for example, when

$$\rho_1+\rho_2-1-\alpha = \rho_2-\rho_1, \qquad (5.6)$$

the series can be summed by Dougall's theorem (4 . 1), giving

$$\int_0^\infty [{}_1F_2(\rho_2-\rho_1;\ \rho_1,\ \rho_2;\ -x^2)]^2\,x^{4\rho_1-3}\,dx$$

$$= \tfrac{1}{2}\,\frac{\Gamma(1+2\rho_2-4\rho_1)\,\Gamma(\rho_2)\,\Gamma(\rho_2)\,\Gamma(2\rho_1-1)}{\Gamma(\rho_2-2\rho_1+1)\,\Gamma(\rho_2-2\rho_1+1)\,\Gamma(2\rho_2-2\rho_1)}, \qquad (5.7)$$

provided that $R(4\rho_1) < R(2\rho_2+1)$ and $R(\rho_1) > \frac{1}{2}$.

The justification of the method, however, is somewhat tedious.

Special cases of (4 . 6) are also of interest. On putting $\rho = \alpha = \nu+\frac{1}{2}$, we get Jolliffe's‡ expansion for the square of a Bessel function.

On multiplying together two expansions of the type (4 . 6) and integrating, we get, with the help of (5 . 4),

$$\int_0^\infty (\tfrac{1}{4}x)^{2\rho+2\sigma}\,{}_1F_2(\alpha;\ \rho+\tfrac{1}{2},\ 2\alpha;\ -\tfrac{1}{4}x^2)\,.\,{}_1F_2(\beta;\ \sigma+\tfrac{1}{2},\ 2\beta;\ -\tfrac{1}{4}x^2)\,\frac{dx}{x} = 0, \qquad (5.8)$$

* When infinities occur in the numerators and denominators of any of the terms of (5 . 3), the term must be replaced by a limit.

† Watson, *loc. cit.*, 370 (1).

‡ Watson, *loc. cit.*

when $\rho-\sigma$ is an odd integer positive or negative, but when $\rho=\sigma$ its value is

$$\frac{1}{4\pi}\Gamma(\rho)\left\{\Gamma(\rho+\tfrac{1}{2})\right\}^2\frac{\Gamma(\alpha+\beta-2\rho)\,\Gamma(\alpha+\tfrac{1}{2})\,\Gamma(\beta+\tfrac{1}{2})}{\Gamma(\alpha+\tfrac{1}{2}-\rho)\,\Gamma(\beta+\tfrac{1}{2}-\rho)\,\Gamma(\alpha+\beta-\rho)}, \qquad (5\,.\,9)$$

obtained by summing the resultant series by (4 . 1). When $\rho=\sigma$ the integral converges at infinity provided that $R(\alpha+\beta-2\rho)>0$ and $R(1+\alpha-\rho)>0$, and it converges at the origin provided that $R(\rho)>0$.

The conditions which follow (5 . 7) and (5 . 8) for the convergence of the integrals in these equations can be obtained by finding the order of the hypergeometric functions in the integrands when x is large by a method such as Barnes's* method†.

* Barnes, *loc. cit.*

† I am greatly indebted to the referees for pointing out some numerical errors in my manuscript and also for the remark in the first footnote to § 5.

ON THE TRANSFORMATION OF HAMILTONIAN SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT OR PERIODIC COEFFICIENTS

By T. M. CHERRY

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THE subject of this paper arose in connexion with the solution by successive approximation of a set of differential equations

$$\dot{x}_r = X_r, \quad (r = 1, \ldots n) \tag{1}$$

where the X_r are power series in $x_1, \ldots, x_n$, vanishing with these variables, and having coefficients which are constants or periodic functions of t. To facilitate the finding of the second and higher approximations it is advantageous to make a homogeneous linear transformation whereby we obtain equations of the same form as (1), but in which the linear terms on the right have a simple "normal" form with constant coefficients. When the equations (1) have the Hamiltonian form it is important that this transformation should be a contact transformation, in order that the transformed system also may be Hamiltonian. The object of this paper is therefore to investigate the reduction of linear Hamiltonian systems to a simple form by means of linear contact transformations with constant or periodic coefficients.

It is convenient to treat first the case in which the coefficients in the equations are constants, proceeding later to the case in which they are periodic functions of t.

Linear Hamiltonian systems with constant coefficients.

In (1) let $$X_r = b_{r1}x_1 + b_{r2}x_2 + \ldots + b_{rn}x_n;$$

then the equation

$$\begin{vmatrix} b_{11}-\lambda, & b_{12}, & \ldots \\ b_{21}, & b_{22}-\lambda, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix} = 0$$

is called the *characteristic equation*, and its roots $\lambda_1, \ldots, \lambda_n$ the *exponents*. The system (1) is supposed to be of Hamiltonian form, so that n is necessarily even, $= 2m$, say, and the variables x_r, x_{m+r} are conjugate. It is well known that there is a linear transformation of non-vanishing modulus to new variables $z_1, \ldots, z_n$ which satisfy equations of simpler form; viz. there is a z corresponding to each exponent, and if k exponents have the value λ' the corresponding z's, say $z_1, \ldots, z_k$, satisfy equations of the form

$$\left.\begin{aligned} \dot{z}_1 &= \lambda' z_1 \\ \dot{z}_2 &= a_{21}z_1 + \lambda' z_2 \\ \ldots \quad & \quad \ldots \quad \ldots \quad \ldots \\ \dot{z}_k &= a_{k1}z_1 + a_{k2}z_2 + \ldots + \lambda' z_k \end{aligned}\right\}. \tag{2}$$

we shall say that equations like this are of the *normal* form, and we proceed to show that the transformation from $x_1, \ldots, x_n$ to $z_1, \ldots, z_n$ may always be supposed* to be a contact transformation. For this the conditions will be taken in the form involving Poisson brackets, viz.,

$$\left.\begin{aligned} (z_r, z_{m+r}) &= 1 \quad (r = 1, \ldots m) \\ (z_r, z_s) &= 0 \quad (r, s = 1, \ldots 2m, \quad |r-s| \neq m) \end{aligned}\right\}, \tag{3}$$

where z_{m+r} is the variable conjugate to z_r, and

$$(z_r, z_s) = \sum_{k=1}^{m} \frac{\partial(z_r, z_s)}{\partial(x_k, x_{m+k})}.$$

* There is a *family* of transformations which bring (1) to the normal form, and it will be shown that this family always includes a contact transformation.

1. Let ϕ, ψ be any functions of the x_r and t; then their Poisson bracket (ϕ, ψ) is a function of the x_r and t, and we have

$$\frac{d}{dt}(\phi, \psi) = \frac{\partial}{\partial t}(\phi, \psi) + \Big((\phi, \psi), H\Big),$$

where H is the Hamiltonian function of the system (1)*. But

$$\frac{\partial}{\partial t}(\phi, \psi) = \Big(\frac{\partial \phi}{\partial t}, \psi\Big) + \Big(\phi, \frac{\partial \psi}{\partial t}\Big),$$

from the commutative property of partial differential coefficients, and

$$\Big((\phi, \psi), H\Big) = \Big((\phi, H), \psi\Big) + \Big(\phi, (\psi, H)\Big),$$

from the identity

$$\Big((\phi, \psi), H\Big) + \Big((\psi, H), \phi\Big) + \Big((H, \phi), \psi\Big) = 0.$$

Hence
$$\frac{d}{dt}(\phi, \psi) = \Big(\frac{\partial \phi}{\partial t} + (\phi, H), \psi\Big) + \Big(\phi, \frac{\partial \psi}{\partial t} + (\psi, H)\Big)$$

$$= \Big(\frac{d\phi}{dt}, \psi\Big) + \Big(\phi, \frac{d\psi}{dt}\Big). \tag{4}$$

2. The z's occurring in the equations (2) are linear functions of the x's, so that the Poisson bracket of any pair of them is a constant. Thus

$$\frac{d}{dt}(z_1, z_2) = 0.$$

But, from (4),

$$\frac{d}{dt}(z_1, z_2) = \Big(\frac{dz_1}{dt}, z_2\Big) + \Big(z_1, \frac{dz_2}{dt}\Big)$$

$$= (\lambda' z_1, z_2) + (z_1, a_{21} z_1 + \lambda' z_2) \text{ from (2)},$$

$$= 2\lambda'(z_1, z_2);$$

thus $\lambda'(z_1, z_2) = 0$, and, supposing at present that $\lambda' \neq 0$, we have

$$(z_1, z_2) = 0. \tag{5}$$

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* *i.e.* $X_r = \frac{\partial H}{\partial x_{r+m}}$, $X_{r+m} = -\frac{\partial H}{\partial x_r}$ $(r = 1, \ldots, m)$.

Next $$0 = \frac{d}{dt}(z_1, z_3) = \left(\frac{dz_1}{dt}, z_3\right) + \left(z_1, \frac{dz_3}{dt}\right) \quad \text{from (4)},$$

$$= 2\lambda'(z_1, z_3) + a_{32}(z_1, z_2),$$

i.e. $$0 = 2\lambda'(z_1, z_3) \quad \text{from (5)}.$$

Hence $$(z_1, z_3) = 0.$$

Proceeding thus we may show that the Poisson bracket of any pair of z's belonging to a non-zero exponent λ' is zero.

3. Let $u_1, \ldots, u_p$ be the z's belonging to a p-fold exponent λ'', satisfying the equations

$$\left.\begin{aligned} \dot{u}_1 &= \lambda'' u_1 \\ \dot{u}_2 &= c_{21}u_1 + \lambda'' u_2 \\ \ldots \quad &\ldots \quad \ldots \end{aligned}\right\}. \tag{6}$$

Then $$\frac{d}{dt}(z_1, u_1) = \left(\frac{dz_1}{dt}, u_1\right) + \left(z_1, \frac{du_1}{dt}\right)$$

$$= (\lambda' z_1, u_1) + (z_1, \lambda'' u_1) \quad \text{from (2), (6)}$$

i.e. $$0 = (\lambda' + \lambda'')(z_1, u_1).$$

Hence $$(z_1, u_1) = 0, \tag{7}$$

provided that $\lambda' + \lambda'' \neq 0$. We then have

$$\frac{d}{dt}(z_1, u_2) = \left(\frac{dz_1}{dt}, u_2\right) + \left(z_1, \frac{du_2}{dt}\right)$$

$$= (\lambda' + \lambda_{''})(z_1, u_2) + c_{21}(z_1, u_1),$$

i.e., using (7), $$0 = (\lambda' + \lambda'')(z_1, u_2).$$

Hence $(z_1, u_2) = 0$. Proceeding similarly we may show that, if z_r and z_s belong respectively to exponents λ', λ'' whose sum is not zero, then

$$(z_r, z_s) = 0.$$

Now we cannot have $(z_r, z_s) = 0$ for all values of the subscript s from 1 to n, for from these n linear equations in $\partial z_r/\partial x_1, \ldots, \partial z_r/\partial x_n$ it would

follow that the modulus of the transformation from the x's to the z's was zero. Hence corresponding to each exponent λ' there is another exponent equal to $-\lambda'$.

4. Suppose at present that no exponent is zero, and let the exponents $\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots$ be of respective multiplicity $k_1, k_1', k_2, k_2', \ldots$. We shall see in a moment that $k_r = k_r'$, but, not assuming this, suppose the notation so arranged that $k_r \geqslant k_r'$, implying that $k_1+k_2+\ldots \geqslant m$. Then any m of the z's belonging to $\lambda_1, \lambda_2, \ldots$, say $z_1, \ldots, z_m$, form a set of m linear functions of the x's in involution. There are therefore m other linear functions of the x's, say $u_1, \ldots, u_m$, such that the transformation from the x's to the z's and u's is a contact transformation,* and the latter variables must satisfy equations of the form

$$\dot{z}_r = \frac{\partial K}{\partial u_r}, \quad \dot{u}_r = -\frac{\partial K}{\partial z_r} \quad (r = 1, \ldots, m); \tag{8}$$

and here we know that the first m equations have the normal form. Thus K can have no terms quadratic in the u's, and it is easily seen that if, by a further contact transformation, we can remove from K the terms quadratic in the z's, leaving the remaining terms (bilinear in the z's and u's) unaltered in form, then the second m equations also will have the normal form. To show how this may be done it will be clearest to take a specific example.

5. Take the case in which $n = 8$ and the first set of equations (8) has the form

$$\left.\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 \\ \dot{z}_2 &= a_{21}z_1+\lambda_1 z_2 \\ \dot{z}_3 &= a_{31}z_1+a_{32}z_2+\lambda_1 z_3 \\ \dot{z}_4 &= \lambda_4 z_4 \end{aligned}\right\}. \tag{9}$$

Then K must have the form

$$K \equiv \lambda_1 z_1 u_1+(a_{21}z_1+\lambda_1 z_2)u_2+(a_{31}z_1+a_{32}z_2+\lambda_1 z_3)u_3+\lambda_4 z_4 u_4$$
$$+\text{a homogeneous quadratic function of } z_1, z_2, z_3, z_4,$$

* See *e.g.* Whittaker, *Analytical Dynamics*, §§ 131, 148.

and the second set of equations (8) are

$$\left.\begin{aligned}
\dot{u}_1 &= -\frac{\partial K}{\partial z_1} = -\lambda_1 u_1 - a_{21}u_2 - a_{31}u_3 + b_{11}z_1 + b_{12}z_2 + b_{13}z_3 + b_{14}z_4 \\
\dot{u}_2 &= -\frac{\partial K}{\partial z_2} = \qquad -\lambda_1 u_2 - a_{32}u_3 + b_{12}z_1 + b_{22}z_2 + b_{23}z_3 + b_{24}z_4 \\
\dot{u}_3 &= -\frac{\partial K}{\partial z_3} = \qquad\qquad -\lambda_1 u_3 + b_{13}z_1 + b_{23}z_2 + b_{33}z_3 + b_{34}z_4 \\
\dot{u}_4 &= -\frac{\partial K}{\partial z_4} = -\lambda_4 u_4 \qquad\qquad + b_{14}z_1 + b_{24}z_2 + b_{34}z_3 + b_{44}z_4
\end{aligned}\right\}. \tag{10}$$

The characteristic equation is easily seen to have the roots λ_1, $-\lambda_1$ (each triple), λ_4, $-\lambda_4$, so that, in the notation of § 4, $k_1 = k_1'$, $k_4 = k_4'$.

To remove from (10) the terms in z_1, make a contact transformation of the form

$$\left.\begin{aligned}
z_r' &= z_r \quad (r = 1, \ldots, 4) \\
u_1' &= u_1 + A_1 z_1 + A_2 z_2 + A_3 z_3 + A_4 z_4 \\
u_2' = u_2 + A_2 z_1, &\quad u_3' = u_3 + A_3 z_1, \quad u_4' = u_4 + A_4 z_1
\end{aligned}\right\}. \tag{11}$$

Then u_1' will satisfy the equation

$$\dot{u}_1' = -\lambda_1 u_1' - a_{21}u_2' - a_{31}u_3', \tag{12}$$

provided that

$$\begin{aligned}
0 = {} & \dot{u}_1 + A_1\dot{z}_1 + A_2\dot{z}_2 + A_3\dot{z}_3 + A_4\dot{z}_4 + \lambda_1(u_1 + A_1z_1 + A_2z_2 + A_3z_3 + A_4z_4) \\
& \qquad + a_{21}(u_2 + A_2z_1) + a_{31}(u_3 + A_3z_1) \\
= {} & b_{11}z_1 + b_{12}z_2 + b_{13}z_3 + b_{14}z_4 + A_1\lambda_1 z_1 + A_2(a_{21}z_1 + \lambda_1 z_2) \\
& + A_3(a_{31}z_1 + a_{32}z_2 + \lambda_1 z_3) + A_4\lambda_4 z_4 + \lambda_1(A_1z_1 + A_2z_2 + A_3z_3 + A_4z_4) \\
& + a_{21}A_2z_1 + a_{31}A_3z_1,
\end{aligned}$$

i.e. provided that

$$\left.\begin{aligned}
(\lambda_1 + \lambda_4)A_4 + b_{14} &= 0 \\
2\lambda_1 A_1 + 2a_{21}A_2 + 2a_{31}A_3 + b_{11} &= 0 \\
2\lambda_1 A_2 + a_{32}A_3 + b_{12} &= 0 \\
2\lambda_1 A_3 + b_{13} &= 0
\end{aligned}\right\}. \tag{13}$$

Since $\lambda_1 \neq 0$, $\lambda_1+\lambda_4 \neq 0$, we may choose A_1, A_2, A_3, A_4 so that these equations are satisfied. Since (11) is a contact transformation, the new variables satisfy equations of Hamiltonian form, of which those for $z_1', \ldots, z_4'$ and u_1' have the forms (9), (12). Thus the new Hamiltonian function must be of the form

$$K' \equiv \lambda_1 z_1' u_1' + (a_{21} z_1' + \lambda_1 z_2') u_2' + (a_{31} z_1' + a_{32} z_2' + \lambda_1 z_3') u_3' + \lambda_4 z_4' u_4' + \text{a homogeneous quadratic function of } z_2', z_3', z_4',$$

and the equations are reduced to a form like (9), (10), but in which

$$b_{11} = b_{12} = b_{13} = b_{14} = 0. \tag{14}$$

By a similar contact transformation we may now reduce the equations to a form like (9), (10), in which, as well as (14), we have

$$b_{22} = b_{23} = b_{24} = 0;$$

and so on. Finally all the terms in the z's are removed from (10), and this set of equations is left in normal form.

The method is clearly general, and we have the theorem that, pro vided that no exponent is zero, a linear Hamiltonian system with constant coefficients may be reduced by a linear contact transformation to the normal form.

6. We have supposed hitherto that no exponent is zero. Suppose now that there are k zero exponents; the corresponding z's will satisfy equations of the form

$$\left.\begin{aligned} \dot{z}_1 &= 0 \\ \dot{z}_2 &= a_{21} z_1 \\ \dot{z}_3 &= a_{31} z_1 + a_{32} z_2 \\ \cdots &\quad \cdots \quad \cdots \end{aligned}\right\}. \tag{15}$$

Pursuing the method of §§ 2, 3 we have, if $k > 2$,

$$0 = \frac{d}{dt}(z_1, z_3) = \left(z_1, \frac{dz_3}{dt}\right) = a_{32}(z_1, z_2),$$

so that either $(z_1, z_2) = 0$ or $a_{32} = 0$. In the latter case, if either (z_1, z_2) or (z_1, z_3) is zero, it is merely a matter of notation to suppose that

$(z_1, z_2) = 0$; if neither is zero, say $(z_1, z_2) = p$, $(z_1, z_3) = q$, and we put $z_2' = qz_2 - pz_3$, we have

$$(z_1, z_2') = q(z_1, z_2) - p(z_1, z_3) = 0,$$

and z_2', moreover, satisfies the equation

$$\dot{z}_2' = (qa_{21} - pa_{31})\, z_1,$$

which has the same form as that satisfied by z_2. Thus the set of z's satisfying (15) may always be supposed such that

$$(z_1, z_2) = 0. \tag{16}$$

If $k \leqslant 4$ we need proceed no further; if $k > 4$ the process is continued. We have

$$0 = \frac{d}{dt}(z_2, z_3) = (a_{21}z_1, z_3) + (z_2, a_{31}z_1 + a_{32}z_2) = a_{21}(z_1, z_3) \text{ from (16).}$$

Hence either $(z_1, z_3) = 0$ or $a_{21} = 0$. In the latter case, if either (z_1, z_3) or (z_2, z_3) is zero it is (since $a_{21} = 0$) merely a matter of notation to suppose that $(z_1, z_3) = 0$; if $(z_1, z_3) = p$ and $(z_2, z_3) = q$, where neither p nor q is zero, and we put $z_1' = pz_2 - qz_1$, we have

$$(z_1', z_3) = p(z_2, z_3) - q(z_1, z_3) = 0,$$

$$(z_1', z_2) = -q(z_1, z_2) = 0 \quad \text{from (16).}$$

Thus we may always suppose the set of z's which satisfy (15) to be such that

$$(z_1, z_2) = (z_1, z_3) = 0. \tag{17}$$

We now have

$$0 = \frac{d}{dt}(z_2, z_4) = (a_{21}z_1, z_4) + (z_2, a_{41}z_1 + a_{42}z_2 + a_{43}z_3)$$

$$= a_{21}(z_1, z_4) + a_{43}(z_2, z_3) \quad \text{from (17);} \tag{18}$$

$$0 = \frac{d}{dt}(z_1, z_5) = (z_1, a_{51}z_1 + a_{52}z_2 + a_{53}z_3 + a_{54}z_4)$$

$$= a_{54}(z_1, z_4) \quad \text{from (17).} \tag{19}$$

From (19) we may show, as before, that we may always suppose the z's to be such that $(z_1, z_4) = 0$, and (18) then gives either

$$(z_2, z_3) = 0 \quad \text{or} \quad a_{43} = 0;$$

in the latter case we may choose z_3', a linear combination of z_3 and z_4, such that $(z_2, z_3') = 0$, and we shall also have $(z_1, z_3') = 0$ in virtue of $(z_1, z_3) = (z_1, z_4) = 0$.

If $k \leqslant 6$ we need proceed no further. If $k > 6$ we continue this process, which is clearly general, until we prove that of the quantities $z_1, \ldots, z_k$ appearing in (15), the first half $z_1, \ldots, z_{k/2}$* may always be supposed to be a set of functions of the x's in involution.

7. Let the non-zero exponents be $\lambda_1, -\lambda_1, \lambda_2, -\lambda_2 \ldots$ of respective multiplicity $k_1, k_1', k_2, k_2', \ldots$, where, as in § 4, we may suppose that $k_1 \geqslant k_1'$, $k_2 \geqslant k_2', \ldots$. Choose the first half-set of z's belonging to the zero exponents, which have just been proved to be in involution, and sufficient of those belonging to $\lambda_1, \lambda_2, \ldots$, to make up a total of m; all will actually be required, since, as we shall see, $k_r = k_r'$. These form a set of m linear functions of the x's in involution—say $z_1, \ldots z_m$. Then there is a conjugate set of linear functions $u_1, \ldots, u_m$, such that the transformation from the x's to the z's and u's is a contact transformation.

From this point the argument proceeds as in §§ 4, 5 with one exception: we cannot, by transformation of the u's belonging to the *zero* exponents, obtain for them equations free from the corresponding z's†, for, in the equations analogous to (13), $\lambda_1 = 0$. This, however, does not affect the truth of the theorem, for the equations belonging to the zero exponents already form a set in normal form; for example, if six exponents are zero, the corresponding equations have the form

$$\left.\begin{aligned}
\dot{z}_1 &= 0 \\
\dot{z}_2 &= a_{21} z_1 \\
\dot{z}_3 &= a_{31} z_1 + a_{32} z_2 \\
\dot{u}_3 &= b_{13} z_1 + b_{23} z_2 + b_{33} z_3 \\
\dot{u}_2 &= b_{12} z_1 + b_{22} z_2 + b_{23} z_3 - a_{32} u_3 \\
\dot{u}_1 &= b_{11} z_1 + b_{12} z_2 + b_{13} z_3 - a_{31} u_3 - a_{21} u_2
\end{aligned}\right\} .$$

It is easily seen, as in § 4, that two non-zero exponents whose sum

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* We do not necessarily suppose k to be even at present, though actually this will be shown below. If k is odd we take $z_1, \ldots z_{(k-1)/2}$.

† The question of removing some, but not all, of the z's is from the present point of view unimportant and is not investigated.

is zero occur with the same multiplicity, and hence that the number of zero exponents is always even.

Linear Hamiltonian systems with periodic coefficients.

8. In (1) let $X_r = b_{r1}x_1 + \ldots + b_{rn}x_n$, where the b_{rs} are periodic functions* of t with (real) period T. The characteristic equation is defined as follows. Let

$$\left.\begin{array}{llllll} x_1 = \phi_{11}(t), & x_2 = \phi_{12}(t), & \ldots & x_n = \phi_{1n}(t) \\ x_1 = \phi_{21}(t), & x_2 = \phi_{22}(t), & \ldots & x_n = \phi_{2n}(t) \\ \ldots \quad \ldots & \ldots & \ldots & \ldots \quad \ldots \\ x_1 = \phi_{n1}(t), & x_2 = \phi_{n2}(t), & \ldots & x_n = \phi_{nn}(t) \end{array}\right\} \tag{20}$$

be a set of n solutions of (1), linearly independent. These must remain solutions if we change t into $t+T$, and these new solutions must be linear combinations of (20). There is therefore a set of n^2 constants A_{rs} such that

$$\left.\begin{array}{l} \phi_{1r}(t+T) = A_{11}\phi_{1r}(t) + A_{12}\phi_{2r}(t) + \ldots + A_{1n}\phi_{nr}(t) \\ \phi_{2r}(t+T) = A_{21}\phi_{1r}(t) + A_{22}\phi_{2r}(t) + \ldots + A_{2n}\phi_{nr}(t) \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \phi_{nr}(t+T) = A_{n1}\phi_{1r}(t) + A_{n2}\phi_{2r}(t) + \ldots + A_{nr}\phi_{nr}(t) \end{array}\right\} (r = 1, \ldots n). \tag{21}$$

The characteristic equation is

$$\begin{vmatrix} A_{11}-e^{\lambda T}, & A_{12}, & \ldots & A_{1n} \\ A_{21}, & A_{22}-e^{\lambda T}, & \ldots & A_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n1}, & A_{n2}, & \ldots & A_{nn}-e^{\lambda T} \end{vmatrix} = 0,$$

and its roots λ are called the *exponents*. By their definition the exponents are indeterminate to an integral multiple of $2\pi i/T$; writing $\lambda = \sigma + i\tau$ we make the definition precise by the condition†

$$-\pi/T < \tau \leqslant \pi/T.$$

* All periodic functions which occur are assumed to be expansible in Fourier series.

† It will appear that the exponents for which $\tau = \pi/T$ fall into pairs, and it will be convenient later to assign to those in each pair the values $\tau = \pi/T$, $-\pi/T$ respectively.

If no exponent is equal to $\pi i/T$, it will be shown that the equations may be reduced by a linear contact transformation with *periodic** coefficients to a normal form precisely the same as that already investigated for systems with constant coefficients, and the coefficients in the reduced equations are *constants*. If there are exponents with the value $\pi i/T$ the normal form is similar, except that the coefficients of certain terms have the form

$$(\text{constant}) \times e^{-2\pi i t/T}.$$

9. Let λ' be a k-fold exponent; then† the equations (1) possess k independent solutions of the form

$$\left.\begin{aligned}
&x_r = e^{\lambda' t}\theta_{1r}(t) && (r = 1, \dots n)\\
&x_r = e^{\lambda' t}\{\theta_{2r}(t) + \mu_1 t\theta_{1r}(t)\} && (r = 1, \dots n)\\
&x_r = e^{\lambda' t}[\theta_{3r}(t) + \mu_2 t\theta_{1r}(t) + \mu_3\{t\theta_{2r}(t) + \tfrac{1}{2}\mu_1 t^2\theta_{1r}(t)\}] && (r = 1, \dots n)\\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots
\end{aligned}\right\}, \tag{22}$$

where the θ's are periodic functions of t, and the μ's are definite constants. In the n independent solutions corresponding to the n exponents $\lambda_1, \dots, \lambda_n$ there occur n^2 functions $\theta_{rs}(t)$, those belonging to x_r being say $\theta_{1r}(t), \dots, \theta_{nr}(t)$. We make the linear transformation with periodic coefficients:

$$\left.\begin{aligned}
x_1 &= z_1\theta_{11}(t) + z_2\theta_{21}(t) + \dots + z_n\theta_{n1}(t)\\
x_2 &= z_1\theta_{12}(t) + z_2\theta_{22}(t) + \dots + z_n\theta_{n2}(t)\\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots\\
x_n &= z_1\theta_{1n}(t) + z_2\theta_{2n}(t) + \dots + z_n\theta_{nn}(t)
\end{aligned}\right\}. \tag{23}$$

This is easily seen to be of non-zero modulus, since the solutions (22) are independent.

If we substitute for $x_1, \dots, x_n$ in (23) the general solution of the equations (1), and solve for $z_1, \dots, z_n$, we must obtain the general solution of the equations satisfied by $z_1, \dots, z_n$. We shall find for $z_1, \dots, z_n$ expressions linear in $c_1 e^{\lambda_1 t}, \dots, c_n e^{\lambda_n t}$ (where $c_1, \dots, c_n$ are the constants of integration) and polynomial in t, and it will thus appear that the z's satisfy linear equations with *constant* coefficients. To prove this consider that part of the general solution of (1) which belongs to the exponent

* Here and below the period of a function, when unspecified, is understood to be T.

† Flocquet, *Annales Sci. de l'Ecole Normale* (2), 12 (1883), 47.

λ' occurring in (22). This part is

$$x_r = c_1 e^{\lambda' t}\theta_{1r}(t) + c_2 e^{\lambda' t}\{\theta_{2r}(t) + \mu_1 t\theta_{1r}(t)\}$$
$$+ c_3 e^{\lambda' t}[\theta_{3r}(t) + \mu_2 t\theta_{1r}(t) + \mu_3\{t\theta_{2r}(t) + \tfrac{1}{2}\mu_1 t^2\theta_{1r}(t)\}] + \ldots \quad (r = 1, \ldots, n).$$

The equations (23) therefore become

$$\theta_{1r}(t)(z_1 - c_1 e^{\lambda' t} - c_2\mu_1 t e^{\lambda' t} - c_3\mu_2 t e^{\lambda' t} - \tfrac{1}{2}c_3\mu_1\mu_3 t^2 e^{\lambda' t} + \ldots)$$
$$+ \theta_{2r}(t)(z_2 - c_2 e^{\lambda' t} - c_3\mu_3 t e^{\lambda' t} + \ldots)$$
$$+ \theta_{3r}(t)(z_3 - c_3 e^{\lambda' t} + \ldots) + \ldots = 0 \quad (r = 1, \ldots, n).$$

Since the determinant of the $\theta_{rs}(t)$ is non-zero, we deduce

$$\left.\begin{aligned} z_1 &= c_1 e^{\lambda' t} + c_2\mu_1 t e^{\lambda' t} + c_3\mu_2 t e^{\lambda' t} + \tfrac{1}{2}c_3\mu_1\mu_3 t^2 e^{\lambda' t} + \ldots \\ z_2 &= c_2 e^{\lambda' t} + c_3\mu_3 t e^{\lambda' t} + \ldots \\ z_3 &= c_3 e^{\lambda' t} + \ldots \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}\right\}.$$

Eliminating the arbitrary constants $c_1, \ldots, c_n$ we find

$$\left.\begin{aligned} \dot{z}_1 &= \lambda' z_1 + \mu_1 z_2 + \mu_2 z_3 + \ldots \\ \dot{z}_2 &= \lambda' z_2 + \mu_3 z_3 + \ldots \\ \dot{z}_3 &= \lambda' z_3 + \ldots \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}\right\}.$$

We have thus proved that there is a linear transformation (23), with periodic coefficients, to variables $z_1, \ldots, z_n$ satisfying linear equations with constant coefficients; to a k-fold exponent λ' belong k to the z's, satisfying equations of the form (2). We proceed to consider the nature of this transformation when the equations (1) have the Hamiltonian form, the method of proof following in its main lines that of §§ 1–7.

10. Suppose (2) are the z-equations belonging to a non-zero k-fold exponent λ'. Then, since the identity (4) has been established when ϕ, ψ are any functions x's and t, we have

$$\frac{d}{dt}(z_1, z_2) = \left(\frac{dz_1}{dt}, z_2\right) + \left(z_1, \frac{dz_2}{dt}\right)$$
$$= (\lambda' z_1, z_2) + (z_1, a_{21} z_1 + \lambda' z_2)$$
$$= 2\lambda'(z_1, z_2),$$

so that $(z_1, z_2) = Ae^{2\lambda' t}$, where A is constant. But from (23) the z's are linear functions of the x's with periodic coefficients, so that (z_1, z_2) must be a periodic function of t with period T. If λ' is not equal to $\pi i/T$, these two statements are consistent only if $A = 0$, *i.e.*

$$(z_1, z_2) = 0. \tag{24}$$

We then have

$$\frac{d}{dt}(z_1, z_3) = \left(\frac{dz_1}{dt}, z_3\right) + \left(z_1, \frac{dz_3}{dt}\right)$$

$$= 2\lambda'(z_1, z_3) \text{ on use of (2), (24).}$$

Hence, as before, $(z_1, z_3) = 0$, and we may similarly show that the Poisson bracket of any pair of z's belonging to an exponent λ' is zero, provided $\lambda' \neq 0$, $\lambda' \neq \pi i/T$.

Now let (6) be the z-equations belonging to another exponent λ''. Then

$$\frac{d}{dt}(z_1, u_1) = \left(\frac{dz_1}{dt}, u_1\right) + \left(z_1, \frac{du_1}{dt}\right)$$

$$= (\lambda' + \lambda'')(z_1, u_1) \quad \text{from (2), (6).}$$

Hence $(z_1, u_1) = Ae^{(\lambda'+\lambda'')t}$, where A is constant. Also, as before (z_1, u_1) is a periodic function of t having the period T. These statements are consistent only if either

$$(z_1, u_1) = 0$$

or

$$\lambda' + \lambda'' \equiv 0 \pmod{2\pi i/T}.$$

We may proceed, as in § 3, to show that, if z_r and z_s belong respectively to exponents λ', λ'', whose sum is neither zero nor $2\pi i/T$, then $(z_r, z_s) = 0$. Also, as before, corresponding to each exponent there must be another whose sum with it is zero.

11. Now suppose that there are k zero exponents, and let (15) be the equations satisfied by the corresponding z's. We have

$$\frac{d}{dt}(z_1, z_2) = \left(\frac{dz_1}{dt}, z_2\right) + \left(z_1, \frac{dz_2}{dt}\right)$$

$$= (z_1, a_{21}z_1) = 0;$$

and so

$$(z_1, z_2) = \text{constant}, = A_{12} \text{ say.}$$

Next (if $k > 2$),

$$\frac{d}{dt}(z_1, z_3) = \left(z_1, \frac{dz_3}{dt}\right) = a_{32}(z_1, z_2) = a_{32} A_{12},$$

so that $(z_1, z_3) = a_{32} A_{12}\, t +$ constant. But, since z_1 and z_3 are linear functions of the x's with coefficients periodic in t, (z_1, z_3) must be a periodic function of t. Hence either $A_{12} = 0$, *i.e.* $(z_1, z_2) = 0$, or $a_{32} = 0$, and, as in § 6, we may show that it is always permissible to suppose the z's occurring in (15) to be such that $(z_1, z_2) = 0$.

From the above the modification required in the argument of § 6 is obvious without further exposition, and we conclude that, of the quantities $z_1, \ldots, z_k$ appearing in (15), the first half may always be supposed to be a set of functions of the x's and t in involution.

12. Suppose at present that no exponent has the value $\pi i/T$. Let the non-zero exponents $\lambda_1, -\lambda_1, \ldots$ be of respective multiplicity $k_1, k_1', \ldots$, where it is merely a matter of notation to suppose that $k_1 \geqslant k_1', \ldots$. Choose the first half-set of z's belonging to the zero exponents and sufficient* of those belonging to $\lambda_1, \lambda_2, \ldots$ to make up a total of m functions of the x's and t (say $z_1, \ldots, z_m$) in involution). Then there is a conjugate set of functions $u_1, \ldots, u_m$ linear in the x's and periodic† in t, such that the transformation from the x's to the z's and u's is a contact transformation. The latter variables must therefore satisfy Hamiltonian equations

$$\dot{z}_r = \frac{\partial K}{\partial u_r}, \quad \dot{u}_r = -\frac{\partial K}{\partial z_r} \quad (r = 1, \ldots, m), \tag{25}$$

and here we know that the first set of m equations has the normal form. Thus K is the sum of a bilinear function of the z's and u's with constant coefficients and a quadratic function of the z's whose coefficients can only be constants or periodic functions of t. We remove these quadratic terms by contact transformations similar to those of § 5.

Suppose, for definiteness, that $n = 8$ and that the equations (25) are given explicitly by (9), (10), where now the coefficients b_{rs} are periodic functions of t. Make the contact transformation (11) where the A's are

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* All will actually be required, since it will appear that $k_1 = k_1', \ldots$.

† Since the z's are periodic in t, the conditions (3) require that the derivatives $\partial u_r/\partial x_s$ should be periodic in t; the statement in the text is thence obvious.

periodic functions of t; then u_1' will satisfy the equation

$$\dot{u}_1' = -\lambda_1 u_1' - a_{21} u_2' - a_{31} u_3' \tag{26}$$

provided that

$$\begin{aligned} 0 &= \dot{u}_1 + A_1\dot{z}_1 + A_2\dot{z}_2 + A_3\dot{z}_3 + A_4\dot{z}_4 + \dot{A}_1 z_1 + \dot{A}_2 z_2 + \dot{A}_3 z_3 + \dot{A}_4 z_4 \\ &\quad + \lambda_1(u_1 + A_1 z_1 + A_2 z_2 + A_3 z_3 + A_4 z_4) + a_{21}(u_2 + A_2 z_1) + a_{31}(u_3 + A_3 z_1) \\ &= b_{11} z_1 + b_{12} z_2 + b_{13} z_3 + b_{14} z_4 + A_1\lambda_1 z_1 + A_2(a_{21} z_1 + \lambda_1 z_2) \\ &\quad + A_3(a_{31} z_1 + a_{32} z_2 + \lambda_1 z_3) + A_4\lambda_4 z_4 + \dot{A}_1 z_1 + \dot{A}_2 z_2 + \dot{A}_3 z_3 + \dot{A}_4 z_4 \\ &\quad + \lambda_1(A_1 z_1 + A_2 z_2 + A_3 z_3 + A_4 z_4) + a_{21} A_2 z_1 + a_{31} A_3 z_1, \end{aligned}$$

i.e., provided that

$$\left.\begin{aligned} \dot{A}_4 + (\lambda_1 + \lambda_4) A_4 + b_{14} &= 0 \\ \dot{A}_3 + 2\lambda_1 A_3 + b_{13} &= 0 \\ \dot{A}_2 + 2\lambda_1 A_2 + a_{32} A_3 + b_{12} &= 0 \\ \dot{A}_1 + 2\lambda_1 A_1 + 2a_{21} A_2 + 2a_{31} A_3 + b_{11} &= 0 \end{aligned}\right\}. \tag{27}$$

These equations may be solved in succession for A_4, A_3, A_2, A_1, and will yield solutions which are periodic in t provided that neither $\lambda_1 + \lambda_4$ nor $2\lambda_1$ is zero or a multiple of $2\pi i/T$; by hypothesis the only one of these cases of exception which can occur is $\lambda_1 = 0$. If $\lambda_1 = 0$, we determine $A_1, \ldots, A_4$ so that u_1' may satisfy the equation

$$\dot{u}_1' = -\lambda_1 u_1' - a_{21} u_2' - a_{31} u_3' + c_{11} z_1 + c_{12} z_2 + c_{13} z_3,$$

where c_{11}, c_{12}, c_{13} are constants. The conditions (27) are replaced by

$$\left.\begin{aligned} \dot{A}_4 + \lambda_4 A_4 + b_{14} &= 0 \\ \dot{A}_3 + b_{13} &= c_{13} \\ \dot{A}_2 + a_{32} A_3 + b_{12} &= c_{12} \\ \dot{A}_1 + 2a_{21} A_2 + 2a_{31} A_3 + b_{11} &= c_{11} \end{aligned}\right\},$$

and, if we use a bar to denote the mean value of a periodic function of t, then $A_1, \ldots, A_4$ can be determined as periodic functions of t provided that

$$\left.\begin{aligned} c_{13} &= \bar{b}_{13} \\ c_{12} &= \bar{b}_{12} + a_{32}\bar{A}_3 \\ c_{11} &= \bar{b}_{11} + 2a_{21}\bar{A}_2 + 2a_{31}\bar{A}_3 \end{aligned}\right\}.$$

The argument is general, and we see therefore, as in § 5, that, by means of linear contact transformations with periodic coefficients, the equations (25) may be reduced to the normal form, the coefficients in the reduced equations being constants. Also, as before, it has been proved that two exponents whose sum is zero* occur to the same multiplicity, and that there is an even number of zero exponents.

13. *Case when there are exponents equal to $\pi i/T$.* It is convenient to write $\pi i/T = \nu$. If there is *one* exponent equal to ν, then there must be more than one. For, as in § 10, if z' is the z belonging to ν, and z'' is the z belonging to any other exponent λ, we must have either $(z', z'') = 0$ or $\nu+\lambda \equiv 0 \pmod{2\nu}$, and the second alternative must be satisfied for at least one exponent λ other than ν itself; since, by definition $-\nu < I(\lambda) \leqq \nu$, there must be another exponent equal to ν. Suppose, then, the z's belonging to ν are $z_1, z_2, \ldots$, satisfying the equations

$$\left.\begin{aligned} \dot{z}_1 &= \nu z_1 \\ \dot{z}_2 &= a_{21}z_1+\nu z_2 \\ \dot{z}_3 &= a_{31}z_1+a_{32}z_2+\nu z_3 \\ \cdots &\quad \cdots \quad \cdots \end{aligned}\right\}, \tag{28}$$

where the a_{rs} are constants. Pursuing the method of § 10, we have

$$\frac{d}{dt}(z_1, z_2) = 2\nu(z_1, z_2),$$

$$(z_1, z_2) = A_{12}e^{2\nu t},$$

where A_{12} is constant; this is consistent with the fact that the z's are periodic functions of t with period T, since $2\nu = 2\pi i/T$. We then have

$$\frac{d}{dt}(z_1, z_3) = a_{32}(z_1, z_2)+\nu(z_1, z_3)+\nu(z_1, z_3)$$

$$= a_{32}A_{12}e^{2\nu t}+2\nu(z_1, z_3),$$

and (z_1, z_3) will not be a periodic function of t unless

$$a_{32}A_{12} = 0.$$

* To an exponent with the imaginary part $\pi i/T$ corresponds one with imaginary part $-\pi i/T$. Cp. the footnote to § 8.

We may thence show, as in § 6, that we may always suppose $(z_1, z_2) = 0$. We may proceed by an argument similar to that of § 6 to show that, of the set of z's appearing in (28), the first half may always be supposed to be a set of functions, linear in the x's and periodic in t, in involution.

Let λ_1, $-\lambda_1$, λ_2, $-\lambda_2$, ... be the exponents which are equal neither to zero nor to ν. Choose the first half-set of z's belonging to the zero exponents, the first half-set belonging to the exponents ν, and sufficient of those belonging to λ_1, λ_2, ... to make up a total of m functions in involution, linear in the x's and periodic in t. Calling these functions $z_1, \ldots, z_m$, there is a conjugate set of functions $u_1, \ldots, u_m$ (linear in the x's and periodic in t) such that the transformation from the x's to the z's and u's is a contact transformation, and that the transformed equations have the form (25); the Hamiltonian function K is the sum of a part K_1 bilinear in the z's and u's with constant coefficients, and a part K_2 quadratic in the z's with periodic coefficients. As in § 12, we may make contact transformations of the form (11), the A's being periodic functions of t, so as to remove from K_2 the terms involving the z's belonging to $\lambda_1, \lambda_2, \ldots$; and we may similarly remove from K_2 the terms linear in the z's belonging to the zero exponents, and reduce to constants the coefficients of the terms quadratic in these z's. There will remain in K_2 the terms quadratic in the z's belonging to ν. For instance, if there are six exponents equal to ν and two zero exponents the corresponding equations of the set (25) will have the form

$$\left.\begin{aligned}
\dot{z}_1 &= \nu z_1 & \dot{u}_1 &= -\nu u_1 - a_{21}u_2 - a_{31}u_3 + c_{11}z_1 + c_{12}z_2 + c_{13}z_3 \\
\dot{z}_2 &= a_{21}z_1 + \nu z_2 & \dot{u}_2 &= -\nu u_2 - a_{32}u_3 + c_{12}z_1 + c_{22}z_2 + c_{23}z_3 \\
\dot{z}_3 &= a_{31}z_1 + a_{32}z_2 + \nu z_3 & \dot{u}_3 &= -\nu u_3 + c_{13}z_1 + c_{23}z_2 + c_{33}z_3 \\
\dot{z}_4 &= 0 & \dot{u}_4 &= bz_4
\end{aligned}\right\},$$

where the a's and b are constants and the c's are periodic functions of t. Make the contact transformation (11) with $A_4 = 0$; then u_1' will satisfy the equation

$$\dot{u}_1' = -\nu u_1' - a_{21}u_2' - a_{31}u_3' + e^{-2\nu t}(d_{11}z_1' + d_{12}z_2' + d_{13}z_3'),$$

where d_{11}, d_{12}, d_{13} are constants, provided that

$$\left.\begin{aligned}
\dot{A}_3 + 2\nu A_3 + c_{13} &= d_{13}e^{-2\nu t} \\
\dot{A}_2 + 2\nu A_2 + a_{32}A_3 + c_{12} &= d_{12}e^{-2\nu t} \\
\dot{A}_1 + 2\nu A_1 + 2a_{21}A_2 + 2a_{31}A_3 + c_{11} &= d_{11}e^{-2\nu t}
\end{aligned}\right\}.$$

In order that A_3, A_2, and A_1 may be periodic functions of t it is necessary that, in these equations, no term in $e^{-2\nu t}$ should appear, *i.e.* d_{13} must be equal to the coefficient of this term in c_{13}, d_{12} to its coefficient in $c_{12}+a_{32}A_3$, and d_{11} to its coefficient in $c_{11}+2a_{21}A_2+2a_{31}A_3$. The argument is general, and we see that, by linear contact transformations with periodic coefficients, K is reducible to the sum of a bilinear function of the z's and u's with constant coefficients, a homogeneous quadratic function of the z's belonging to the zero exponents with constant coefficients, and a homogeneous quadratic function of the z's belonging to the exponents ν with coefficients of the form $e^{-2\nu t}$ multiplied by a constant. The equations are thus reduced to the normal form, as stated in § 8. The exponents fall into pairs, the sum of those in any pair being zero, so that, of those whose imaginary part is congruent to ν (mod 2ν), half are to be regarded as having the imaginary part $-\nu$. Cp. the footnote to § 8.

The foregoing theory is valid, with a few obvious modifications of exposition, when the period T is not real.

Examples of the transformation to normal form.

Example I. The Hamiltonian system whose Hamiltonian function is

$$H \equiv xy+\mu zu+ax^3+by^3+cx^2u+dy^2z$$

(where μ, a, b, c, d are constants) has the periodic solution (with imaginary period)

$$\left.\begin{aligned} x &= ae^{\tau}-ba^2e^{-2\tau}+\dots \\ y &= ae^{-\tau}-aa^2e^{2\tau}+\dots \\ z &= \frac{ca^2e^{2\tau}}{2-\mu}+\dots \\ u &= \frac{da^2e^{-2\tau}}{2-\mu}+\dots \end{aligned}\right\},$$

where a is constant and

$$\tau = t\left\{1+2a^2\left(\frac{cd}{2-\mu}-3ab\right)+\dots\right\}.$$

For the variational equations belonging to this periodic solution we obtain

the Hamiltonian set

$$\left.\begin{aligned}
\dot{\xi} &= \xi+\eta\left(6bae^{-\tau}-6aba^2e^{2\tau}+\frac{2cda^2e^{2\tau}}{2-\mu}+\ldots\right)+\zeta(2dae^{-\tau}-2ada^2e^{2\tau}+\ldots)\\
\dot{\eta} &= -\eta-\xi\left(6aae^{\tau}-6aba^2e^{-2\tau}+\frac{2cda^2e^{-2\tau}}{2-\mu}+\ldots\right)\\
&\qquad -\omega(2cae^{\tau}-2bca^2e^{-2\tau}+\ldots)\\
\dot{\zeta} &= \mu\zeta+\xi(2cae^{\tau}-2bca^2e^{-2\tau}+\ldots)\\
\dot{\omega} &= -\mu\omega-\eta(2dae^{-\tau}-2ada^2e^{2\tau}+\ldots)
\end{aligned}\right\}.$$

The linear contact transformation with periodic coefficients, whereby these equations are reduced to the normal form, is

$$\begin{aligned}
\xi &= \frac{p}{\sqrt{2}}\left\{e^{\tau}+2bae^{-2\tau}+2aba^2e^{\tau}-\frac{2cda^2e^{\tau}}{(2-\mu)^2}+\ldots\right\}\\
&\quad+\frac{q}{\sqrt{2}}\left\{e^{\tau}-2bae^{-2\tau}+2aba^2e^{\tau}-\frac{2cda^2e^{\tau}}{(2-\mu)^2}+\ldots\right\}\\
&\quad+r\left\{-\frac{2dae^{-\tau}}{2-\mu}-\frac{2ada^2e^{2\tau}}{1+\mu}+\ldots\right\}+s\left\{\frac{12bca^2}{(2-\mu)(1+\mu)}+\ldots\right\}\\
\eta &= \frac{p}{\sqrt{2}}\left\{-e^{-\tau}-2aae^{2\tau}-2aba^2e^{-\tau}+\frac{2cda^2e^{-\tau}}{(2-\mu)^2}+\ldots\right\}\\
&\quad+\frac{q}{\sqrt{2}}\left\{e^{-\tau}-2aae^{2\tau}+2aba^2e^{-\tau}-\frac{2cda^2e^{-\tau}}{(2-\mu)^2}+\ldots\right\}\\
&\quad+r\left\{\frac{12ada^2}{(2-\mu)(1+\mu)}+\ldots\right\}+s\left\{-\frac{2cae^{\tau}}{2-\mu}-\frac{2bca^2e^{-2\tau}}{1+\mu}+\ldots\right\}\\
\zeta &= \frac{p}{\sqrt{2}}\left\{\frac{2cae^{2\tau}}{2-\mu}-\frac{2bca^2e^{-\tau}}{1+\mu}+\ldots\right\}+\frac{q}{\sqrt{2}}\left\{\frac{2cae^{2\tau}}{2-\mu}+\frac{6bca^2e^{-\tau}}{1+\mu}+\ldots\right\}\\
&\quad+r\left\{1-\frac{2cda^2}{(2-\mu)^2}+\ldots\right\}+sO(a^3)\\
\omega &= \frac{p}{\sqrt{2}}\left\{-\frac{2dae^{-2\tau}}{2-\mu}+\frac{2ada^2e^{\tau}}{1+\mu}+\ldots\right\}+\frac{q}{\sqrt{2}}\left\{\frac{2dae^{-2\tau}}{2-\mu}+\frac{6ada^2e^{\tau}}{1+\mu}+\ldots\right\}\\
&\quad+rO(a^3)+s\left\{1-\frac{2cda^2}{(2-\mu)^2}+\ldots\right\};
\end{aligned}$$

and the new variables p, q, r, s satisfy the Hamiltonian equations

$$\frac{dp}{dt} = \frac{\partial G}{\partial q}, \quad \frac{dq}{dt} = -\frac{\partial G}{\partial p}, \quad \frac{dr}{dt} = \frac{\partial G}{\partial s}, \quad \frac{ds}{dt} = -\frac{\partial G}{\partial r},$$

where

$$G \equiv q^2\left\{2a^2\left(\frac{cd}{2-\mu} - 3ab\right) + \ldots\right\} + rs\left(\mu - \frac{4cda^2}{2-\mu} + \ldots\right).$$

Two exponents are zero and the other two are

$$\pm\left(\mu - \frac{4cda^2}{2-\mu} + \ldots\right).$$

Example II. The Hamiltonian system

$$\left.\begin{aligned} \dot{x} &= \lambda y + (\lambda - \nu)(x \sin 2\nu t + y \cos 2\nu t) \\ \dot{y} &= -\lambda x + (\lambda - \nu)(x \cos 2\nu t - y \sin 2\nu t) \end{aligned}\right\},$$

where λ, ν are constants, will be found to have its exponents congruent to $i\nu$ (mod $2i\nu$). If we apply the contact transformation

$$\left.\begin{aligned} x &= z(1 - \cos 2\nu t + i \sin 2\nu t)/2i - u(1 + \cos 2\nu t + i \sin 2\nu t)/2 \\ y &= z(1 + \cos 2\nu t - i \sin 2\nu t)/2 - u(1 - \cos 2\nu t - i \sin 2\nu t)/2i \end{aligned}\right\}$$

we find for z, u the Hamiltonian equations in the normal form

$$\left.\begin{aligned} \dot{z} &= i\nu z \\ \dot{u} &= -i\nu u + 2(\nu - \lambda) z e^{-2i\nu t} \end{aligned}\right\}.$$

SOME PROPERTIES OF METHODS OF EVALUATION OF DIVERGENT SEQUENCES

By W. A. HURWITZ.

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1. A number of investigations* have been made of those methods of evaluation of divergent† sequences which assign to a sequence $x_1, x_2, \ldots$, as generalized limit the actual limit of the sequence $y_1, y_2, \ldots$, obtained by a linear transformation of the form

$$y_n = a_{n,1}x_1 + a_{n,1}x_2 + \ldots + a_{n,n}x_n ; \tag{A}$$

such a transformation is completely characterized by the matrix $\| a_{n,k} \|$ of the coefficients of (A). If (A) is such that it carries every convergent sequence (x_n) into a sequence (y_n) converging to the same value, then the transformation or definition is said to be *regular*. According to the Silverman-Toeplitz theorem‡, a necessary and sufficient condition that (A) should be regular is that

$$\lim_{n\to\infty} a_{n,k} = 0 \quad (k = 1, 2, \ldots); \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1; \tag{2}$$

$$\sum_{k=1}^{n} |a_{n,k}| \text{ is bounded for all } n. \tag{3}$$

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* For bibliography see Smail's *History and Synopsis of the Theory of Summable Infinite Processes*.

† In the terminology of this paper a sequence is *convergent* if it approaches a finite limit, otherwise *divergent*.

‡ Silverman, *Dissertation* (Missouri), 1910, and *University of Missouri Studies* (1913); Toeplitz, *Prace matematycznofizychne*, 22 (1911), 113.

2. In case the sequences and transformations are real, it is of interest to know in what circumstances the limit-preserving property of a regular transformation is maintained even for the improper limits $+\infty$, $-\infty$; that is, when it will be true that* to every x-sequence for which $x_n \to +\infty$ there corresponds by (A) a y-sequence for which $y_n \to +\infty$. That this need not always hold is clear from such an example as

$$y_n = \frac{2}{n}(x_1+x_2+\ldots+x_n)-x_n,$$

which is obviously regular, but which nevertheless transforms the sequence $x_n = n$, having the limit $+\infty$, into the sequence $y_n = 1$, having the limit 1.

We define a real transformation (A) to be *totally regular* if it is regular and if it transforms every real sequence which becomes positively infinite into a real sequence which becomes positively infinite. We may now state

THEOREM I.—*A necessary and sufficient condition that the regular transformation* (A) *should be totally regular is that*

$$a_{n,k} \geqslant 0 \tag{4}$$

except for at most a finite set of values of k.

We first prove the sufficiency† of the condition. Let (A) be regular, and $a_{n,k} \geqslant 0$ for all sufficiently great values of k. Let $x_1, x_2, \ldots$ be any sequence such that $x_n \to +\infty$. For any $M > 0$ we can find p such that when $n > p$, $x_n > M$, and when $k > p$, $a_{n,k} \geqslant 0$. Then, for $n > p$,

$$y_n = \sum_{k=1}^{n} a_{n,k}x_k$$

$$= M\sum_{k=1}^{n} a_{n,k} + \sum_{k=1}^{p} a_{n,k}(x_k-M) + \sum_{k=p+1}^{n} a_{n,k}(x_k-M).$$

* It evidently suffices to state the condition for $+\infty$ alone, since a change of sign of every x_n produces a change of sign of every y_n.

† The proof has essentially been given by Silverman and, for more general forms of definition, by Smail, James, and Schur.

But, since (A) is regular,

$$M \sum_{k=1}^{n} a_{n,k} \to M, \quad \sum_{k=1}^{p} a_{n,k}(x_k - M) \to 0,$$

and the choice of p insures that

$$\sum_{k=p+1}^{n} a_{n,k}(x_k - M) \geqslant 0.$$

Hence $$\liminf_{n\to\infty} y_n \geqslant M;$$

as M is an arbitrary constant, it follows that

$$\lim_{n\to\infty} y_n = +\infty.$$

In order to prove the necessity of (4), we show that, if (A) satisfies (1), (2), (3), but not (4), we can find a sequence which becomes positively infinite, and such that the transformed sequence does not become positively infinite. Suppose, then, that $a_{n,k} < 0$ for an infinite set of values of k. Choose at random one such negative element

$$a_{n_1,k_1} < 0 \quad (k_1 \leqslant n_1).$$

Define $$x_k = 0 \quad (k \leqslant n_1, \ k \neq k_1),$$

$$x_{k_1} = -\frac{1}{a_{n_1,k_1}}.$$

Since by (1) and (2),

$$\lim_{n\to\infty} \sum_{k=1}^{n_1} a_{n,k}\, x_k = 0, \quad \lim_{n\to\infty} \sum_{k=n_1+1}^{n} a_{n,k} = 1,$$

it follows that for all sufficiently great n

$$\sum_{k=1}^{n_1} a_{n,k}\, x_k < \tfrac{1}{2}, \quad \sum_{k=n_1+1}^{n} a_{n,k} < \tfrac{3}{2};$$

and by our hypothesis there must be among the values of n satisfying these relations a number k_2 such that with some n_2, $a_{n_2,k_2} < 0$. We have therefore

$$\sum_{k=1}^{n_1} a_{n_2,k}\, x_k < \tfrac{1}{2}, \quad \sum_{k=n_1+1}^{n_2} a_{n_2,k} < \tfrac{3}{2}, \quad a_{n_2,k_2} < 0.$$

Define $$x_k = 1, \quad n_1 < k \leqslant n_2 \quad (k \neq k_2),$$

$$x_{k_2} = 1 - \frac{2}{a_{n_2, k_2}}.$$

Similarly it can be shown that there exist n_3, k_3, such that

$$\sum_{k=1}^{n_2} a_{n_3, k} x_k < \tfrac{1}{3}, \qquad \sum_{k=n_2+1}^{n_3} a_{n_3, k} < 1\tfrac{4}{3}, \quad a_{n_3, k_3} < 0.$$

Define $$x_k = 2, \quad n_2 < k \leqslant n_3 \quad (k \neq k_3)$$

$$x_{k_3} = 2 - \frac{3}{a_{n_3, k_3}}.$$

In general we choose n_p, k_p, such that

$$k_1 \leqslant n_1 < k_2 \leqslant n_2 < k_3 \leqslant n_3 < \ldots$$

$$\sum_{k=1}^{n_{p-1}} a_{n_p, k} x_k < \frac{1}{p}, \qquad \sum_{k=n_{p-1}+1}^{n_p} a_{n_p, k} < 1 + \frac{1}{p}, \quad a_{n_p, k_p} < 0,$$

and we define $$x_k = p-1, \quad n_{p-1} < k \leqslant n_p \quad (k \neq k_p),$$

$$x_{k_p} = p - 1 - \frac{p}{a_{n_p, k_p}}.$$

Since $$x_k \geqslant p-1, \quad n_{p-1} < k \leqslant n_p,$$

it follows that $x_n \to +\infty$.

Now let us examine certain elements of the transformed sequence y_n. We have

$$y_{n_1} = a_{n_1, k_1} x_1 = -1 < 0,$$

and in general

$$\begin{aligned} y_{n_p} &= \sum_{k=1}^{n_p} a_{n_p, k} x_k \\ &= \sum_{k=1}^{n_{p-1}} a_{n_p, k} x_k + \sum_{k=n_{p-1}+1}^{n_p} a_{n_p, k} x_k \\ &< \frac{1}{p} + (p-1)\left(1 + \frac{1}{p}\right) + a_{n_p, k_p}\left(-\frac{p}{a_{n_p, k_p}}\right) = 0, \end{aligned}$$

therefore $$y_{n_p} < 0 \quad (p = 1, 2, \ldots),$$

and $$\liminf_{n \to \infty} y_n \leqslant 0.$$

Thus the necessity of the condition is established.

3. In the proof of the sufficiency of (4) it was shown that if (4) holds, then in case (for any fixed M) ultimately $x_n > M$, it will be true that

$$\liminf_{n\to\infty} y_n \geqslant M.$$

Suppose now that (4) holds and that a given sequence (x_n) has a finite inferior limit,

$$\liminf_{n\to\infty} x_n = l;$$

then, however small a positive number ϵ may be, ultimately $x_n > l-\epsilon$; hence

$$\liminf_{n\to\infty} y_n \geqslant l-\epsilon;$$

and, since ϵ is arbitrary,

$$\liminf_{n\to\infty} y_n \geqslant l.$$

Therefore, if (4) is true,

$$\liminf_{n\to\infty} y_n \geqslant \liminf_{n\to\infty} x_n,$$

whenever the right-hand limit is finite.* It will now be shown that the converse of this result is true, so that we may state

THEOREM II′.—*In order that a regular transformation* (A) *may be such that*

$$\liminf_{n\to\infty} y_n \geqslant \liminf_{n\to\infty} x_n,$$

whenever the right-hand limit is finite, (4) *is necessary and sufficient.*

The sufficiency has been proved; to prove the necessity we assume (4) to be untrue, and construct a sequence for which

$$\liminf_{n\to\infty} y_n < \liminf_{n\to\infty} x_n.$$

First choose n_1, k_1, so that

$$a_{n_1, k_1} < 0.$$

Then take n_2, k_2, so that

$$a_{n_2, k_1} < -\tfrac{1}{2} a_{n_1, k_1}, \quad a_{n_2, k_2} < 0,$$

* Obviously also if it is $+\infty$ or $-\infty$; these cases are intentionally omitted here and relegated to Theorem III′.

n_3, k_3, so that

$$a_{n_3, k_1} < -\tfrac{1}{4} a_{n_1, k_1}, \quad a_{n_3, k_2} < -\tfrac{1}{4} a_{n_2, k_2}, \quad a_{n_3, k_3} < 0,$$

and in general n_p, k_p, so that

$$a_{n_p, k_q} < -\frac{1}{2(p-1)} a_{n_q, k_q} \quad (q = 1, 2, \ldots, p-1);$$

$$a_{n_p, k_p} < 0.$$

Such choice is always possible by (1) and the denial of (4). Now define

$$x_k = 0 \quad (k \neq k_1, k_2, k_3, \ldots),$$

$$x_{k_p} = -\frac{1}{a_{n_p, k_p}}.$$

Then $$\liminf_{n \to \infty} x_n = 0.$$

But

$$y_{n_p} = \sum_{k=1}^{n_p} a_{n_p, k} x_k$$

$$= \sum_{q=1}^{p} a_{n_p, k_q} x_{k_q}$$

$$= -1 + \sum_{q=1}^{p-1} a_{n_p, k_q} x_{k_q}$$

$$< -1 + (p-1)\left[\frac{1}{2(p-1)}\right] = -\tfrac{1}{2},$$

and our assertion is proved.

We have similarly

THEOREM II″.—*In order that a regular transformation* (A) *may be such that*

$$\limsup_{n \to \infty} y_n \leqslant \limsup_{n \to \infty} x_n,$$

whenever the right-hand limit is finite, (4) *is necessary and sufficient.*

The following two theorems are obvious:

THEOREM III′.—*In order that a regular transformation* (A) *may be such that*

$$\liminf_{n \to \infty} y_n \geqslant \liminf_{n \to \infty} y_n,$$

when the right-hand limit is $+\infty$, (4) *is necessary and sufficient; when it is* $-\infty$, *no condition beyond regularity is required.*

THEOREM III″.—*In order that a regular transformation* (A) *may be such that*

$$\limsup_{n\to\infty} y_n \leqslant \limsup_{n\to\infty} x_n,$$

when the right-hand limit is $-\infty$, (4) *is necessary and sufficient; when it is* $+\infty$, *no condition beyond regularity is required.*

4. If, however, we wish to demand that *both* $\limsup y_n \leqslant \limsup x_n$ *and* $\liminf y_n \geqslant \liminf x_n$, we find in some cases a different condition, the character of the result depending on the finiteness or non-finiteness of the x-limits.

THEOREM IV.—*In order that a regular transformation* (A) *may be such that*

$$\limsup_{n\to\infty} y_n \leqslant \limsup_{n\to\infty} x_n$$

and

$$\liminf_{n\to\infty} y_n \geqslant \liminf_{n\to\infty} x_n,$$

whenever the x-limits are finite, it is necessary and sufficient that

$$\lim_{n\to\infty} \sum_{k=1}^{n} |a_{n,k}| = 1. \tag{5}$$

Put $b_{n,k} = \frac{1}{2}[|a_{n,k}|+a_{n,k}]$, $c_{n,k} = \frac{1}{2}[|a_{n,k}|-a_{n,k}]$,

so that $a_{n,k} = b_{n,k}-c_{n,k}$, $|a_{n,k}| = b_{n,k}+c_{n,k}$.

It follows from (1) that, for each k,

$$\lim_{n\to\infty} b_{n,k} = 0, \quad \lim_{n\to\infty} c_{n,k} = 0;$$

and from (2) and (5) that

$$\lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k} = 1, \quad \lim_{n\to\infty} \sum_{k=1}^{n} c_{n,k} = 0.$$

To prove the sufficiency of (5), write

$$\limsup_{n\to\infty} x_n = L,$$

and suppose* that for all n, $|x_n| < X$. For a given positive ϵ choose p so that when $k > p$, then $x_k \leqslant L+\epsilon$. For $n > p$,

$$y_n = \sum_{k=1}^{n} (b_{n,k} - c_{n,k})\, x_k$$

$$= (L+\epsilon) \sum_{k=1}^{n} b_{n,k} - \sum_{k=1}^{n} c_{n,k}\, x_k$$

$$+ \sum_{k=1}^{p} b_{n,k}(x_k - L - \epsilon) + \sum_{k=p+1}^{n} b_{n,k}(x_k - L - \epsilon).$$

The last term of the four just written is necessarily less than or equal to zero; increasing the second and third terms, we have

$$y_n \leqslant (L+\epsilon) \sum_{k=1}^{n} b_{n,k} + X \sum_{k=1}^{n} c_{n,k} + (X + |L| + \epsilon) \sum_{k=1}^{p} b_{n,k}.$$

Since the right-hand side has the limit $L+\epsilon$ as $n \to \infty$,

$$\limsup_{n \to \infty} y_n \leqslant L+\epsilon;$$

and since ϵ is an arbitrary positive number

$$\limsup_{n \to \infty} y_n \leqslant L.$$

A similar proof holds for the inferior limit.

To prove the necessity of the condition, we suppose that for some $\lambda > 0$, repeatedly

$$\sum_{k=1}^{n} |a_{n,k}| > 1 + 3\lambda,$$

and show that a sequence exists for which the statement of the theorem is violated. Choose n_1 so that

$$\sum_{k=1}^{n_1} |a_{n_1,k}| > 1 + 3\lambda,$$

and n_2 so that

$$\sum_{k=1}^{n_1} |a_{n_2,k}| < \lambda, \quad \sum_{k=1}^{n_2} |a_{n_2,k}| > 1 + 3\lambda,$$

* This supposition makes use of the finiteness of both superior and inferior limits.

and, generally, n_p so that

$$\sum_{k=1}^{n_{p-1}} |a_{n_p,k}| < \lambda, \quad \sum_{k=1}^{n_p} |a_{n_p,k}| > 1+3\lambda.$$

Define* $$x_k = (-1)^p \operatorname{sgn} a_{n_p,k} \quad (n_{p-1} < k \leqslant n_p).$$

Then† $$\limsup_{n\to\infty} x_n = 1, \quad \liminf_{n\to\infty} x_n = -1.$$

But $$y_{n_p} = \sum_{k=1}^{n_p} a_{n_p,k} x_k$$

$$= \sum_{k=1}^{n_{p-1}} a_{n_p,k} x_k + (-1)^p \sum_{k=n_{p-1}+1}^{n_p} |a_{n_p,k}|$$

$$= \sum_{k=1}^{n_{p-1}} a_{n_p,k} x_k - (-1)^p \sum_{k=1}^{n_{p-1}} |a_{n_p,k}| + (-1)^p \sum_{k=1}^{n_p} |a_{n_p,k}|;$$

thus $$(-1)^p y_{n_p} = (-1)^p \sum_{k=1}^{n_{p-1}} a_{n_p,k} x_k - \sum_{k=1}^{n_{p-1}} |a_{n_p,k}| + \sum_{k=1}^{n_p} |a_{n_p,k}|$$

$$> -2 \sum_{k=1}^{n_{p-1}} |a_{n_p,k}| + \sum_{k=1}^{n_p} |a_{n_p,k}|$$

$$> -2\lambda + (1+3\lambda) = 1+\lambda.$$

Therefore, for p even, $$y_{n_p} > 1+\lambda,$$

and, for p odd, $$y_{n_p} < -1-\lambda;$$

so that $$\limsup_{n\to\infty} y_n > 1, \quad \liminf_{n\to\infty} y_n < -1.$$

THEOREM V′.—*In order that a regular transformation* (A) *may be such that*

$$\liminf_{n\to\infty} y_n \geqslant \liminf_{n\to\infty} x_n,$$

whenever the right-hand limit is finite and $\limsup x_n = +\infty$, (4) *is necessary and sufficient.*

The sufficiency of (4) is merely a special case of Theorem II′. To prove the necessity, we suppose $a_{n,k} < 0$ for certain arbitrarily great k,

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* To secure complete generality, we assume $n_0 = 0$.

† There will be an infinite set of positive values and an infinite set of negative values, by (2) and the denial of (5).

and proceed to construct an appropriate sequence; we shall, in fact, make

$$\limsup x_n = +\infty, \quad \liminf x_n = 0, \quad \limsup y_n = +\infty, \quad \liminf y_n = -\infty.$$

Choose n_1, k_1 so that $a_{n_1, k_1} < 0$;

define $$x_k = 0, \quad k \leqslant n_1, \quad k \neq k_1;$$

$$x_{k_1} = 1 + \frac{1}{|a_{n_1, k_1}|}.$$

Generally, choose n_p, k_p, so that

$$\operatorname{sgn} a_{n_p, k_p} = (-1)^p, \quad \left| \sum_{k=1}^{n_{p-1}} a_{n_p, k} x_k \right| < 1;$$

and define $$x_k = 0, \quad n_{p-1} < k \leqslant n_p, \quad k \neq k_p;$$

$$x_{k_p} = p + \frac{p}{|a_{n_p, k_p}|}.$$

Obviously $$\liminf_{n \to \infty} x_n = 0, \quad \limsup_{n \to \infty} x_n = +\infty.$$

But
$$y_{n_p} = \sum_{k=1}^{n_{p-1}} a_{n_p, k} x_k + \sum_{k=n_{p-1}+1}^{n_p} a_{n_p, k} x_k$$

$$= \sum_{k=1}^{n_{p-1}} a_{n_p, k} x_k + a_{n_p, k_p} \left[\frac{p}{|a_{n_p, k_p}|} \right];$$

$$(-)^p y_{n_p} = (-1)^p \sum_{k=1}^{n_{p-1}} a_{n_p, k} x_k + p |a_{n_p, k_p}| + p$$

$$> -1 + 0 + p = p - 1.$$

Thus $$\liminf_{n \to \infty} y_n = -\infty, \quad \limsup_{n \to \infty} y_n = +\infty.$$

Similarly we have

THEOREM V″.—*In order that a regular transformation* (A) *may be such that*

$$\limsup_{n \to \infty} y_n \leqslant \limsup_{n \to \infty} x_n,$$

whenever the right-hand limit is finite and $\liminf x_n = -\infty$, (4) *is necessary and sufficient.*

5. In the cases not covered by the preceding theorems the conditions are easily seen to be trivial. We proceed to tabulate all the results, including these trivial ones. Each item in the following table gives necessary and sufficient conditions that the superior limit L be not increased or the inferior limit l not decreased or both when the transformation (A) is applied to *all* sequences of the type indicated :—

(A) *satisfies* (1), (2), (3):

(1)	L *finite*:	(4)
(2)	$L = +\infty$:	*nothing additional*
(3)	l *finite*:	(4)
(4)	$l = -\infty$:	*nothing additional*
(5)	$L = l$ *finite*:	*nothing additional*
(6)	$L = l = +\infty$:	(4)
(7)	$L = l = -\infty$:	(4)
(8)	$L > l$, *both finite*:	(5)
(9)	$L = +\infty$, l *finite*:	(4)
(10)	L *finite*, $l = -\infty$:	(4)
(11)	$L = +\infty$, $l = -\infty$:	*nothing additional*

6. Especial interest attaches to those transformations (A) which have the algebraic property of permutability with the arithmetic mean transformation

$$(M) \qquad y_n = \frac{1}{n}(x_1 + x_2 + \ldots + x_n),$$

for the reasons (among others) that all the Hölder and Cesàro means are such transformations, and that all such transformations give definitions of summability which are mutually consistent. Transformations permutable with (M) were first studied in a joint paper by Silverman and the writer*; it was shown that any such transformation must have the

* *Trans. Amer. Math. Soc.*, 18 (1917), 1–20.

form $$a_{n,k} = \sum_{h=k}^{n} (-1)^{h-k} \frac{(n-1)!}{(n-h)!\,(h-k)!\,(k-1)!} f\left(\frac{1}{h}\right),$$

and that the transformation is regular provided that $f(z)$ is analytic for $|z-\frac{1}{2}| \leqslant \frac{1}{2}$ and $f(1) = 1$.

Starting independently from a similar point of view, Hausdorff* has given an elegant treatment of these transformations, and he has enunciated a very simple form of necessary and sufficient condition for regularity; this condition is, of course, more general than the sufficient condition previously stated†.

Hausdorff shows that, for a transformation permutable with M to carry convergent sequences into convergent sequences (irrespective of relationship between the limits), (3) alone is necessary and sufficient, and that, for such transformations, (3) is equivalent to the condition‡

$$\lim_{n\to\infty} \sum_{k=1}^{n} |a_{n,k}| \text{ exists};$$

we shall call this limit **A**. Another form of necessary and sufficient condition is that there exists a function $A(t)$ of bounded variation, $0 \leqslant t \leqslant 1$, such that*

$$a_{n,k} = \frac{(n-1)!}{(n-k)!\,(k-1)!} \int_0^1 t^{k-1}(1-t)^{n-k}\, dA(t),$$

the integral being a Stieltjes integral. It will be true that every $a_{n,k} \geqslant 0$ if and only if $A(t)$ is monotonically increasing.

For regularity (*i.e.* the equality of old and new limits) the adjunction of condition (1) for the single value $k = 1$ and of condition (2) is necessary and sufficient. Since, for transformations permutable with (M), $\Sigma a_{n,k}$ is independent of n, an alternative form of (2) is

$$\sum_{k=1}^{n} a_{n,k} = 1,$$

or even $$a_{1,1} = 1.$$

In terms of the function $A(t)$ the condition for regularity is that $A(t)$ be continuous at $t = 0$ and that $A(1)-A(0) = 1$.

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* *Math. Zeitschrift*, 9 (1921), 74–109.

† From Hausdorff's condition it may be deduced that $f(z)$ is analytic for $|z-\frac{1}{2}| < \frac{1}{2}$, and so the greater generality permits only singularities on the boundary of this circle.

‡ The notation of the present paper differs from that of Hausdorff in numbering the subscripts from 1 instead of from 0.

Hausdorff shows further that any transformation (A) permutable with (M) and satisfying (3) can be expressed as the difference of two transformations (P), (Q),

$$a_{n,k} = p_{n,k} - q_{n,k},$$

each permutable with (M) and satisfying (3), and, furthermore, such that

$$p_{n,k} \geqslant 0, \quad q_{n,k} \geqslant 0, \quad \sum_{k=1}^{n} (p_{n,k} + q_{n,k}) = \mathbf{A}.$$

By means of these results we can easily examine the effect of conditions (4) and (5) on regular transformations permutable with (M). For such a transformation to satisfy (5) we must have $\mathbf{A} = 1$, so that

$$\sum_{k=1}^{n} (p_{n,k} + q_{n,k}) = \sum_{k=1}^{n} (p_{n,k} - q_{n,k}),$$

and hence

$$\sum_{k=1}^{n} q_{n,k} = 0;$$

but since $q_{n,k} \geqslant 0$, this implies $q_{n,k} = 0$, $a_{n,k} = p_{n,k}$ and therefore $a_{n,k} \geqslant 0$. Thus even the stronger condition (4) must hold, and, in fact, not merely for an infinite set of values of k, but for all n, k. We have

THEOREM VI.—*A regular transformation* (A) *permutable with* (M) *can satisfy* (4) *or* (5) *if and only if* $a_{n,k} \geqslant 0$ *for all values of* n *and* k. *The corresponding function* $A(t)$ *of Hausdorff is then monotonically increasing.*

In the special case given by Silverman and the writer, let $f(z)$, which is analytic at $z = 0$, be given by

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots,$$

and define the associated function

$$F(z) = a_1 + a_2 z + a_3 \frac{z^2}{2!} + a_4 \frac{z^3}{3!} + \ldots;$$

$F(z)$ will be an integral function. These two functions are connected by the formulae

$$F(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t^2} e^{z/t}\, dt,$$

where C is a contour surrounding $t = 0$;

$$f(z) = a_0 + \int_0^\infty e^{-u/z} F(u)\, du \quad (0 < z \leqslant 1).$$

Hausdorff's $A(t)$ is then given by the formulae

$$A(0) = 0,$$

$$A(t) = \int_0^t F(-\log u)\, du \quad (0 < t < 1),$$

$$A(1) = f(0) + \int_0^1 F(-\log u)\, du = 1.$$

For $A(t)$ to be monotonically increasing, it is easily seen to be necessary and sufficient that $f(0) \geqslant 0$ and $F(x) \geqslant 0$ when $x \geqslant 0$.

THEOREM VII.—*An analytically regular* transformation* (A) *permutable with* (M) *can satisfy* (4) *or* (5) *if and only if the generating function* $f(z)$ *and its associated function* $F(z)$ *satisfy the conditions*

$$f(0) \geqslant 0, \quad F(x) \geqslant 0 \quad \textit{when} \quad x \geqslant 0.$$

7. The general criteria of Theorems I–V and the special ones of Theorem VI, VII may readily be applied to common types of definition. For the Hölder mean (M^r) of order r, and for the Cesàro mean (C_r) of order r, obviously $a_{n,k} > 0$, so that (4) holds. For the transformation† $M^r C_r^{-1}$

$$f(z) = \frac{\{1+z\}\{1+2z\}\ldots\{1+(r-1)z\}}{r!};$$

as $f(z)$ is a polynomial with positive coefficients, $F(z)$ is a polynomial with positive coefficients, and by Theorem VII, (4) is satisfied. But* for $C_r M^{-r}$,

$$f(z) = \frac{r!}{\{1+z\}\{1+2z\}\ldots\{1+(r-1)z\}},$$

* Hurwitz and Silverman, *loc. cit.*, 13.

† Hurwitz and Silverman, *loc. cit.*, 7 (Lemma 5) and 10 (Corollary).

‡ *Ibid.*

which is clearly (when $r>1$) a decreasing function for positive real z; thus

$$a_{n,n-1}=(n-1)\left[f\left(\frac{1}{n-1}\right)-f\left(\frac{1}{n}\right)\right]<0.$$

Therefore (4) fails, and, by Theorem VI, (5) also fails. While C_r and M^r are equivalent for finite limits, they are not, for $r>1$, totally equivalent; we may say that M^r *totally includes* C_r, but C_r includes M^r only non-totally*—there can exist sequences evaluated by M^r to $+\infty$ and not evaluated by C_r to $+\infty$. Also the limits of indetermination obtained on transforming a sequence by M^r may be spread further apart on transforming by C_r.

Again†, for $C_{r+h}C_r^{-1}$, where r and h need not be integers,

$$A'(t)=\frac{\Gamma(r+h+1)}{\Gamma(h)\,\Gamma(r+1)}\,t^r(1-t)^{h-1}\quad(0<t<1);$$

evidently $A(t)$ is an increasing function if $r>-1$, $h>0$; by Theorem VI, (4) holds. Thus for $r>-1$, $h>0$, C_{r+h} totally includes C_r; and the oscillation of a sequence, when transformed by C_r, does not become more violent when the sequence is transformed by C_{r+h}.

An example of a transformation (A) not permutable with (M) is the mean of de la Vallée-Poussin‡:

$$y_n=u_0+\frac{n}{n+1}\,u_1+\frac{n(n-1)}{(n+1)(n+2)}\,u_2+\ldots+\frac{n(n-1)\ldots1}{(n+1)(n+2)\ldots(2n)}\,u_n,$$

where the original series is

$$u_0+u_1+u_2+\ldots.$$

Since the coefficient of u_k decreases as k increases, it is clear that, when y_n is written in the form

$$y_n=a_{n,0}\,x_0+a_{n,1}\,x_1+\ldots+a_{n,n}\,x_n,$$

we have $a_{n,k}>0$ always. Thus the definition is totally regular.

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* This fact has been noted by I. Schur, *Math. Annalen*, 74 (1913), 453.

† Hausdorff, *loc. cit.*, 87.

‡ *Bulletin de la classe des sciences de l'Académie Royale de Belgique* (1908), 233.

8. A type of transformation more general than (A) is frequently useful for summing divergent series and sequences. Let* a point set T be given in space of any number of dimensions, real or complex, having a limit point t_0 (actual or symbolic†) not belonging to T, and let the functions $a_k(t)$ $[k = 1, 2, \ldots]$ be defined in T; then if the sequence (x_n) is such that

$$(G) \qquad y(t) = \sum_{k=1}^{\infty} a_k(t)\, x_k$$

converges for each t in T, and if

$$\lim_{t \to t_0(T)} y(t) = l,$$

the sequence (x_n) is said to be summable G to the value l.

Conditions for regularity in this case are due to Carmichael,‡ Hildebrandt,§ and Schur‖; it is necessary and sufficient that

(6) *for each* k, $$\lim_{t \to t_0(T)} a_k(t) = 0;$$

(7) *for each* t *in* T, $\sum_{k=1}^{\infty} |a_k(t)|$ *converges, and for all* t *in* T, $\sum_{k=1}^{\infty} |a_k(t)|$ *is bounded;*

$$(8) \qquad \lim_{t \to t_0(T)} \sum_{k=1}^{\infty} a_k(t) = 1.$$

In case the functions $a_k(t)$ are all real and the transformation is applied to real sequences, we may again inquire when the preservation of existing limits extends to the improper limits $+\infty$, $-\infty$; and when the violence of oscillation of a sequence is not increased by the transformation. It is easy to show that conditions analogous to the earlier ones are still sufficient; I state the two most important theorems of this kind, which may be proved by methods similar to those used for Theorems I and IV:

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* This formulation is quoted from the author's "Report on Topics in the Theory of Divergent Series", *Bull. Amer. Math. Soc.*, 28 (1922), 18–19. It is possible to make use also of repeated limits; see, *e.g.*, Hardy and Chapman, *Quart. J. of Math.*, 42 (1911), 181–215.

† *I.e.* having one or more of its coordinates infinite.

‡ *Bull. Amer. Math. Soc.*, 25 (1918–19), 118.

§ *Bull. Amer. Math. Soc.*, 29 (1923), 316.

‖ *Journal für Math.*, 151 (1920), 82.

THEOREM VIII.—*The transformation* (G), *if regular, will be totally regular if*

(9) $a_k(t) \geqslant 0$, *except for at most a finite set of values of* k.

THEOREM IX.—*In order that the regular transformation* (G) *be such that*

$$\limsup_{t \to t_0(T)} y(t) \leqslant \limsup_{n \to \infty} x_n,$$

and

$$\liminf_{t \to t_0(T)} y(t) \geqslant \liminf_{n \to \infty} x_n,$$

whenever the x*-limits are finite, it is sufficient that*

(10) $$\lim_{t \to t_0(T)} \sum_{k=1}^{\infty} |a_k(t)| = 1.$$

9. For the Euler-Abel power-series method, T is the set of real numbers t in some interval $c \leqslant t < 1$, $t_0 = 1$, and

$$y(t) = u_1 + u_2 t + u_3 t^2 + \ldots$$

$$= (1-t)(x_1 + x_2 t + x_3 t^2 + \ldots);$$

thus $a_k(t) = t^{k-1}(1-t) > 0$; (9) and (10) are satisfied.

For the Borel means, T is the set of positive real numbers, $t_0 = +\infty$ and

$$y(t) = e^{-t}\left(x_1 + x_2 t + x_3 \frac{t^2}{2!} + \ldots\right);$$

$a_k(t) = t^{k-1} e^{-t}/(k-1)! > 0$; (9) and (10) hold.

For the Le Roy mean, T is the set of numbers t in an interval $c \leqslant t < 1$, $t_0 = 1$,

$$y(t) = \sum_{k=0}^{\infty} \frac{\Gamma(kt+1)}{k!} u_k,$$

and

$$a_k(t) = \frac{\Gamma(kt+1)}{k!} - \frac{\Gamma(kt+t+1)}{(k+1)!} > 0,$$

so that again (9) and (10) hold.

For the Riesz mean of order $r > 0$ and type (λ_n), where (λ_n) is an increasing sequence of positive numbers becoming positively infinite,

T is the set of positive real numbers, $t_0 = +\infty$, and

$$y(t) = t^{-r} \sum_{\lambda_k < t} (t-\lambda_k)^r u_k ;$$

thus

$$a_k(t) = \begin{cases} (t-\lambda_k)^r - (t-\lambda_{k+1})^r, & \lambda_{k+1} < t ; \\ (t-\lambda_k)^r, & \lambda_k < t \leqslant \lambda_{k+1} ; \\ 0, & \lambda_k \geqslant t ; \end{cases}$$

and in all cases $a_k(t) \geqslant 0$, so that (9) and (10) are verified.

10. For the generalization of functional limits definitions similar to (A) and (G) exist.* In these cases theorems similar to Theorems VIII and IX can, with ease, be formulated and proved.

* Silverman, *Trans. Amer. Math. Soc.*, 17 (1916), 284, and *Bull. Amer. Math. Soc.*, 22 (1915–16), 459; Kojima, *Tôhoku Math. J.*, 14 (1918), 64, and 18 (1920), 37.

APOLAR TRIADS ON A CUBIC CURVE

By W. SADDLER.

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Grassmann* has shown that a perfectly general cubic curve may be generated by a point, if the lines joining it to three fixed points meet, separately, three fixed lines in three points which are collinear : the line of these three points also envelops a curve of the third class. Apolar properties with reference to such a curve and envelope are here discussed.

Apolar triangles are such that the bipolar of any two vertices with respect to the given cubic curve passes through the third vertex†. Thus, if the cubic be given by $a_x^3 = b_x^3 = \ldots = 0$, while the vertices are $u_y u_z u_t = 0$, then yzt is an apolar triad if $a_y a_z a_t = 0$. Corresponding definitions hold for the cubic envelope.

Let the two fundamental triangles be ABC, $A'B'C'$, and let $B'C'$, $C'A'$, $A'B'$ meet BC, CA, AB in A'', B'', C''. Then it is known† that, starting with 3 arbitrary points on the cubic curve which occupy the role of $A''B''C''$ in Grassmann's construction, the curve may be described in 12 ways following the Grassmann method, there being four different triangles ABC whose sides pass through $A''B''C''$.

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* Clebsch-Lindemann, *Leçons sur la géometrie*, 2 (1880), 272.

† W. P. Milne, "Apolar triads on a cubic curve", *Proc. Edinburgh Math. Soc.*, 30 (1911–12), 9.

We shall establish the following theorems :

I. *The apolarity of one of the triangles ABC, A'B'C', A"B"C" with respect to the cubic curve entails the apolarity of the others.*

II. *All such triangles are apolar to the same member of the Hessian pencil, if one has this property.*

III. (Corollary.) *If A"B"C" be collinear and apolar to* $f+6\lambda H=0$, *then the line A"B"C" will touch* $6P+\lambda Q=0$, *where f is the Grassmann cubic and H, P, Q are the Hessian, Pippian (Cayleyan), and Quippian respectively. In particular, if ABC be apolar to the cubic, A"B"C" will touch its Cayleyan, and, if ABC be apolar to the Hessian, A"B"C" will touch the Quippian.*

IV. *If one of the triangles be apolar to the Grassmann cubic, all are apolar to the Grassmann envelope.*

V. *The determining of a sixth point by ruler construction, when 5 points are taken in general position, and such that the six behave as ABC, A'B'C' in the Grassmann construction and are apolar triangles to the cubic curve and envelope.*

VI. *If ABC be apolar to the Hessian of the Grassmann cubic, it is apolar to the Hessian of the Grassmann envelope and reciprocally.*

1. Proofs of the results I–IV were first obtained by algebraical methods from first principles starting from the apolarity condition $a_y a_z a_t = 0$; but the proofs of (I–III) are sketched briefly by the use of elliptic functions.

I and II. Thus consider Theorem II, of which I is a special case.

Let the cubic curve be

$$f \equiv y^2 - 4x^3 + g_2 x + g_3 = 0,$$

and denote the elliptic arguments of the points A'', B'', C'' by a'', b'', c'', and similarly for A, B, etc.

If $f+6\lambda H=0$ be the Hessian pencil, then the condition that $A''B''C''$ be apolar to the same member is*

$$m-\lambda\rho=0,$$

where $$m=\xi(a''-b'')+\xi(b''-c'')+\xi(c''-a''),$$

$$\rho=\wp'(a''-b'')+\wp'(b''-c'')+\wp'(c''-a'')+2m^2;$$

or $$F(a''-b'',\ b''-c'',\ c''-a'')=0.$$

Now $$a''+b+c=0,\quad b''+c+a=0,\quad c''+a+b=0,$$

whence it follows at once that

$$c=-\tfrac{1}{2}(a''+b''-c'')+\pi_1,$$

and similarly for a and b, where $2\pi_1$ is the elliptic period.

From these equations we obtain

$$b-c=b''-c'',\quad c-a=c''-a'',\quad a-b=a''-b'',$$

and hence $$F(b-c,\ c-a,\ a-b)=0,$$

which proves the theorem. Similarly for $a'b'c'$ instead of abc.

Theorem I follows by putting $\lambda=0$.

Consider now Theorem III. It is known that, when three points X, Y, Z are a collinear apolar triad on a cubic curve, this line will touch the Cayleyan of the cubic†.

Suppose now that $A''B''C''$ are collinear, and that ABC forms an apolar triad. Since both triads are then apolar to the same member of the Hessian pencil, $f+6\lambda H=0$, it follows that $A''B''C''$ touches

$$6P+\lambda Q=0,$$

namely, the Cayleyan of this member of the pencil, where P and Q are the Pippian (Cayleyan) and Quippian respectively. This proves the theorem, and the special cases enunciated follow when $\lambda=0$ and ∞.

* W. P. Milne and D. G. Taylor, "The significance of apolar triangles in elliptic function theory", *Proc. London Math. Soc.* (2), 18 (1919), 375.

† Milne, *loc cit.*

2. Theorem IV may be regarded as a projective property of two coplanar triangles, algebraically closely akin to Desargues's theorem on perspective triangles, but geometrically more complicated.

Let ABC and $A'B'C'$ be two such triangles, and let the coordinates of these points be represented by corresponding small letters and the opposite sides by Greek letters.

Thus A has coordinates (a_1, a_2, a_3) and BC coordinates $(\alpha_1, \alpha_2, \alpha_3)$. Then

$$(aa' . bb' . cc') = 0 \quad \text{necessitates} \quad (a\alpha' . \beta\beta' . \gamma\gamma') = 0^*,$$

and *vice versa*, which is Desargues' theorem.

Now the Grassmann cubic has the equation

$$(xa . b'c' \,|\, xb . c'a' \,|\, xc . a'b') = 0, \tag{i}$$

where $(xa . b'c')_i$ are the coordinates of the point of intersection of xA with $B'C'$, and $(\lambda \,|\, \mu \,|\, \nu)$ is the determinant $\Sigma \pm \lambda_1 \mu_2 \nu_3$. Similarly, the Grassmann envelope has the equation

$$(u\alpha . \beta'\gamma' \,|\, u\beta . \gamma'\alpha' \,|\, u\gamma . \alpha'\beta') = 0 \tag{ii}$$

and is the reciprocal of the other.

The triangle ABC is apolar to (i) if

$$(ba . b'c' \,|\, cb . c'a' \,|\, ac . a'b') + (ca . b'c' \,|\, ab . c'a' \,|\, bc . a'b') = 0,$$

or

$$(\gamma . \alpha' \,|\, \alpha . \beta' \,|\, \beta . \gamma') - (\beta . \alpha' \,|\, \gamma . \beta' \,|\, \alpha . \gamma') = 0, \tag{iii}$$

while the triangle ABC is apolar to (ii) if

$$(c . a' \,|\, a . b' \,|\, b . c') - (b . a' \,|\, c . b' \,|\, a . c') = 0. \tag{iv}$$

Now it is easily verified, by expanding (iii), that it is the same as (iv) save for a non-vanishing factor, $(abc)(a'b'c')$. Thus Theorem IV is established.

If, in addition, $A''B''C''$ are collinear, or

$$(a\alpha' . \beta\beta' . \gamma\gamma') = 0 \quad \text{or} \quad (aa' . bb' . cc') = 0,$$

then it follows that the centre of perspective of the two triangles will be on the Cayleyan of the cubic envelope.

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* E. Study, "Vectorenrechnung", *Die Wissenschaft*, 71 (1924), 62; Grace and Young, *Algebra of invariants*, 280.

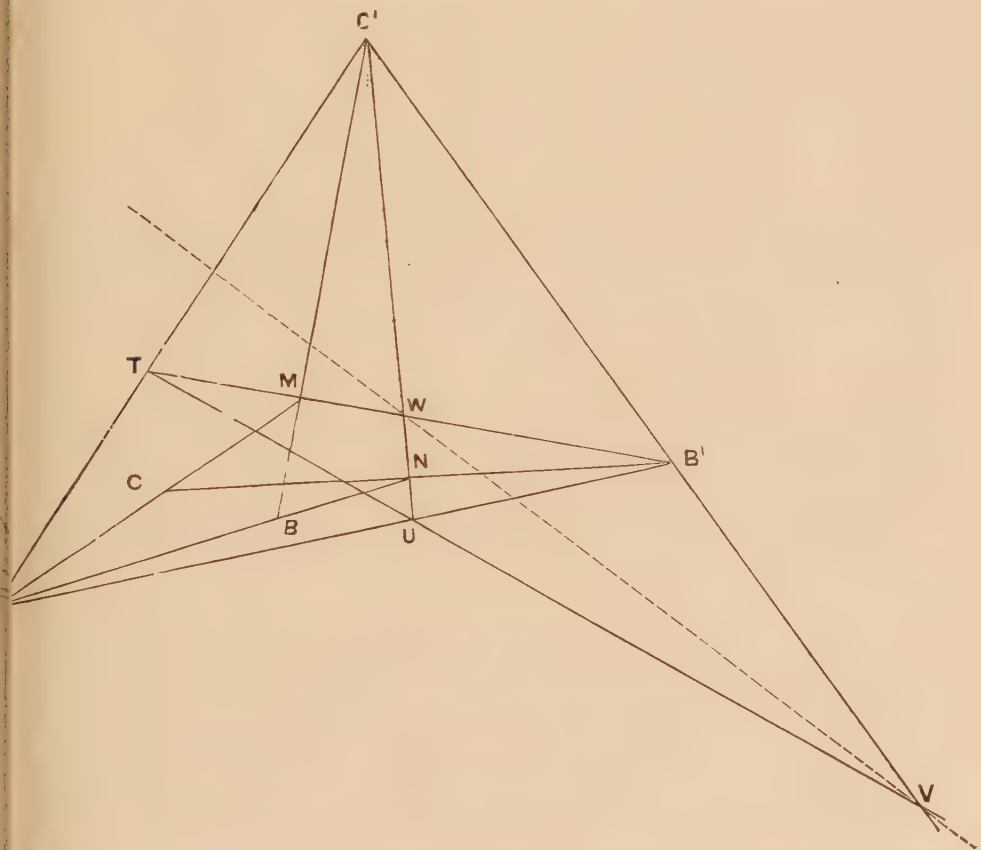
3. *Construction for Theorem* V. Consider the 5 points in general position as B, C, A', B', C'. Determine A to satisfy relation (iv) or

$$(ca' \,.\, ab' \,.\, bc') - (ba' \,.\, cb' \,.\, ac') = 0.$$

Thus A will lie on the line given by

$$(ca' \,.\, xb' \,.\, bc') - (ba' \,.\, cb' \,.\, xc') = 0.$$

This locus is given in the Figure I, by the line VW, which is therefore such that any point taken on this line, say A, will, with B and C and



the other triangle $A'B'C'$, generate apolar triads to the Grassmann cubic constructed from ABC, $A'B'C'$.

If $A'B'C'$ be taken as the triangle of reference, the condition (iv) reduces to

$$2a_1 b_2 c_3 - a_3 b_1 c_2 - a_2 b_3 c_1 = 0. \qquad \text{(v)}$$

Then

N, the intersection of $B'C$, $A'B$, has coordinates $(c_1 b_3,\ c_3 b_2,\ b_3 c_3)$,

M, „ „ $C'B$, $A'C$, „ „ $(b_1 c_2,\ b_2 c_2,\ b_2 c_3)$,

T, „ „ BM, $A'C'$, „ „ $(b_1 c_2,\ 0,\ b_2 c_3)$,

U, „ „ CN, $A'B'$, „ „ $(c_1 b_3,\ c_3 b_2,\ 0)$.

Hence the equation of UT is

$$-x_1 b_2 c_3 + x_2 b_3 c_1 + x_3 b_1 c_2 = 0,$$

so that the equation of VW is

$$-2x_1 b_2 c_3 + x_2 b_3 c_1 + x_3 b_1 c_2 = 0.$$

In particular, given two points B, C and three lines intersecting in $A'B'C'$, there is a unique point, say A, on any other line of the plane, namely, its point of intersection of the line VW, such that the two triangles ABC, $A'B'C'$ will be apolar triangles to the Grassmann cubic.

4. If three points $A'B'C'$ are taken as the vertices of the triangle of reference, they will form an apolar triad to a cubic curve if the coefficient of $x_1 x_2 x_3$ in the equation to the curve vanishes.

Reciprocally, the sides of the triangle of reference will form an apolar triad to a cubic envelope if the coefficient of $u_1 u_2 u_3$ in the equation to the envelope vanishes.

Thus $p_x = 0$, $q_x = 0$, $r_x = 0$, will be an apolar triangle to $u_a^3 = 0$ if $p_a q_a r_a = 0$; but for the special case of $p_x = x_1$, $q_x = x_2$, $r_x = x_3$, $p_a q_a r_a = 0$ reduces to $\alpha_1 \alpha_2 \alpha_3 = 0$ or the coefficient of $u_1 u_2 u_3$ in $u_a^3 = 0$ vanishes.

Consider now Theorem VI. The equation of the Grassmann envelope in § 2 (ii) reduces to Σ, where Σ stands for

$$u_2^2 u_3 \alpha_1(\beta_3\gamma_1 - \beta_1\gamma_3) + u_2 u_3^2 \alpha_1(\beta_1\gamma_2 - \beta_2\gamma_1) + u_3^2 u_1 \beta_2(\gamma_1\alpha_2 - \gamma_2\alpha_1)$$

$$+ u_3 u_1^2 \beta_2(\gamma_2\alpha_3 - \gamma_3\alpha_2) + u_1^2 u_2 \gamma_3(\alpha_2\beta_3 - \alpha_3\beta_2) + u_1 u_2^2 \gamma_3(\alpha_3\beta_1 - \alpha_1\beta_3)$$

$$+ u_1 u_2 u_3(2\alpha_1\beta_2\gamma_3 - \alpha_2\beta_3\gamma_1 - \alpha_3\beta_1\gamma_2) = 0,$$

when $A'B'C'$ is taken as the triangle of reference; or to

$$u_2u_3(b_2c_3-b_3c_2)(a_2u_2+a_3u_3)+u_3u_1(c_3a_1-c_1a_3)(b_1u_1+b_3u_3)$$

$$+u_1u_2(a_1b_2-a_2b_1)(c_1u_1+c_2u_2)+u_1u_2u_3(2a_1b_2c_3-a_3b_1c_2-a_2b_3c_1)=0,$$

by using the relations

$$a_1=(b_2c_3-b_3c_2),\ \text{etc.};\quad (a_2\beta_3-a_3\beta_2)=c_1(abc),\ \text{etc.}$$

Again, the cubic curve f, with the same triangle of reference, has the equation

$$x_2^2x_3a_1(b_3c_1-b_1c_3)+x_2x_3^2a_1(b_1c_2-b_2c_1)+x_1^2x_3b_2(c_2a_3-c_3a_2)$$

$$+x_1x_3^2b_2(c_1a_2-c_2a_1)+x_1^2x_2c_3(a_2b_3-a_3b_2)+x_1x_2^2c_3(a_3b_1-a_1b_3)$$

$$+x_1x_2x_3(2a_1b_2c_3-a_3b_1c_2-a_2b_3c_1)=0.$$

Now $A'B'C'$ will be apolar to the Hessian of the cubic envelope Σ when this triad of points is apolar to the Hessian of the cubic curve f, if the coefficient of $u_1u_2u_3$ in the Hessian of Σ is equal to the coefficient of $x_1x_2x_3$ in the Hessian of f.

This is the case, for the coefficient of $x_1x_2x_3$ in the Hessian of a cubic given by

$$A_1x_2^2x_3+A_2x_2x_3^2+B_1x_1^2x_3+B_2x_1x_3^2+C_1x_1^2x_2+C_2x_1x_2^2+Dx_1x_2x_3=0,$$

is found to be

$$24(A_1B_2C_1+A_2C_2B_1)-8(A_2C_1+A_1B_1+B_2C_2)D+2D^3=0.$$

By the application of this result to the cubics f and Σ, it is found that the two coefficients mentioned above have the same value, which proves the theorem.

5. By consideration of the geometry of the Grassman cubic, it is found that AA', BB', CC' are conjugate pairs of points for the curve and thus, if $B'C$ meets BC' in λ, $C'A$ meets CA' in μ, $A'B$ meets AB' in ν, we find, from algebraical considerations, that λ, μ, ν will be an apolar triad if

ABC is one. Again, λ is the conjugate point to A''. Now let $A\lambda$ meet the cubic in P; then $B'C'P$ forms an apolar triad (since the projection of an apolar triad is an apolar triad*). Hence the points $A'PA''$ are collinear, and this line is the bipolar of B', C' with respect to the cubic. This may also be proved quite directly from the equation f to the curve.

Similarly $A''A$ can be shown to be the bipolar of B, C, etc. Thus we have an independent proof† of the Theorem II : that all the triangles are apolar to the same member of the Hessian pencil, if one of them has the property.

[I have to thank the referee and Prof. H. W. Turnbull for their valuable criticism and interesting suggestions during the preparation of this paper.]

* Milne, *loc. cit.*

† H. Hilton, *Plane algebraic curves* (1920), 238.

SOME TRANSFORMATIONS OF GENERALIZED HYPERGEOMETRIC SERIES

By F. J. W. WHIPPLE.

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1. *Introduction.*

In recent papers some of the properties of generalized hypergeometric series, such as

$$ {}_4F_3\left[\begin{matrix} a, b, c, d; \pm 1 \\ 1+a-b, 1+a-c, 1+a-d \end{matrix}\right], $$

have been investigated. In these series, to which the name "well-poised" series has been given*, the parameters occur in pairs, each pair with the same sum; the first of these pairs consists of the a, which is shown explicitly, and unity, which may be regarded as the leading parameter for the lower row, the general term of the series being

$$ \frac{(a)_r (b)_r (c)_r (d)_r}{(1)_r (1+a-b)_r (1+a-c)_r (1+a-d)_r} (\pm 1)^r. $$

In the series first considered in the present note, all but one of the pairs of parameters have the same sum. Such series I call nearly-poised.

The most general series to be dealt with is

$$ {}_4F_3\left[\begin{matrix} a, b, c, d \\ \kappa-b, \kappa-c, \kappa-d \end{matrix}\right]. $$

The identity in which this series occurs is found by Thomae's method†, in which, with the help of Gauss's theorem, the series is

* Whipple, **1**.

† Thomae, **1**.

equated to a double series, the terms of which can be rearranged. An alternative proof, not reproduced here, has been found. It is by Dougall's method*, in which the identity of two polynomials of degree n is deduced from their equality for more than n values of the variable, each polynomial being symmetrical in $-n$ and the variable.

A number of identities are deduced as special cases; the most important is a quadratic transformation of the series

$$ {}_3F_2\left[\begin{matrix} a, b, c; \zeta \\ 1+a-b, 1+a-c \end{matrix}\right]. $$

Just as the definition of a well-poised series can be modified to give nearly-poised series, so another obvious modification gives half-poised series. Transformations are known for three kinds of half-poised series, but the search for a general transformation has not been successful. One of the known transformations is due to Cayley. A new proof of this transformation and of two similar transformations discovered by Orr is given in a companion paper†.

2. *Transformation of a nearly-poised series* ${}_3F_2(-1)$.

By the definition of the series we have

$$ (2.1)\quad {}_3F_2\left[\begin{matrix} a, b, c; -1 \\ \kappa-b, \kappa-c \end{matrix}\right] $$

$$ = M \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(1-a-n)\,\Gamma(1-b-n)\,\Gamma(1-c-n)\,\Gamma(\kappa-b+n)\,\Gamma(\kappa-c+n)}, $$

where

$$ (2.11)\qquad M = \Gamma(1-a)\,\Gamma(1-b)\,\Gamma(1-c)\,\Gamma(\kappa-b)\,\Gamma(\kappa-c). $$

Now we know, by Gauss's theorem, that

$$ (2.2)\qquad \frac{\Gamma(\kappa+2n)\,\Gamma(\kappa-b-c)}{\Gamma(\kappa-b+n)\,\Gamma(\kappa-c+n)} = {}_2F_1\left[\begin{matrix} b+n, c+n \\ \kappa+2n \end{matrix}\right], $$

* Dougall, 1.

† Whipple, 4.

and hence that

$$(2.21)\quad \frac{1}{\Gamma(1-b-n)\,\Gamma(1-c-n)\,\Gamma(\kappa-b+n)\,\Gamma(\kappa-c+n)}$$

$$= \frac{1}{\Gamma(\kappa-b-c)} \sum_{p=n}^{\infty} \frac{1}{\Gamma(1-b-p)\,\Gamma(1-c-p)\,\Gamma(1+p-n)\,\Gamma(\kappa+p+n)}.$$

Accordingly

$$(2.22)\quad {}_3F_2\left[\begin{matrix} a, b, c; \\ \kappa-b, \kappa-c \end{matrix} -1\right]$$

$$= \frac{M}{\Gamma(\kappa-b-c)} \sum_{n=0}^{\infty} \left[\frac{1}{n!\,\Gamma(1-a-n)} \times \sum_{p=n}^{\infty} \frac{1}{\Gamma(1-b-p)\,\Gamma(1-c-p)\,\Gamma(1+p-n)\,\Gamma(\kappa+p+n)}\right].$$

A change in the order of summation gives

$$(2.3)\quad {}_3F_2 = \frac{M}{\Gamma(\kappa-b-c)\,\Gamma(1-a)} \times \sum_{p=0}^{\infty} \frac{1}{\Gamma(1-b-p)\,\Gamma(1-c-p)\,\Gamma(1+p)\,\Gamma(\kappa+p)}\ {}_2F_1\left[\begin{matrix} a, -p \\ \kappa+p \end{matrix}\right].$$

By another application of Gauss's theorem we find that

$$(2.4)\quad {}_3F_2 = \frac{M}{\Gamma(\kappa-b-c)\,\Gamma(1-a)} \times \sum_{p=0}^{\infty} \frac{\Gamma(\kappa+2p-a)}{\Gamma(1-b-p)\,\Gamma(1-c-p)\,\Gamma(1+p)\,\Gamma(\kappa+p-a)\,\Gamma(\kappa+2p)}.$$

This is equivalent to

$$(2.5)\quad {}_3F_2\left[\begin{matrix} a, b, c; \\ \kappa-b, \kappa-c \end{matrix} -1\right] = \frac{\Gamma(\kappa-b)\,\Gamma(\kappa-c)}{\Gamma(\kappa-b-c)\,\Gamma(\kappa)}\ {}_4F_3\left[\begin{matrix} b, c, \frac{1}{2}\kappa-\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a \\ \kappa-a, \frac{1}{2}\kappa, \frac{1}{2}+\frac{1}{2}\kappa \end{matrix}\right].$$

The change in the order of summation is legitimate, since the condition $R(\kappa-b-c) > 0$, which is required to ensure the convergence of the ${}_4F_3$ in (2.5), also makes the ${}_2F_1$ of (2.2) absolutely convergent.

3. *Transformation of a nearly-poised series* ${}_4F_3(+1)$.

The analysis for the transformation of the ${}_4F_3$ follows the same lines as far as the analogue of (2 . 3), viz.,

$$(3\,.\,1)\quad {}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d \\ \kappa-b,\ \kappa-c,\ \kappa-d \end{matrix}\right] = \sum_{p=0}^{\infty} \frac{N}{\Gamma(1-c-p)\,\Gamma(1-d-p)\,\Gamma(1+p)\,\Gamma(\kappa+p)}\, {}_3F_2\left[\begin{matrix} a,\ b,\ -p\,;\ -1 \\ \kappa-b,\ \kappa+p \end{matrix}\right],$$

in which $$N = \frac{\Gamma(1-c)\,\Gamma(1-d)\,\Gamma(\kappa-c)\,\Gamma(\kappa-d)}{\Gamma(\kappa-c-d)}.$$

The ${}_3F_2$ on the right can be transformed by the previous proposition (2 . 5), $-p$ being substituted for c.

Accordingly

$$(3\,.\,2)\quad {}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d \\ \kappa-b,\ \kappa-c,\ \kappa-d \end{matrix}\right] = \sum_{p=0}^{\infty} \frac{N\,.\,\Gamma(\kappa-b)}{\Gamma(1-c-p)\,\Gamma(1-d-p)\,\Gamma(1+p)\,\Gamma(\kappa-b+p)\,\Gamma(\kappa)} \times {}_4F_3\left[\begin{matrix} b,\ -p,\ \frac{1}{2}\kappa-\frac{1}{2}a,\ \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a \\ \kappa-a,\ \frac{1}{2}\kappa,\ \frac{1}{2}+\frac{1}{2}\kappa \end{matrix}\right]$$

$$= \sum_{p=0}^{\infty} \sum_{n=0}^{p} \frac{N\,.\,\Gamma(\kappa-b)\,\Gamma(1-b)\,\Gamma(\kappa-a+2n)}{n!\ \Gamma(1-b-n)\,\Gamma(1+p-n)\,\Gamma(1-c-p)} \times \Gamma(1-d-p)\,\Gamma(\kappa-b+p)\,\Gamma(\kappa-a+n)\,\Gamma(\kappa+2n).$$

If we change the order of summation, this becomes

$$(3\,.\,3)\quad {}_4F_3 = \sum_{n=0}^{\infty} \frac{N\,.\,\Gamma(\kappa-b)\,\Gamma(1-b)\,\Gamma(\kappa-a+2n)}{n!\ \Gamma(1-b-n)\,\Gamma(1-c-n)\,\Gamma(1-d-n)} \times \Gamma(\kappa-b+n)\,\Gamma(\kappa-a+n)\,\Gamma(\kappa+2n) \times {}_2F_1\left[\begin{matrix} c+n,\ d+n \\ \kappa-b+n \end{matrix}\right].$$

Another application of Gauss's theorem leads to the equivalent form

$$(3\,.\,4)\quad {}_4F_3 = \sum_{n=0}^{\infty} \frac{N\,.\,\Gamma(\kappa-b)\,\Gamma(1-b)\,\Gamma(\kappa-a+2n)\,\Gamma(\kappa-b-c-d-n)}{n!\ \Gamma(1-b-n)\,\Gamma(1-c-n)\,\Gamma(1-d-n)} \times \Gamma(\kappa-a+n)\,\Gamma(\kappa+2n)\,\Gamma(\kappa-b-c)\,\Gamma(\kappa-b-d),$$

so that finally

$$(3.5)\quad {}_4F_3\left[\begin{matrix}a, b, c, d\\ \kappa-b, \kappa-c, \kappa-d\end{matrix}\right]$$

$$= \frac{\Gamma(\kappa-b)\,\Gamma(\kappa-c)\,\Gamma(\kappa-d)\,\Gamma(\kappa-b-c-d)}{\Gamma(\kappa-c-d)\,\Gamma(\kappa-b-c)\,\Gamma(\kappa-b-d)\,\Gamma(\kappa)}$$

$$\times {}_5F_4\left[\begin{matrix}b, c, d, \frac{1}{2}\kappa-\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a\\ \kappa-a, \frac{1}{2}\kappa, \frac{1}{2}+\frac{1}{2}\kappa, b+c+d-\kappa+1\end{matrix}\right].$$

The demonstration of this identity is valid only if one of the parameters b, c, and d is a negative integer. Otherwise the series

$${}_2F_1\left[\begin{matrix}c+n, d+n\\ \kappa-b+n\end{matrix}\right],$$

which occurs in (3 . 3), will be divergent for all values of n exceeding $R(\kappa-b-c-d)$. If b, c, or d is a negative integer, all the series which occur in the demonstration terminate, on account of the appearance of $\Gamma(0)$ in the denominators of the higher terms.

4. The identity (3 . 5) is the most general one that has been found involving "nearly-poised" series. It is an extension of the known proposition* that

$$(4.1)\quad {}_4F_3\left[\begin{matrix}a, b, c, d\\ 1+a-b, 1+a-c, 1+a-d\end{matrix}\right]$$

$$= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-b-c-d)}{\Gamma(1+a-c-d)\,\Gamma(1+a-b-c)\,\Gamma(1+a-b-d)\,\Gamma(1+a)}$$

$$\times {}_4F_3\left[\begin{matrix}\frac{1}{2}, b, c, d\\ 1+\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a, b+c+d-a\end{matrix}\right],$$

which is also subject to the restriction that b, c, or d is a negative integer.

It should be noticed† that the series ${}_5F_4$ in (3 . 5) is Saalschützian; *i.e.* it is a terminating series and the sum of the parameters in the lower row exceeds by unity the sum of those in the upper. The identity (2 . 5)

* Whipple, **1**, Equation (9 . 4).

† Whipple, **2**.

is also an extension of a known proposition* concerning well-poised series, viz.,

$$(4.2)\quad {}_3F_2\left[\begin{matrix}a, b, c; -1\\ 1+a-b, 1+a-c\end{matrix}\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)} \times {}_3F_2\left[\begin{matrix}\frac{1}{2}, b, c\\ 1+\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a\end{matrix}\right].$$

5. *Some identities involving hypergeometric series of lower orders.*

In (3.5) let $a \to 0$. The right-hand side becomes unity, and we have Saalschütz's identity

$$(5.1)\quad {}_3F_2\left[\begin{matrix}b, c, d\\ \kappa, b+c+d-\kappa+1\end{matrix}\right] = \frac{\Gamma(\kappa)\,\Gamma(\kappa-b-c)\,\Gamma(\kappa-c-d)\,\Gamma(\kappa-b-d)}{\Gamma(\kappa-b)\,\Gamma(\kappa-c)\,\Gamma(\kappa-d)\,\Gamma(\kappa-b-c-d)},$$

which is valid if b, c, or d is a negative integer.

If in (3.5) we write $c = \frac{1}{2}\kappa$, $d = -n$, we find that

$$(5.2)\quad {}_3F_2\left[\begin{matrix}a, b, -n\\ \kappa-b, \kappa+n\end{matrix}\right] = \frac{\Gamma(\kappa-b)\,\Gamma(\frac{1}{2}\kappa)\,\Gamma(\kappa+n)\,\Gamma(\frac{1}{2}\kappa-b+n)}{\Gamma(\kappa)\,\Gamma(\frac{1}{2}\kappa-b)\,\Gamma(\frac{1}{2}\kappa+n)\,\Gamma(\kappa-b+n)} \times {}_4F_3\left[\begin{matrix}b, -n, \frac{1}{2}\kappa-\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a\\ \kappa-a, \frac{1}{2}+\frac{1}{2}\kappa, b-\frac{1}{2}\kappa-n+1\end{matrix}\right].$$

The series on the right is Saalschützian, and is therefore transformable by the formulae† appropriate to Saalschützian ${}_4F_3$'s. The ${}_3F_2$ on the left is also capable‡ of numerous transformations, so that many new results could be deduced from (5.2).

From (2.5), by putting $c = \frac{1}{2}\kappa$, we derive the formula

$$(5.3)\quad {}_2F_1\left[\begin{matrix}a, b; -1\\ \kappa-b\end{matrix}\right] = \frac{\Gamma(\kappa-b)\,\Gamma(\frac{1}{2}\kappa)}{\Gamma(\frac{1}{2}\kappa-b)\,\Gamma(\kappa)}\,{}_3F_2\left[\begin{matrix}b, \frac{1}{2}\kappa-\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a\\ \kappa-a, \frac{1}{2}+\frac{1}{2}\kappa\end{matrix}\right],$$

which is an extension of Kummer's identity§ :

$$(5.31)\qquad {}_2F_1\left[\begin{matrix}a, b; -1\\ 1+a-b\end{matrix}\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-b)\,\Gamma(1+a)}.$$

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* Whipple, **1**, Equation (9.3).
† Whipple, **2**.
‡ Whipple, **3**.
§ Kummer, **1**.

In (5 . 3) we may write $\kappa - a = c$, and the identity becomes

$$(5.32)\quad {}_2F_1\left[\begin{matrix}a, b; \\ c\end{matrix} -1\right] = \frac{\Gamma(c)\,\Gamma(\frac{1}{2}b+\frac{1}{2}c)}{\Gamma(\frac{1}{2}c-\frac{1}{2}b)\,\Gamma(b+c)} \times {}_3F_2\left[\begin{matrix}b, \frac{1}{2}(b+c-a), \frac{1}{2}(1+b+c-a) \\ b+c-a, \frac{1}{2}(1+b+c)\end{matrix}\right].$$

It will be noticed that the series ${}_3F_2$ terminates if $b+c-a$ is a negative integer.

Again, in (5 . 3) let $\kappa - a \to 0$. Then

$$(5.33)\quad {}_2F_1\left[\begin{matrix}a, b; \\ a-b\end{matrix} -1\right] = \tfrac{1}{2}\frac{\Gamma(a-b)\,\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b)\,\Gamma(a)}\left\{1+{}_2F_1\left[\begin{matrix}b, \frac{1}{2} \\ \frac{1}{2}+\frac{1}{2}a\end{matrix}\right]\right\}$$

$$= \tfrac{1}{2}\frac{\Gamma(a-b)}{\Gamma(a)}\left[\frac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b)} + \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a)}{\Gamma(\frac{1}{2}+\frac{1}{2}a-b)}\right].$$

Since the ${}_2F_1$ on the left of (5 . 32) is symmetrical in a and b, we can write down the equation

$$(5.4)\quad \frac{1}{2^b\,\Gamma(\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}c)\,\Gamma(\frac{1}{2}c-\frac{1}{2}b)}\,{}_3F_2\left[\begin{matrix}b, \frac{1}{2}(b+c-a), \frac{1}{2}(1+b+c-a) \\ b+c-a, \frac{1}{2}(1+b+c)\end{matrix}\right]$$

$$= \frac{1}{2^a\,\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}c)\,\Gamma(\frac{1}{2}c-\frac{1}{2}a)}\,{}_3F_2\left[\begin{matrix}a, \frac{1}{2}(a+c-b), \frac{1}{2}(1+a+c-b) \\ a+c-b, \frac{1}{2}(1+a+c)\end{matrix}\right].$$

This result may be thrown into a more interesting form by writing

$$a = t-x, \quad b = t-y, \quad c = x+y,$$

when it becomes

$$(5.41)\quad \frac{2^y}{\Gamma(\frac{1}{2}+\frac{1}{2}t+\frac{1}{2}x)\,\Gamma(\frac{1}{2}x+y-\frac{1}{2}t)}\,{}_3F_2\left[\begin{matrix}t-y, x, x+\frac{1}{2} \\ 2x, \frac{1}{2}+\frac{1}{2}t+\frac{1}{2}x\end{matrix}\right]$$

$$= \frac{2^x}{\Gamma(\frac{1}{2}+\frac{1}{2}t+\frac{1}{2}y)\,\Gamma(\frac{1}{2}y+x-\frac{1}{2}t)}\,{}_3F_2\left[\begin{matrix}t-x, y, y+\frac{1}{2} \\ 2y, \frac{1}{2}+\frac{1}{2}t+\frac{1}{2}y\end{matrix}\right].$$

It may be remarked that this is *not* a particular case of the transformations* which apply to any series ${}_3F_2(+1)$.

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* Thomae, **1**; Whipple, **3**.

6. *Reversal of "nearly-poised" series.*

The terms of a terminating generalized hypergeometric series can be written in the reverse order. The result in the case of a nearly-poised series is to effect an apparent change of type.

Starting with (3 . 5) and writing $d = -n$, we have

$$(6\,.\,1)\quad {}_4F_3\left[\begin{matrix} a,\ b,\ c,\ -n \\ \kappa-b,\ \kappa-c,\ \kappa+n \end{matrix}\right] = \frac{\Gamma(\kappa-b)\,\Gamma(\kappa-c)\,\Gamma(\kappa+n)\,\Gamma(\kappa-b-c+n)}{\Gamma(\kappa)\,\Gamma(\kappa-b-c)\,\Gamma(\kappa-c+n)\,\Gamma(\kappa-b+n)}$$

$$\times {}_5F_4\left[\begin{matrix} b,\ c,\ -n,\ \frac{1}{2}\kappa-\frac{1}{2}a,\ \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a \\ \kappa-a,\ \frac{1}{2}\kappa,\ \frac{1}{2}+\frac{1}{2}\kappa,\ b+c-\kappa-n+1 \end{matrix}\right].$$

Now the reversal of the series on the left gives the identity

$$(6\,.\,2)\quad {}_4F_3\left[\begin{matrix} a,\ b,\ c,\ -n \\ \kappa-b,\ \kappa-c,\ \kappa+n \end{matrix}\right]$$

$$= (-1)^n\,\frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c+n)\,\Gamma(\kappa-b)\,\Gamma(\kappa-c)\,\Gamma(\kappa+n)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(\kappa-b+n)\,\Gamma(\kappa-c+n)\,\Gamma(\kappa+2n)}$$

$$\times {}_4F_3\left[\begin{matrix} 1-2n-\kappa,\ 1-n-\kappa+b,\ 1-n-\kappa+c,\ -n \\ 1-n-b,\ 1-n-c,\ 1-n-a \end{matrix}\right].$$

Similarly

$$(6\,.\,3)\quad {}_5F_4\left[\begin{matrix} b,\ c,\ -n,\ \frac{1}{2}\kappa-\frac{1}{2}a,\ \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a \\ \kappa-a,\ \frac{1}{2}\kappa,\ \frac{1}{2}+\frac{1}{2}\kappa,\ b+c-\kappa-n+1 \end{matrix}\right]$$

$$= (-1)^n\,\frac{\Gamma(b+n)\,\Gamma(c+n)\,\Gamma(\kappa-a+2n)\,\Gamma(\kappa)\,\Gamma(b+c-\kappa-n+1)}{\Gamma(b)\,\Gamma(c)\,\Gamma(\kappa-a+n)\,\Gamma(\kappa+2n)\,\Gamma(b+c-\kappa+1)}$$

$$\times {}_5F_4\left[\begin{matrix} 1-n-\kappa+a,\ 1-n-\frac{1}{2}\kappa,\ \frac{1}{2}-n-\frac{1}{2}\kappa,\ \kappa-b-c,\ -n \\ 1-n-b,\ 1-n-c,\ 1-n-\frac{1}{2}\kappa+\frac{1}{2}a,\ \frac{1}{2}-n-\frac{1}{2}\kappa+\frac{1}{2}a \end{matrix}\right].$$

The result of rewriting (6 . 1) and omitting the common factors is therefore

$$(6\,.\,4)\quad {}_4F_3\left[\begin{matrix} 1-2n-\kappa,\ 1-n-\kappa+b,\ 1-n-\kappa+c,\ -n \\ 1-n-b,\ 1-n-c,\ 1-n-a \end{matrix}\right]$$

$$= \frac{\Gamma(1-n-a)\,\Gamma(\kappa-a+2n)}{\Gamma(1-a)\,\Gamma(\kappa-a+n)}$$

$$\times {}_5F_4\left[\begin{matrix} 1-n-\kappa+a,\ 1-n-\frac{1}{2}\kappa,\ \frac{1}{2}-n-\frac{1}{2}\kappa,\ \kappa-b-c,\ -n \\ 1-n-b,\ 1-n-c,\ 1-n-\frac{1}{2}\kappa+\frac{1}{2}a,\ \frac{1}{2}-n-\frac{1}{2}\kappa+\frac{1}{2}a \end{matrix}\right],$$

which is equivalent to

$$(6.5)\quad {}_4F_3\left[\begin{matrix} x, y, z, -n \\ 1+x-y, 1+x-z, w \end{matrix}\right] = \frac{\Gamma(w)\,\Gamma(w-x+n)}{\Gamma(w+n)\,\Gamma(w-x)} \times {}_5F_4\left[\begin{matrix} 1+x-w, \frac{1}{2}(1+x), \frac{1}{2}x, 1+x-y-z, -n \\ 1+x-y, 1+x-z, 1+\frac{1}{2}(x-w-n), \frac{1}{2}(1+x-w-n) \end{matrix}\right].$$

The ${}_5F_4$ is, of course, Saalschützian. The ${}_4F_3$ is "nearly-poised", but the breakdown in the equality of sums of parameters now occurs with the last pair.

In (6 . 5) let $z \to \infty$. The number of terms on each side of the equation is finite, so that we find at once

$$(6.51)\quad {}_3F_2\left[\begin{matrix} x, y, -n; -1 \\ 1+x-y, w \end{matrix}\right] = \frac{\Gamma(w)}{\Gamma(w+n)}\,\frac{\Gamma(w-x+n)}{\Gamma(w-x)} \times {}_4F_3\left[\begin{matrix} 1+x-w, \frac{1}{2}(1+x), \frac{1}{2}x, -n \\ 1+x-y, 1+\frac{1}{2}(x-w-n), \frac{1}{2}(1+x-w-n) \end{matrix}\right].$$

By putting $z = \frac{1}{2}(1+x)$ in (6 . 5), we get

$$(6.52)\quad {}_3F_2\left[\begin{matrix} x, y, -n \\ 1+x-y, w \end{matrix}\right] = \frac{\Gamma(w)}{\Gamma(w+n)}\,\frac{\Gamma(w-x+n)}{\Gamma(w-x)} \times {}_4F_3\left[\begin{matrix} 1+x-w, \frac{1}{2}x, \frac{1}{2}+\frac{1}{2}x-y, -n \\ 1+x-y, 1+\frac{1}{2}(x-w-n), \frac{1}{2}(1+x-w-n) \end{matrix}\right].$$

Hence, if $y \to \infty$,

$$(6.53)\quad {}_2F_1\left[\begin{matrix} x, -n; -1 \\ w \end{matrix}\right] = \frac{\Gamma(w)\,\Gamma(w-x+n)}{\Gamma(w+n)\,\Gamma(w-x)} \times {}_3F_2\left[\begin{matrix} 1+x-w, \frac{1}{2}x, -n \\ 1+\frac{1}{2}(x-w-n), \frac{1}{2}(1+x-w-n) \end{matrix}\right].$$

These last three relations could be derived, of course, from (2 . 5), (5 . 2), and (5 . 3) respectively.

[*Added February 28th*, 1927.

The formula (6 . 5) was established on the understanding that x was not a negative integer between 0 and $-2n$, for the κ of (6 . 1) must not be a negative integer. However, since the functions involved are continuous, (6 . 5) must still be valid when x is a negative integer. If we write $-m$ for x, m being a positive integer, and then proceed to

apply Carlson's theorem*, we obtain the equation

$$(6.6)\quad {}_4F_3\left[\begin{matrix}-m,\ y,\ z,\ u\\ 1-m-y,\ 1-m-z,\ w\end{matrix}\right]=\frac{\Gamma w\,\Gamma(w+m-u)}{\Gamma(w+m)\,\Gamma(w-u)}$$

$$\times {}_5F_4\left[\begin{matrix}1-m-w,\ \tfrac{1}{2}(1-m),\ -\tfrac{1}{2}m,\ 1-m-y-z,\ u\\ 1-m-y,\ 1-m-z,\ 1+\tfrac{1}{2}(u-m-w),\ \tfrac{1}{2}(1+u-m-w)\end{matrix}\right].$$

Equations corresponding to (6 . 51), (6 . 52), and (6 . 53) follow at once, but in the last two cases the parameter which takes the place of x must be an even negative number.

Equation (6 . 6) has an application in the theory of the confluent hypergeometric function. The following identity, discovered by Ramanujan, has been proved† by C. T. Preece.

$$(6.7)\qquad {}_1F_1\left(\begin{matrix}\alpha;\ x\\ \rho\end{matrix}\right){}_1F_1\left(\begin{matrix}\alpha;\ -x\\ \rho\end{matrix}\right)={}_2F_3\left(\begin{matrix}\alpha,\ \rho-\alpha;\ \tfrac{1}{4}x^2\\ \rho,\ \tfrac{1}{2}\rho,\ \tfrac{1}{2}\rho+\tfrac{1}{2}\end{matrix}\right).$$

Now it is well known that

$$(6.71)\qquad e^x\,{}_1F_1\left(\begin{matrix}\alpha;\ -x\\ \rho\end{matrix}\right)={}_1F_1\left(\begin{matrix}\rho-\alpha;\ x\\ \rho\end{matrix}\right).$$

Accordingly

$$(6.72)\qquad {}_1F_1\left(\begin{matrix}\alpha;\ x\\ \rho\end{matrix}\right){}_1F_1\left(\begin{matrix}\rho-\alpha;\ x\\ \rho\end{matrix}\right)=e^x\,{}_2F_3\left(\begin{matrix}\alpha,\ \rho-\alpha;\ \tfrac{1}{4}x^2\\ \rho,\ \tfrac{1}{2}\rho,\ \tfrac{1}{2}\rho+\tfrac{1}{2}\end{matrix}\right).$$

This equation appears to have been overlooked by Preece, who denies‡ in fact the possibility that a product of two functions ${}_1F_1$ may be equal to a function of the type $e^{px}\,{}_2F_3$.

Equating powers of x^n in (6 . 72) gives

$$(6.73)\quad \frac{(\rho-\alpha)_n}{(\rho)_n\,n!}\,{}_3F_2\left(\begin{matrix}-n,\ 1-n-\rho,\ \alpha;\ -1\\ \rho,\ 1-n-\rho+\alpha\end{matrix}\right)$$

$$=\frac{1}{n!}\,{}_4F_3\left(\begin{matrix}\alpha,\ \rho-\alpha,\ -\tfrac{1}{2}n,\ \tfrac{1}{2}(1-n)\\ \rho,\ \tfrac{1}{2}\rho,\ \tfrac{1}{2}(1+\rho)\end{matrix}\right),$$

and this is a particular case of (6 . 6).]

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* Cf. Whipple, **2**, § 13.

† Preece, **1**, Equation 10.

‡ Preece, **1**, p. 379.

7. *A quadratic transformation.*

By writing $w = -n/\xi$ in (6 . 5), and supposing that $n \to \infty$ through integral values, we are led to the identity

$$(7\,.\,1)\quad {}_3F_2\left[\begin{matrix} x, y, z; \xi \\ 1+x-y, 1+x-z \end{matrix}\right] = (1-\xi)^{-x}\, {}_3F_2\left[\begin{matrix} 1+x-y-z, \tfrac{1}{2}x, \tfrac{1}{2}+\tfrac{1}{2}x; -4\xi/(1-\xi)^2 \\ 1+x-y, 1+x-z \end{matrix}\right].$$

The passage to the limit is somewhat troublesome to justify. A simpler proof of (7 . 1) is found by multiplying both sides by $(1-\xi)^x$. The coefficients of ξ^n are then seen to be equal in virtue of (6 . 5) itself, w in that formula being put equal to $1+x-n$.

Since $-4\xi/(1-\xi)^2$ is unaltered by the substitution of $1/\xi$ for ξ, we learn that

$$(7\,.\,2)\quad {}_3F_2\left[\begin{matrix} x, y, z; 1/\xi \\ 1+x-y, 1+x-z \end{matrix}\right] = \left(1-\frac{1}{\xi}\right)^{-x} {}_3F_2\left[\begin{matrix} 1+x-y-z, \tfrac{1}{2}x, \tfrac{1}{2}+\tfrac{1}{2}x; -4\xi/(1-\xi)^2 \\ 1+x-y, 1+x-z \end{matrix}\right].$$

As the limiting case of (7 . 1) and (7 . 2) we have

$$(7\,.\,3)\quad {}_3F_2\left[\begin{matrix} x, y, z; -1 \\ 1+x-y, 1+x-z \end{matrix}\right] = 2^{-x}\, {}_3F_2\left[\begin{matrix} 1+x-y-z, \tfrac{1}{2}x, \tfrac{1}{2}+\tfrac{1}{2}x \\ 1+x-y, 1+x-z \end{matrix}\right],$$

a known transformation* of a well-poised series.

Alternatively, by writing

$$\frac{n}{Z} = \frac{w}{1-Z} = \frac{w+n}{1}$$

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* Whipple, **1**, equation 9·5 with $e = \frac{1}{2}a$. (There is a misprint in this equation : the first of the lower parameters should be $1+a-c$.)

in (6 . 5), and making $n \to \infty$, we get

$$(7\,.\,4)\quad {}_3F_2\left[\begin{matrix} x,\ y,\ z\ ;\ Z/(Z-1) \\ 1+x-y,\ 1+x-z \end{matrix}\right] = (1-Z)^x\,{}_3F_2\left[\begin{matrix} 1+x-y-z,\ \frac{1}{2}x,\ \frac{1}{2}+\frac{1}{2}x\ ;\ 4Z-4Z^2 \\ 1+x-y,\ 1+x-z \end{matrix}\right].$$

The ${}_3F_2$ on the right is convergent if the representative point in the Z-plane lies within either loop of a certain lemniscate*. The domain of the identity is the interior of the loop which contains the origin. The domain of the identity obtained by inverting the $Z/(Z-1)$ is the other loop which surrounds $Z=1$. The limiting case (7 . 3) refers to the node.

By equating z to $\frac{1}{2}(1+x)$ we deduce, from (7 . 1), the well known formula

$$(7\,.\,5)\quad {}_2F_1\left[\begin{matrix} x,\ y\ ;\ \xi \\ 1+x-y \end{matrix}\right] = (1-\xi)^{-x}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}x-y,\ \frac{1}{2}x\ ;\ -4\xi/(1-\xi)^2 \\ 1+x-y \end{matrix}\right].$$

The domain of this identity is the same as that of (7 . 1) itself.

On the other hand, by making $z \to -\infty$ and changing the sign of ξ, we get

$$(7\,.\,51)\quad {}_2F_1\left[\begin{matrix} x,\ y\ ;\ \xi \\ 1+x-y \end{matrix}\right] = (1+\xi)^c\,{}_2F_1\left[\begin{matrix} \frac{1}{2}x,\ \frac{1}{2}+\frac{1}{2}x\ ;\ 4\xi/(1+\xi)^2 \\ 1+x-y \end{matrix}\right],$$

a formula discovered† by Gauss.

A generalization of Gauss's quadratic transformation‡

$$(7\,.\,6)\quad {}_2F_1\left[\begin{matrix} 2\alpha,\ 2\beta\ ;\ \xi \\ \frac{1}{2}+\alpha+\beta \end{matrix}\right] = {}_2F_1\left[\begin{matrix} \alpha,\ \beta\ ;\ 4\xi(1-\xi) \\ \frac{1}{2}+\alpha+\beta \end{matrix}\right]$$

can be derived from (6 . 52) by using one of Thomae's formulae for transforming the ${}_3F_2$. If in (6 . 52) we write $\frac{1}{2}x=\alpha$, $\frac{1}{2}+\frac{1}{2}x-y=\beta$, we have

$$(7\,.\,7)\quad {}_3F_2\left[\begin{matrix} 2\alpha,\ \frac{1}{2}+\alpha-\beta,\ -n \\ \frac{1}{2}+\alpha+\beta,\ w \end{matrix}\right] = \frac{\Gamma(w)\,\Gamma(w-2\alpha+n)}{\Gamma(w+n)\,\Gamma(w-2\alpha)} \times {}_4F_3\left[\begin{matrix} \alpha,\ \beta,\ 1+2\alpha-w,\ -n \\ \frac{1}{2}+\alpha+\beta,\ 1+\alpha-\frac{1}{2}w-\frac{1}{2}n,\ \frac{1}{2}+\alpha-\frac{1}{2}w-\frac{1}{2}n \end{matrix}\right],$$

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* Cf. Whittaker and Watson, *Modern Analysis*, Ch. 14, Ex. 4.

† Gauss, **1**, Equation 100.

‡ Gauss, **1**, Equation 102.

and therefore*

$$(7.71)\quad {}_3F_2\left[\begin{matrix}2\alpha,\ 2\beta,\ \frac{1}{2}+\alpha+\beta+n\\ \frac{1}{2}+\alpha+\beta,\ 2\beta+w+n\end{matrix}\right]=\frac{\Gamma(2\beta+w+n)\,\Gamma(w-2\alpha+n)}{\Gamma(2\beta+w+n-2\alpha)\,\Gamma(w+n)}$$

$$\times\,{}_4F_3\left[\begin{matrix}\alpha,\ \beta,\ 1+2\alpha-w,\ -n\\ \frac{1}{2}+\alpha+\beta,\ 1+\alpha-\frac{1}{2}w-\frac{1}{2}n,\ \frac{1}{2}+\alpha-\frac{1}{2}w-\frac{1}{2}n\end{matrix}\right].$$

That this is the required generalization is seen by writing $n/(w+n)=\xi$ and making $n\to\infty$, when we obtain Gauss's formula (7.6).

8. Another transformation of hypergeometric series in Gauss's classical paper† is

$$(8.1)\quad (1+y)^{2\alpha}\,{}_2F_1\left[\begin{matrix}\alpha,\ \alpha+\frac{1}{2}-\beta\\ \beta+\frac{1}{2}\end{matrix};\ y^2\right]={}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ 2\beta\end{matrix};\ 4y/(1+y)^2\right].$$

If both sides of (8.1) are multiplied by $(1+y)^{-2\alpha}$, comparison of the coefficients of y^n on the right and left of the equation gives, when n is an even integer,

$$(8.2)\quad {}_3F_2\left[\begin{matrix}2\alpha+n,\ -n,\ \beta\\ \alpha+\frac{1}{2},\ 2\beta\end{matrix}\right]$$

$$=\frac{\Gamma(\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})\,\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta-\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}n)\,\Gamma(\alpha+\frac{1}{2}n+\frac{1}{2})\,\Gamma(\beta+\frac{1}{2}+\frac{1}{2}n)\,\Gamma(\beta-\alpha-\frac{1}{2}n+\frac{1}{2})},$$

which is the equivalent of a formula given by Watson‡. This proof, like Watson's, requires n to be an even integer. The formula is valid, however, without that restriction§ as long as the series is convergent.

Conversely, Watson's formula, for which alternative proofs are available, may be used to establish (8.1).

Further, if both sides of (8.1) are multiplied by $(1+y)^{-2\gamma}$, comparison

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* In the notation of Whipple, **3**,

$$Fp(0;\ 4,\ 5)=Fp(0;\ 1,\ 4).$$

† Gauss, **1**, Equation 101.

‡ Watson, **1**.

§ Whipple, **3**, 113.

of powers of y leads to

$$(8.3)\quad {}_4F_3\left[\begin{matrix} a,\ 2\gamma+n,\ -n,\ \beta \\ \gamma,\ \gamma+\frac{1}{2},\ 2\beta \end{matrix}\right] = \frac{\Gamma(2\gamma)\,\Gamma(2\gamma-2a+n)}{\Gamma(2\gamma+n)\,\Gamma(2\gamma-2a)} \times {}_4F_3\left[\begin{matrix} a,\ a+\frac{1}{2}-\beta,\ -\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}n \\ \beta+\frac{1}{2},\ a-\gamma-\frac{1}{2}n+\frac{1}{2},\ a-\gamma-\frac{1}{2}n+1 \end{matrix}\right].$$

This equation is a generalization of (8 . 1), since it reduces to that form when

$$\frac{y}{-n} = \frac{1}{2\gamma+n} = \frac{1+y}{2\gamma}$$

and $n \to \infty$.

9. *Half-poised series.*

The present paper originated in an attempt to demonstrate a proposition of Cayley concerning the products of certain hypergeometric series. Cayley's proposition* is equivalent to the statement of an identity between two generalized hypergeometric series of the types ${}_4F_3(+1)$ and ${}_3F_2(+1)$. The ${}_4F_3(+1)$ is not, however, of the class to which the name nearly-poised series has been given. It is an example of what may be called a half-poised series,

$${}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d \\ 1+a-b,\ \kappa-c,\ \kappa-d \end{matrix}\right].$$

I have not been able to discover any general transformation of series of this class. Cayley's case may be written

$$(9.1)\quad {}_4F_3\left[\begin{matrix} -n,\ \frac{1}{2}-\gamma-n,\ a,\ \beta \\ \gamma+\frac{1}{2},\ 1-n+\beta-\gamma,\ 1-n+a-\gamma \end{matrix}\right] = \frac{\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)\,\Gamma(\gamma-a-\beta+n)\,\Gamma(\gamma+n)}{\Gamma(\gamma-a+n)\,\Gamma(\gamma-\beta+n)\,\Gamma(\gamma-a-\beta)\,\Gamma(\gamma)} \times {}_3F_2\left[\begin{matrix} -n,\ 2a,\ 2\beta, \\ 2\gamma,\ 1+a+\beta-\gamma-n \end{matrix}\right].$$

* Cayley, **1**.

Another case occurs implicitly in the demonstration by Edwardes* of Cayley's proposition. It is

$$(9.2)\quad {}_4F_3\left[\begin{matrix}-n,\ \frac{1}{2}-\gamma-n,\ \alpha,\ \beta\\ \gamma+\frac{1}{2},\ \frac{1}{2}-n-\alpha,\ \frac{1}{2}-n-\beta\end{matrix}\right]$$

$$=\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})\,\Gamma(\alpha+\beta+n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+\alpha+\frac{1}{2})\,\Gamma(n+\beta+\frac{1}{2})\,\Gamma(\alpha+\beta+\frac{1}{2})}$$

$$\times {}_4F_3\left[\begin{matrix}-n,\ \gamma,\ 2\alpha,\ 2\beta\\ 2\gamma,\ \alpha+\beta+\frac{1}{2},\ \frac{1}{2}-n\end{matrix}\right].$$

There is also the example found in the previous paragraph. To bring out the analogy we change the notation and write the equation (8 3) in the form

$$(9.3)\quad {}_4F_3\left[\begin{matrix}-n,\ \frac{1}{2}-\gamma-n,\ -\frac{1}{2}m,\ \frac{1}{2}-\frac{1}{2}m\\ \gamma+\frac{1}{2},\ 1-n-\delta-\frac{1}{2}m,\ \frac{1}{2}-n-\delta-\frac{1}{2}m\end{matrix}\right]$$

$$=\frac{\Gamma(2\delta+m)\,\Gamma(2\delta+2n)}{\Gamma(2\delta)\,\Gamma(2\delta+2n+m)}\ {}_4F_3\left[\begin{matrix}-n,\ \gamma,\ -m,\ 2\delta+m\\ 2\gamma,\ \delta,\ \delta+\frac{1}{2}\end{matrix}\right].$$

Here m is integral, n is not necessarily integral.

The formulae of Cayley and Edwardes are included in

$$(9.4)\quad {}_4F_3\left[\begin{matrix}-n,\ \frac{1}{2}-\gamma-n,\ \alpha,\ \beta\\ \gamma+\frac{1}{2},\ \kappa-\alpha,\ \kappa-\beta\end{matrix}\right]$$

$$=\frac{\Gamma(1-\kappa)\,\Gamma(1-\kappa+\alpha-n)\,\Gamma(1-\kappa+\beta-n)\,\Gamma(\alpha+\beta+1-\kappa)}{\Gamma(1-\kappa-n)\,\Gamma(1-\kappa+\alpha)\,\Gamma(1-\kappa+\beta)\,\Gamma(\alpha+\beta+1-n-\kappa)}$$

$$\times {}_4F_3\left[\begin{matrix}-n,\ \gamma,\ 2\alpha,\ 2\beta\\ 2\gamma,\ \kappa,\ \alpha+\beta+1-n-\kappa\end{matrix}\right],$$

but this equality is not generally valid. Obviously it does not include (9 . 3).

The question whether any general transformation of a half-poised series exists must be left open.

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* Edwardes, **1**, Equation 10.

References.

F. J. W. Whipple, **1**. "On well-poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum", *Proc. London Math. Soc.* (2), 24 (1925), 247–263.

—— **2**. "Well-poised series and other generalized hypergeometric series", *Proc. London Math. Soc.* (2), 25 (1926), 525-544.

—— **3**. "A group of generalized hypergeometric series; relations between 120 allied series of the type $F\left[\begin{matrix} a, b, c \\ e, f \end{matrix}\right]$", *Proc. London Math. Soc.* (2), 23 (1924), 104–114.

—— **4**. "Algebraic proofs of the theorems of Cayley and Orr concerning the products of certain hypergeometric series", *Journal London Math. Soc.*, 2 (1927), 85–90.

J. Thomae, **1**. "Ueber die Funktionen u.s.w.", *Journal für Math.*, 87 (1879), 26–73.

J. Dougall, **1**. "On Vandermonde's theorem and some more general expansions", *Proc. Edinburgh Math. Soc.*, 25 (1907), 114–132.

E. E. Kummer, **1**. "Ueber die hypergeometrische Reihe", *Journal für Math.*, 15 (1836), 39.

C. T. Preece, **1**. "The product of two generalized hypergeometric functions", *Proc. London Math. Soc.* (2), 22 (1924), 370-380.

C. F. Gauss, **1**. *Disquisitiones generales circa seriem infinitem, etc. Werke*, 3 (1866.) I quote from the translation by H. Simon (Berlin, 1888). The first part of the *Disquisitiones* was published in 1813, *Comm. Soc. R. Göttingensis*, 2 (1813). Gauss's intention to publish in the same form further researches on the subject was not fulfilled until 53 years later, when the extended thesis was published after his death in his collected works. The formulae to which I refer are in the supplementary part of the thesis.

G. N. Watson, **1**. "A note on generalized hypergeometric series", *Proc. London Math. Soc.*, Records for 14 February, 1924.

A. Cayley, **1**. "On a theorem relating to hypergeometric series", *Phil. Mag.* (4), 16 (1858), 356. [Collected Papers, 3, 268.]

D. Edwardes, **1**. "An expansion in factorials similar to Vandermonde's theorem, and applications", *Messenger of Math.*, 52 (1923), 129–136.

NOTES ON THE THEORY OF SERIES (IV): ON THE STRONG SUMMABILITY OF FOURIER SERIES

By G. H. HARDY *and* J. E. LITTLEWOOD.

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1. We proved in 1913* that *the Fourier series of an integrable function $f(t)$ is strongly summable $(C, 1)$ at any point of continuity x, i.e.* that, if $s_m = s_m(x)$ is the sum of the first $m+1$ terms of the series for $t = x$, then

$$\sum_0^n |s_m - f(x)| = o(n). \tag{1.1}$$

More generally and more precisely, we proved that, if $f(t)$ is of integrable square in a neighbourhood of x, and

$$\int_0^t \{\phi(u)\}^2 du = o(t), \tag{1.2}$$

where

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$

then

$$\sum_0^n (s_m - s)^2 = o(n), \tag{1.3}$$

and *a fortiori*

$$\sum_1^n |s_m - s| = o(n). \tag{1.31}$$

In particular when $f(t)$ is of integrable square in $(-\pi, \pi)$ the series is strongly summable, to sum $f(x)$, for almost all values of x.

The classical theorem of Fejér and its generalization by Lebesgue† show that, in these circumstances,

$$\sum_0^n (s_m - s) = o(n).$$

* Hardy and Littlewood, **4.**

† Lebesgue, **9.**

The interest of our theorem is that it shows (for example) that, when the Fourier series of a continuous function is not convergent, its summability is not merely a consequence of the cancelling of the various deviations summed in Fejér's mean, but rather of the comparative smallness of the deviations.

Our original results were completed and simplified in various ways by Fejér* and Fekete†. A wider extension was made in 1922 by Carleman‡, who showed that, if

$$\int_0^t \{\phi(u)\}^2 du = O(t) \tag{1.4}$$

and

$$\int_0^t |\phi(u)|\, du = o(t), \tag{1.5}$$

then

$$\sum_0^n |s_m - s|^q = o(n) \tag{1.6}$$

for all positive values of q. The conditions (1.4) and (1.5) are certainly satisfied if (1.2) is satisfied, and (1.6), by Hölder's inequality, says the more the larger q is, so that Carleman's theorem generalizes ours in every direction. His proof is also more direct.

More recently, Sutton§ succeeded in simplifying Carleman's proof considerably by the use of certain inequalities due to Hausdorff. He also replaced Carleman's condition (1.4) by the less restrictive condition

$$\int_0^t |\phi(u)|^p du = O(t), \tag{1.7}$$

where p may be any number greater than 1. He also stated another theorem, viz. that, if Carleman's second condition (1.5) is replaced by

$$\int_0^t \phi(u)\, du = o(t), \tag{1.8}$$

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* Fejér, **2.**
† Fekete, **3.**
‡ Carleman, **1.**
§ Sutton, **12.**

then

$$\sum_0^n |\sigma_m - s|^q = o(n) \tag{1.9}$$

for every q, σ_m being now the m-th Fejér mean.

The state in which the theorem has been left by all these investigations is, however, still not entirely satisfactory. In § 2 we prove a theorem which has more the air of finality. Roughly, we show that Sutton's most general hypotheses (1.7) and (1.8) involve actually not merely (1.9), but Carleman's conclusion (1.6). We then, in § 3, prove the corresponding theorem for the allied series. Finally, in §§ 4–7, we prove two theorems which may be called the "reciprocals" or "transforms" of the two earlier theorems.

2. THEOREM 1. *If* $p > 1$ *and*

$$\int_0^t |\phi(u)|^p du = O(t), \quad \int_0^t \phi(u)\, du = o(t), \tag{2.1}$$

then

$$\sum_0^n |s_m - s|^q = o(n)$$

for every positive q.

We begin by making certain standard simplifications of the data. We suppose that

$$f(t) \sim \sum_1^\infty a_n \cos nt$$

is an even function with zero mean value, and that $x = 0$, $s = 0$, so that

$$\phi(t) = f(t), \quad s_m = a_1 + a_2 + \ldots + a_m.$$

If the theorem is proved in this case it will (as is usual in the theory of Fourier series) be true generally.

Following Carleman and Sutton, we write

$$\begin{aligned} \pi s_n &= \int_0^\pi \frac{\sin(n+\frac{1}{2})t}{\sin \frac{1}{2}t} f(t)\, dt \\ &= \int_0^{k/n} \sin nt \cot \tfrac{1}{2}t\, f(t)\, dt + \int_{k/n}^\pi \sin nt \cot \tfrac{1}{2}t\, f(t)\, dt + \int_0^\pi \cos nt\, f(t)\, dt \\ &= a_n + \beta_n + \gamma_n, \end{aligned} \tag{2.2}$$

say. It is plain that $\gamma_n \to 0$, so that

$$\left(\sum_0^n |\gamma_m|^q\right)^{1/q} < \epsilon n^{1/q}, \tag{2.3}$$

for $n > n_1(\epsilon)$.

Next, if $$\eta = \frac{k}{n}, \quad F(t) = \int_0^t f(u)\,du,$$

we have

$$\alpha_n = \sin n\eta \cot \tfrac{1}{2}\eta\, F(\eta) - \int_0^\eta (n \cos nt \cot \tfrac{1}{2}t - \tfrac{1}{2} \sin nt \operatorname{cosec}^2 \tfrac{1}{2}t)\, F(t)\,dt$$

$$= o(1) + \int_0^\eta o\left\{\left(\frac{n}{t} + \frac{nt}{t^2}\right)t\right\} dt = o(1) \to 0,$$

when k is fixed. It follows that, when k is fixed, we have

$$\left(\sum_0^n |\alpha_m|^q\right)^{1/q} < \epsilon n^{1/q} \tag{2.4}$$

for $n > n_2(\epsilon) = n_2(\epsilon, k)$. So far we have made no use of the first of our hypotheses (2.1).

In discussing β_n we assume that*

$$q > 2, \quad q' = \frac{q}{q-1} < p,$$

this being legitimate because the conclusion is stronger the larger be q. We denote by $c_n(\tau)$ the n-th Fourier sine coefficient of the odd function $\chi(t)$ which is equal to $f(t)$ in $(0, \tau)$ and to zero in (τ, π). We have then

$$\beta_n = \int_\eta^\pi \sin nt \cot \tfrac{1}{2}t\, f(t)\,dt = \int_\eta^\pi \cot \tfrac{1}{2}t \left(\frac{d}{dt}\int_0^t \sin mu\, f(u)\,du\right) dt \tag{2.5}$$

$$= -\tfrac{1}{2}\pi \cot \tfrac{1}{2}\eta\, c_m(\eta) + \tfrac{1}{4}\pi \int_\eta^\pi \operatorname{cosec}^2 \tfrac{1}{2}t\, c_m(t)\,dt.$$

From (2.5) it follows, by Minkowski's inequality, that

$$\left(\sum_1^n |\beta_m|^q\right)^{1/q} \leqslant \tfrac{1}{2}\pi \cot \tfrac{1}{2}\eta \left(\sum_1^n |c_m(\eta)|^q\right)^{1/q} + \tfrac{1}{4}\pi \int_\eta^\pi \operatorname{cosec}^2 \tfrac{1}{2}t \left(\sum_1^n |c_m(t)|^q\right)^{1/q} dt.$$

* We write generally r' for $r/(r-1)$ when $r > 1$.

But, by Hausdorff's inequality*,

$$\left(\sum_1^n |c_m(t)|^q\right)^{1/q} \leqslant \left(\frac{1}{\pi}\int_{-\pi}^{\pi} |\chi(u)|^{q'}\,du\right)^{1/q'}$$

$$= \left(\frac{1}{\pi}\int_{-t}^{t} |f(u)|^{q'}\,du\right)^{1/q'} \leqslant At^{1/q'},$$

where A is a constant, since the first condition (2 . 1) is satisfied *a fortiori* when p is replaced by the smaller index q'. Hence

$$(2.6)\qquad \left(\sum_1^n |\beta_m|^q\right)^{1/q} \leqslant \frac{A}{\eta}\cdot\eta^{1/q'} + A\int_\eta^\pi \frac{1}{t^2}\cdot t^{1/q'}\,dt \leqslant A\eta^{-1/q}$$

$$= Ak^{-1/q}n^{1/q} = \epsilon_k n^{1/q},$$

where ϵ_k is a function of k only which tends to zero when $k\to\infty$.

From (2 . 2), (2 . 3), (2 . 4), and (2 . 6) it follows that

$$\pi\left(\sum_1^n |s_n|^q\right)^{1/q} \leqslant \left(\sum_1^n |\alpha_n|^q\right)^{1/q} + \left(\sum_1^n |\beta_n|^q\right)^{1/q} + \left(\sum_1^n |\gamma_n|^q\right)^{1/q}$$

$$< (\epsilon_k + 2\epsilon)\,n^{1/q},$$

if n is greater than n_1 or n_2, and our conclusion follows by choice first of k and then of n.

3. The corresponding theorem for the allied series is :

THEOREM 2. *If $\bar{s}_m$ is the sum of the first $m+1$ terms of the series allied to the Fourier series of $f(t)$; if*

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\},$$

and

$$\int_0^t |\psi(u)|^p\,du = O(t) \quad (p > 1);$$

and if

$$\frac{1}{\pi}\int_0^\pi \psi(t)\cot\tfrac{1}{2}t\,dt$$

exists, as a Cauchy integral†, and has the value $\bar{s}$; then

$$\sum_0^n |\bar{s}_m - \bar{s}|^q = o(n)$$

for every positive q.

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* Hausdorff, **8**.

† See Hardy and Littlewood, **7**, 212.

We may reduce the problem in the standard manner to the special case in which

$$f(t) \sim \sum_1^\infty b_n \sin nt$$

is an odd function, $x = 0$, and $s = 0$. The allied series is then Σb_n, and $\psi(t) = f(t)$. We have

$$\pi \bar{s}_n = \pi(b_1+b_2+\ldots+b_n) = \tfrac{1}{2}\int_{-\pi}^{\pi} \frac{\cos\frac{1}{2}t - \cos(n+\frac{1}{2})t}{\sin\frac{1}{2}t} f(t)\,dt$$

$$= -\int_0^{\pi} \frac{\cos(n+\frac{1}{2})t}{\sin\frac{1}{2}t} f(t)\,dt$$

$$= -\int_0^{k/n} \cos nt \cot\tfrac{1}{2}t\, f(t)\,dt - \int_{k/n}^{\pi} \cos nt \cot\tfrac{1}{2}t\, f(t)\,dt + \int_0^{\pi} \sin nt\, f(t)\,dt$$

$$= \alpha_n + \beta_n + \gamma_n,$$

say. It is plain that $\gamma_n \to 0$.

Writing, as before, η for k/n, we observe first that $\cos nt$ has less than Ak turning values in $(0, \eta)$, so that we may divide the first interval into less than Ak pieces, in each of which $\cos nt$ is monotonic. If we do this, and apply the second mean value theorem to the integral over each piece, we see that $|\alpha_n|$ is not greater than the sum of Ak terms of the type

$$\left| \int_\zeta^{\zeta'} \cot\tfrac{1}{2}t\, f(t)\,dt \right|,$$

where $0 \leqslant \xi < \xi' \leqslant \eta$. It follows that $\alpha_n \to 0$ when k is fixed and $n \to \infty$. Thus α_n and γ_n behave as in § 2. The remaining term β_n may be dealt with just as in § 2, the only difference being that $c_n(\tau)$ must now be defined as the Fourier cosine coefficient of an even function; so that the theorem requires no further proof. In particular, the allied series is strongly summable at any point at which $f(t)$ is continuous and the integral which expresses its sum exists.

4. We shall now apply to Theorems 1 and 2 a process of formal transformation which is fruitful in the generation of new theorems in the theory of trigonometrical series. The genesis of the principle is naturally to be found in the theory of "Fourier transforms". Taking Theorem 1 in the reduced form

(A) "*if* $\int_0^t |f|^p\,du = O(t)$, $F(t) = \int_0^t f\,du = o(t)$, *then* $\sum_1^n |s_m|^q = o(n)$",

we replace t by n, $f(t)$ by a_n, integration over $(0, t)$ by summation over $(1, n)$, and conversely, so that s_n becomes $F(t)$. We thus obtain

(B) "*if* $\sum_1^n |a_m|^p = O(n)$, $\sum_1^n a_m = o(n)$, *then* $\int_0^t |F|^q du = o(t)$".

If now $f \sim \Sigma a_n \cos nt$, then

$$F = \Sigma \frac{a_n}{n} \sin nt = \Sigma b_n \sin nt,$$

say; and, when we express (B) in terms of b_n and F, we obtain

(C) "*if* $\sum_1^n (m|b_m|)^p = O(n)$, $\sum_1^n m b_m = o(n)$, *then* $\int_0^t |F|^q du = o(t)$".

Finally, passing from a "reduced" to a "general" form, we are led to formulate

THEOREM 3. *If* $p > 1$ *and*

$$B_n = b_n \cos nx - a_n \sin nx, \tag{4.1}$$

$$\sum_1^n (m|B_m|)^p = O(n), \quad \sum_1^n m B_m = o(n), \tag{4.2}$$

then there is a function $f(t)$, *belonging to the class* L^q *for every* q, *whose Fourier series is*

$$\Sigma A_n = \Sigma (a_n \cos nt + b_n \sin nt); \tag{4.3}$$

and $f(t)$ *satisfies the equation*

$$\int_0^t |\psi(u)|^q du = \int_0^t |\tfrac{1}{2}\{f(x+u) - f(x-u)\}|^q du = o(t) \tag{4.4}$$

for every q.

The last of our tentative enunciations was, in substance, the special case of Theorem 3 in which $x = 0$. The hypotheses of the theorem are fulfilled, for example, whenever $na_n = o(1)$ and $nb_n = o(1)$, or, again, whenever $na_n = O(1)$, $nb_n = O(1)$, and the allied series ΣB_n is convergent or summable.

It may help to elucidate the relation of Theorem 3 to known theorems if we return for a moment to the reduced form (C), and suppose, in particular, that $nb_n = o(1)$, when the conditions are obviously satisfied. The conclusion involves *a fortiori*

$$\int_0^t F du = \Sigma b_n \frac{1 - \cos nt}{n} = o(t)$$

or (changing t to $2t$ and writing a_n again for nb_n)

$$\Sigma a_n \frac{\sin^2 nt}{n^2} = o(t).$$

That this is so whenever $a_n = o(1)$ is a familiar theorem of Riemann*. This theorem is the "transform" of Fejér's theorem, and Theorem 3 is a development of the one as Theorem 1 is of the other.

5. In proving Theorem 3 we may confine ourselves to the case in which $x = 0$, $B_n = b_n$. We require some simple lemmas.

LEMMA α. *If*

$$\sum_1^n (m|c_m|)^p = O(n) \tag{5.1}$$

for any positive p, then the same equation is true for any smaller positive p.

This is an immediate corollary of Hölder's inequality.

LEMMA β. *If* (5.1) *is true for a $p \geq 1$, then $\Sigma|c_n|^q$ is convergent for every $q > 1$, and*

$$\sum_n^\infty |c_m|^q = O(n^{1-q}) \tag{5.2}$$

for $1 < q \leqslant p$.

If

$$\gamma_{n,m} = \sum_n^m (\nu|c_\nu|)^q,$$

then $\gamma_{n,m} = O(m)$ if $q < p$, by Lemma α. And

$$\sum_n^N |c_m|^q = \sum_n^{N-1} \gamma_{n,m} \Delta m^{-q} + \gamma_{n,N} N^{-q}$$

$$= O\left(\sum_n^N m^{-q}\right) + O(N^{1-q}) = O(n^{1-q}).$$

This proves (5.2). If the series is convergent for any q, it is naturally convergent for any larger q.

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* See, *e.g.*, E. W. Hobson, *Theory of functions of a real variable*, second edition (1926), 2, 647.

LEMMA γ. *If c_n is real and satisfies (5.1) for a $p>1$, then there is an $f(t)$, belonging to every Lebesgue class L^q, of which c_n is the Fourier (sine or cosine) coefficient.*

This is an immediate corollary of Lemma β and Hausdorff's theorem*.

6. Passing to the proof of Theorem 3, we may suppose, as in § 2, that $q>2$ and $q'<p$ (or $q>p'$). By Lemma γ,

$$f \sim \Sigma b_n \sin nt$$

exists and belongs to L^q for every q. And if

$$f_n = \sum_1^n b_m \sin mt,$$

then, by Hausdorff's inequality and Lemma β,

$$\int_0^\pi |f-f_n|^q\,dt \leqslant A\Big(\sum_{n+1}^\infty |b_n|^{q'}\Big)^{q-1} \leqslant \frac{A}{n}, \tag{6.1}$$

the A's here being positive numbers depending only on q†.

We suppose that $0 \leqslant u \leqslant t$, $0<h<k$‡, and write

$$n_1 = h/t, \quad n_2 = k/t,$$

$$f(u) = \sum_1^{n_1} b_m \sin mu + \sum_{n_1+1}^{n_2} b_m \sin mu + \{f(u)-f_{n_2}(u)\}$$

$$= S_1+S_2+S_3,$$

say. We have first

$$|S_1| \leqslant u \sum_1^{n_1} m|b_m| \leqslant An_1u \leqslant Ah,$$

$$\int_0^t |S_1|^q\,du \leqslant Ah^q t;$$

and next

$$\int_0^t |S_3|^q\,du \leqslant \frac{A}{n_2} = \frac{At}{k},$$

by (6.1). Finally

$$S_2 = u \sum_{n_1+1}^{n_2} mb_m \frac{\sin mu}{mu}.$$

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* The original form due to Young (**14**) would be sufficient.

† It will not be necessary to distinguish between A's which depend or do not depend on q.

‡ h is to be thought of as small, k as large.

The range of integration may be divided into at most Ak pieces in which the last factor is monotonic. Applying the second mean value theorem to each piece, and using the second condition (4 . 2), we see that

$$S_2 = o(un_2) = o(1),$$

when h and k are fixed and $t \to 0$. It follows that

$$|S_2| < \epsilon, \quad \int_0^t |S_2|^q \, du < \epsilon^q \, t$$

for $0 < t \leqq t_1 = t_1(\epsilon, h, k)$.

Hence

$$\left(\int_0^t |f(u)|^q \, du\right)^{1/q} \leqslant \left(\int_0^t |S_1|^q \, du\right)^{1/q} + \left(\int_0^t |S_2|^q \, du\right)^{1/q} + \left(\int_0^t |S_3|^q \, du\right)^{1/q}$$

$$< t^{1/q}(Ah + Ak^{-1/q} + \epsilon),$$

for $0 < t \leqq t_1$. Our conclusion follows by choice of ϵ, h, k, and t_1.

7. The theorem which corresponds to Theorem 2 as Theorem 3 corresponds to Theorem 1 is

THEOREM 4. *If*

$$\sum_1^n (m|A_m|)^p = O(n) \tag{7 . 1}$$

and ΣA_n is convergent to sum s, then there is an $f(t)$, belonging to L^q for every q, whose Fourier series is ΣA_n; and $f(t)$ satisfies

$$\int_0^t |\phi(u)|^q \, du = o(t) \tag{7 . 2}$$

for every q.

If we suppose that $x = 0$, $A_n = a_n$, and replace (7 . 1) by the stronger hypothesis $na_n = O(1)$ and (7 . 2) by the weaker conclusion

$$\int_0^t \phi(u) \, du = o(t),$$

we obtain the theorem that

"*if $na_n = O(1)$ and Σa_n is convergent to sum s, then*

$$\Sigma a_n \frac{\sin nt}{nt} \to s$$

when $t \to 0$".

This theorem, which is in its turn a generalization of a theorem of Fatou, we proved in our paper **5***. It is the "transform" of Young's theorem concerning the summability of the allied series†.

In proving Theorem 4 we need consider only the reduced case in which $x = 0$, $\phi = f$, and we may suppose also $s = 0$. We now write

$$f \sim \Sigma a_n \cos nt, \quad f_n = \sum_1^n a_m \cos mt,$$

$$f(u) = \sum_1^{n_2} a_m - \sum_1^{n_1} a_m(1-\cos mu) - \sum_{n_1+1}^{n_2} a_m(1-\cos mu) + f(u) - f_{n_2}(u)$$

$$= S_0 + S_1 + S_2 + S_3.$$

Here $S_0 = o(1)$ and

$$|S_2| \leqslant \tfrac{1}{2}u^2 \sum_1^{n_1} m^2 |a_m| \leqslant Au^2 n_1^2 \leqslant Ah^2;$$

and the rest of the proof follows the same lines as that of Theorem 3.

8. There is another lemma which enables us to generalize the conditions of Theorems 3 and 4, but which we include rather on account of its intrinsic interest.

LEMMA δ. *If c_n satisfies* (5.1), *and Σc_n is summable* (C), *then Σc_n is convergent.*

The conclusion is indeed true whenever Σc_n is summable by Abel's limit, and the lemma may be deduced, in this stronger form, from R. Schmidt's extension‡ of Littlewood's converse of Abel's theorem. As stated, it is a generalization of Hardy's theorem concerning Cesàro summability, and it is interesting to give a direct proof. In order that Σc_n should be summable (C, r), where $r \geqslant 0$, it is necessary and sufficient that Σb_n, where now

$$b_n = \frac{c_n}{n} + \frac{c_{n+1}}{n+1} + \ldots \quad (C, r-1), \tag{8.1}$$

* Hardy and Littlewood, **5**.

† See Hardy and Littlewood, **7**, 212.

‡ Schmidt, **11**. For a more direct proof, see Vijayaraghavan, **13**.

should be summable $(C, r-1)$*. In the present case the series (8 . 1) is (absolutely) convergent, and if $q < p$,

$$|b_n| \leqslant \left(\sum_n^\infty |c_m|^q\right)^{1/q} \left(\sum_n^\infty m^{-q'}\right)^{1/q'} = O\left(\frac{1}{n}\right)$$

by Lemma β. Hence Σb_n is convergent and Σc_n is summable $(C, 1)$.

Also, if $s_n = c_1 + \ldots + c_n$, we have

$$|s_{n+\delta n} - s_n| \leqslant \sum_n^{n+\delta n} |c_m| \leqslant \left(\sum_n^{n+\delta n} m^p |c_m|^p\right)^{1/p} \left(\sum_n^{n+\delta n} m^{-p'}\right)^{1/p} < \epsilon_\delta,$$

where ϵ_δ is a function of δ only which tends to zero when $\delta \to 0$. The convergence of Σc_n now follows by the argument usual in the proof of Hardy's theorem. It is of interest to observe that the result is not true when $p = 1$†.

It follows that, in Theorems 3 and 4, we may replace the hypotheses

$$\text{"}\sum_1^n m b_m = o(n)\text{"}, \quad \text{"}\Sigma A_m \textit{ is convergent}\text{"},$$

by "ΣB_m (*or* ΣA_m) *is summable by some Cesàro mean*".

Additional note (8 *March*, 1927).

We have recently found an alternative proof of Theorems 1 and 2, which seems to us in some ways preferable to that given in the text.

We take Theorem 1 in the "reduced" form, and suppose, as we may, that $q > 2$ and $q > p'$. It is plainly sufficient to show that

$$\mathrm{S}_n^q = \sum_1^n |\mathrm{s}_m|^q = o(n),$$

where

$$\mathrm{s}_n = \int_0^\pi \frac{f(t)}{t} \sin nt \, dt.$$

We write‡

$$c_m = |\mathrm{s}_m|^{q-1} \operatorname{sgn} \bar{\mathrm{s}}_m, \qquad C_n(t) = \sum_1^n c_m \sin mt, \qquad \Gamma_n = \sum_1^n |c_m|.$$

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* Hardy and Littlewood, **6**, 75.

† This may be shown by an example due to Neder (**1**, 180).

‡ $\operatorname{sgn} z = z/|z|$ if $z \neq 0$; $\operatorname{sgn} 0 = 0$; and $\bar{z}$ is the conjugate of z. Note that s_n (which alone occurs in the sequel) is a different symbol from s_n.

Then

$$S_n' = \int_0^\pi \frac{f(t)}{t}\, C_n(t)\, dt = \int_0^{k/n} + \int_{k/n}^\pi = J_1 + J_2, \tag{1}$$

say. We have clearly

$$|C_n(t)| \leqslant \Gamma_n, \quad |C_n(t)| \leqslant nt\Gamma_n, \quad |C_n'(t)| \leqslant n\Gamma_n.$$

Hence

$$J_1 = \left[\frac{F(t)\,C_n(t)}{t}\right]_0^{k/n} - \int_0^{k/n} F(t) \left\{\frac{C_n'(t)}{t} - \frac{C_n(t)}{t^2}\right\} dt \tag{2}$$

$$= o(\Gamma_n) + \int_0^{k/n} o(t) \left\{O\left(\frac{n\Gamma_n}{t}\right) + O\left(\frac{nt\Gamma_n}{t^2}\right)\right\} dt$$

$$= o(\Gamma_n) = o(n^{1/q} S_n^{q/q'}),$$

by Hölder's inequality.

On the other hand

$$|J_2| \leqslant \left(\int_0^\pi |C_n(t)|^q\, dt\right)^{1/q} \left(\int_{k/n}^\pi \left|\frac{f(t)}{t}\right|^{q'} dt\right)^{1/q'} \tag{3}$$

$$\leqslant \left(\pi \sum_1^n |c_m|^{q'}\right)^{1/q'} \left(\int_{k/n}^\pi \frac{\psi}{t^{q'}}\, dt\right)^{1/q'} \leqslant \pi S_n^{q/q'} I^{1/q'},$$

where $\psi = |f|^{q'}$, so that

$$\Psi(t) = \int_0^t \psi(u)\, du = \int_0^t |f(u)|^{q'} du \leqslant At,$$

where A is a constant. But

$$I = \int_{k/n}^\pi \frac{\psi}{t^{q'}}\, dt = \left[\frac{\Psi(t)}{t^{q'}}\right]_{k/n}^\pi + q' \int_{k/n}^\pi \frac{\Psi(t)\, dt}{t^{q'+1}} \tag{4}$$

$$\leqslant \frac{\Psi(\pi)}{\pi^{q'}} + \frac{A_{q'}}{q'-1} \left(\frac{n}{k}\right)^{q'-1} < \{o(1) + \epsilon_k\} n^{q'-1},$$

where ϵ_k is a function of k only which tends to zero with k. From (1)–(4) we deduce

$$S_n^{1/q} \leqslant |J_1|^{1/q} + |J_2|^{1/q} \leqslant \{\epsilon_k + o(1)\}\, n^{1/q} S_n^{1/q'},$$

$$S_n < \{\epsilon_k + o(1)\}\, n^{1/q},$$

and the theorem follows. Theorem 2 may be proved similarly.

Memoirs referred to.

1. T. Carleman, "A theorem concerning Fourier series", *Proc. London Math. Soc.* (2), 21 (1923), 483–492.
2. L. Fejér, "Fourierreihe und Potenzreihe", *Monatshefte für Math.*, 28 (1917), 64–76.
3. M. Fekete, "Vizsgálatok a Fourier sorokrol", *Math. és termesz. ert.*, 34 (1916), 759–786.
4. G. H. Hardy and J. E. Littlewood, "Sur la série de Fourier d'une fonction à carré sommable", *Comptes rendus*, 28 April, 1913.
5. ——, "Abel's theorem and its converse", *Proc. London Math. Soc.* (2), 18 (1920), 205–235.
6. ——, "Solution of the Cesàro summability problem for power series and Fourier series", *Math. Zeitschrift*, 19 (1923), 67–96.
7. ——, "The allied series of a Fourier series", *Proc. London Math. Soc.* (2), 24 (1925), 211–246.
8. F. Hausdorff, "Eine Ausdehnung der Parsevalschen Satzes über Fourierreihen", *Math. Zeitschrift*, 16 (1923), 163–169.
9. H. Lebesgue, "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, 61 (1903), 251–280.
10. L. Neder, "Über Taubersche Bedingungen", *Proc. London Math. Soc.* (2), 23 (1925), 172–184.
11. R. Schmidt, "Über divergente Folgen und lineare Mittelbildungen", *Math. Zeitschrift*, 22 (1925), 89–152.
12. O. G. Sutton, "On a theorem of Carleman", *Proc. London Math. Soc.* (2), 23 (1925), xlviii–li (*Records* for 24 April, 1924).
13. T. Vijayaraghavan, "A Tauberian theorem", *Journal London Math. Soc.*, 1 (1926), 113–120.
14. W. H. Young, "On the determination of the summability of a function by means of its Fourier constants", *Proc. London Math. Soc.* (2), 12 (1912), 71–88.

NOTES ON THE THEORY OF SERIES (V): ON PARSEVAL'S THEOREM

G. H. HARDY *and* J. E. LITTLEWOOD.

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1. Suppose that $f(\theta)$ is a real, periodic, and integrable function of θ, and that

$$(1.1) \qquad \tfrac{1}{2}a_0+\sum_1^\infty(a_n\cos n\theta+b_n\sin n\theta), \quad \sum_1^\infty(b_n\cos n\theta-a_n\sin n\theta)$$

are its Fourier series and the conjugate or allied series. We say that f belongs to L^p, where $p\geq 1$, if $|f|^p$ is integrable. A fundamental theorem of M. Riesz* asserts that the allied series is then also a Fourier series, and that the function $-g(\theta)$ of which it is the Fourier series, and which we call the function conjugate to $f(\theta)$, itself belongs to L^p. The complex function

$$(1.2) \qquad H(z)=\Sigma c_n z^n=\Sigma(a_n-ib_n)r^n e^{ni\theta}$$

is then an analytic function of z regular for $r=|z|<1$, and $h(\theta)=f(\theta)-ig(\theta)$ is the "boundary function" of $H(z)$, that is to say the function defined, for almost all θ, by the values assumed by $H(z)$ when z approaches the unit circle radially. Also

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}h(\theta)\,e^{-ni\theta}\,d\theta;$$

and

$$(1.3) \qquad \int_{-\pi}^{\pi}|H(re^{i\theta})|^p\,d\theta \quad (r<1)$$

is bounded, and indeed increases steadily to $\int|h|^p\,d\theta$ when $r\to 1$.

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* M. Riesz, 5.

It is naturally often convenient to consider the two functions f and g together, in the form $h = f - ig$. We write

$$h(\theta) \sim \sum_0^\infty c_n e^{ni\theta}. \tag{1.4}$$

Here h is not an arbitrary complex function, its real and imaginary parts being determined by one another.

We may, on the other hand, start from a quite arbitrary complex and integrable h. Such a function will not usually be the boundary function of an analytic function. If

$$c_n = \tfrac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\, e^{-ni\theta}\, d\theta, \tag{1.5}$$

we write

$$h(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{ni\theta}, \tag{1.6}$$

and call this series the Fourier series of h. It will be convenient to suppose throughout, as plainly we may without real loss of generality, that $c_0 = 0$. In these circumstances $h = f - ig$, where f and g are real functions whose Fourier series are

$$\sum_1^\infty (a_n \cos n\theta + \beta_n \sin n\theta), \qquad \sum_1^\infty (\delta_n \cos n\theta - \gamma_n \sin n\theta), \tag{1.7}$$

with

$$2a_n = a_n + a_{-n}, \quad 2\beta_n = b_n - b_{-n}, \quad 2\gamma_n = a_n - a_{-n}, \quad 2\delta_n = b_n + b_{-n}.$$

These series are conjugate only when $a_n = \gamma_n$, $\beta_n = \delta_n$, *i.e.* when $a_n = b_n = 0$ for all negative n.

We shall call (1.6) a *general* Fourier series, and a Fourier series of the type (1.4), or a similar series extended only over negative values of n, a *Fourier* (positive or negative) *power series*. It follows from Riesz's theorem that, if the series (1.7) are the (ordinary) Fourier series of functions of L^p, where $p > 1$, then so are the series

$$\sum_1^\infty (\gamma_n \cos n\theta + \delta_n \sin n\theta), \qquad \sum_1^\infty (\beta_n \cos n\theta - a_n \sin n\theta),$$

and therefore, by addition, the series (1.1). In other words, Riesz's theorem may be stated, from our present point of view, as follows: *if* $\Sigma c_n e^{ni\theta}$ *is the general Fourier series of a function of* L^p, *then the two*

halves of the series corresponding to positive and negative values of n are the Fourier power series (positive or negative) *of functions of* L^p.

The upshot of Riesz's theorem is then, to put the matter roughly, that, so long as we are concerned with functions of a Lebesgue class greater than 1, it is indifferent whether we consider Fourier power series or general Fourier series. There are, however, many parts of our recent researches where the distinction is essential. For example, if $\Sigma c_n e^{ni\theta}$ is a Fourier power series, its integrated series is absolutely convergent, while the corresponding theorem for general Fourier series is false.

2. The classical form of "Parseval's theorem" may be stated as follows. If

$$f \sim \sum_{-\infty}^{\infty}{}' a_n e^{ni\theta}, \qquad g \sim \sum_{-\infty}^{\infty}{}' b_n e^{ni\theta} \tag{2.1}$$

are functions of L^2, then

$$\Sigma a_n b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, g(-\theta)\, d\theta. \tag{2.2}$$

The convergence of the series is absolute, since $\Sigma|a_n|^2$ and $\Sigma|b_n|^2$ are convergent.

Riesz, in the investigations which we have quoted, gives a very beautiful extension of Parseval's theorem. He supposes that f and g belong respectively to L^p and $L^{p'}$, p and p' being "conjugate indices" satisfying

$$p > 1, \quad p' = \frac{p}{p-1} > 1, \quad \frac{1}{p} + \frac{1}{p'} = 1 \tag{2.3}$$

(a notation which we adopt throughout, and for letters other than p), and he proves that (2.2) is still true under these conditions*. There is, however, an essential difference between the general case and the special case $p = 2$, in that the series (2.2) is not generally *absolutely* convergent.

Our recent researches have led us to consider Parseval's theorem in a still more general form. We suppose that f and g belong to L^p and L^q, p and q being greater than 1 but no longer conjugate. The series

* It had previously been proved by Young (7) that (2.2) holds when the series is summed $(C, 1)$.

which is then relevant is not (2 . 2) but

$$\Sigma |n|^{-\mu} e^{-\frac{1}{2}\mu\pi i \operatorname{sgn} n} a_n b_n, \tag{2.4}$$

where

$$\mu = \frac{1}{p} + \frac{1}{q} - 1; \tag{2.41}$$

the exponential factor, though important if we wish to write down a formula for the sum of the series, is irrelevant to its convergence, since it is constant for positive and for negative values of n.

We may suppose that $p \leqslant q$, and there are then four possibilities as regards the magnitudes of p and q, viz.

$$(a)\ p \leqslant q \leqslant 2\ (\leqslant p'), \qquad (b)\ (p <)\ 2 < q \leqslant p',$$

$$(c)\ (p \leqslant)\ 2 \leqslant p' < q, \qquad (d)\ (p' <)\ 2 < p \leqslant q.$$

Theorems which we have proved recently, together with Riesz's theorem, enable us to settle the problem in all cases.

3. THEOREM 1. *In case* (*a*) *the series* (2 . 4) *is absolutely convergent. In case* (*b*) *it is convergent, but not necessarily absolutely convergent. In cases* (*c*) *and* (*d*) *it is not necessarily convergent. In cases* (*a*) *and* (*b*) *the number* μ *is the best possible; the series may cease to converge when* μ *is replaced by any smaller number.*

We use, besides Riesz's theorem, two of our own. The first* is

THEOREM *a*. *If* f *belongs to* L^p, *and* $p \leqslant q \leqslant p'$, *then*

$$\Sigma |n^{-\mu} a_n|^q \tag{3.1}$$

is convergent.

The second requires a word of preliminary explanation. Following Weyl†, we define $f_\alpha(\theta)$, the integral of $f(\theta)$ of order α, where $0 < \alpha < 1$, by

$$f_\alpha(\theta) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\theta} f(t)(\theta - t)^{\alpha-1}\, dt.$$

The Fourier series of $f_\alpha(\theta)$ is

$$\Sigma |n|^{-\alpha} e^{-\frac{1}{2}\alpha\pi i \operatorname{sgn} n} a_n e^{ni\theta}.$$

* Hardy and Littlewood, **3** (Theorem 10).

† Weyl, **6.**

The theorem which we require is*

THEOREM β. *If f belongs to L^p, and $0<\alpha<1/p$, then f_α belongs to $L^{p/(1-p\alpha)}$.*

With the help of these theorems and Riesz's theorem, it is very easy to prove Theorem 1. The simplicity of the proof is naturally somewhat illusory, since the theorems on which it depends are difficult.

If one of (a) or (b) holds, then $p \leqslant q \leqslant p'$, and conversely. Suppose first that $p \leqslant q \leqslant 2$ [case (a)]. Since g belongs to L^q, the series $\Sigma|b_n|^{q'}$ is convergent, by Hausdorff's theorem†. But, by Hölder's inequality,

$$\Sigma|n^{-\mu}a_n b_n| \leqslant (\Sigma|n^{-\mu}a_n|^q)^{1/q}(\Sigma|b_n|^{q'})^{1/q'},$$

and therefore, by Theorem α, the series on the left is convergent.

Next, suppose that we are in either case (a) or case (b). Since $q>1$, $\mu<1/p$, and we may take $\alpha=\mu$ in Theorem β. Hence f_μ belongs to $L^{q'}$. The convergence of (2 . 4) then follows from Riesz's theorem.

The remainder of the proof requires only the production of appropriate "Gegenbeispiele". We observe first that if

$$f=\sum_1^\infty n^{-1-\delta+1/p}e^{ni\theta}, \qquad g=\sum_1^\infty n^{-1-\delta+1/q}e^{ni\theta} \quad (\delta>0), \tag{3 . 2}$$

then f belongs to L^p and g to L^q. The series $\Sigma n^{-\nu}a_n b_n$ is then $\Sigma n^{-1-2\delta+\mu-\nu}$, which diverges if $\nu<\mu$ and δ is sufficiently small. There is therefore in no case any possibility of replacing μ by a smaller index.

Next we take

$$f=g=\sum_1^\infty 2^{-\delta n}\cos 2^n\theta \quad (\delta>0), \tag{3 . 3}$$

so that f and g are continuous. The series (2 . 4) is then $\Sigma 2^{-(\mu+2\delta)n}$, which diverges if $\mu<0$ and δ is sufficiently small. We can therefore only infer convergence if $q\leqslant p'$, and so $p\leqslant p'$ or $p\leqslant 2$. This shows that (a) and (b) are the only cases in which the series is necessarily convergent.

Finally we choose f as in (3 . 2), but take

$$g=\sum_1^\infty n^{-\frac{1}{2}-\delta}e^{in\log n+ni\theta} \quad (\delta>0).$$

* Hardy and Littlewood, **2** (Theorem 3). No proof is given there, but a proof is included in a memoir which we are preparing for publication.

† Hausdorff, **4**.

Then g is continuous*, while f, as before, belongs to L^p. The series of moduli is now

$$\Sigma n^{-\frac{1}{2}-2\delta-1/q},$$

and diverges if $q>2$ and δ is sufficiently small. Thus we cannot assert *absolute* convergence in case (b).

The sum of the series, in all the cases in which Theorem 1 asserts its convergence, is given by the formula

$$\Sigma|n|^{-\mu}e^{-\frac{1}{2}\mu\pi i\operatorname{sgn}n}a_n b_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_\mu(\theta)\,g(-\theta)\,d\theta$$

(or the corresponding formula in which f and g are interchanged). This requires no further proof, as it follows directly from Riesz's theorem, at the point at which we appeal to this theorem above.

4. We may observe that the first paragraph of our proof proves more than is actually incorporated in Theorem 1, and the position may become clearer if we embody the additional information which we have obtained in an explicit theorem, though this theorem really adds nothing to Theorem a.

THEOREM 2. *The series* (2 . 4) *is absolutely convergent whenever* f *belongs to* L^p, *where* $p\leqslant 2$, *and the series* $\Sigma|b_n|^{q'}$, *where* $p\leqslant q\leqslant p'$, *is convergent.*

In case (a) this asserts more than Theorem 1, since $\Sigma|b_n|^{q'}$ is convergent whenever g belongs to L^q, while the converse proposition is false. In case (b) the hypotheses are *stronger* than those of Theorem 1†, and the conclusion (to absolute convergence) also stronger.

One further question remains. Since the hypotheses are now unsymmetrical, the assumption that $p\leqslant q$ entails a loss of generality, and it may be asked whether we have really exhausted the question. It is however easy to see that the hypothesis $p\leqslant q\leqslant p'$ is essential to Theorem 2. The conclusion involves the convergence of (3 . 1). Take now

$$f=\sum_{2}^{\infty}n^{-1+1/p}(\log n)^{-\alpha}e^{ni\theta}\quad(\alpha>0).$$

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* Hardy and Littlewood, **1.**

† The converse inference being valid and not the direct one.

Then f is regular except for $\theta = 0$, where it behaves like a multiple of

$$|\theta|^{-1/p}\left(\log\frac{1}{|\theta|}\right)^{-\alpha}.$$

Thus f belongs to L^p if $p\alpha > 1$. The series (3 . 1), on the other hand, is convergent only when $q\alpha > 1$. If $q < p$, the first hypothesis does not involve the second. We must therefore have $p \leqslant q$. Finally, the example (3 . 3) still shows that we must have $q \leqslant p'$.

5. So far we have supposed p and q to be greater than 1. Our theorems may, however, be extended, with appropriate reservations, to all positive values of p and q, and here the distinctions of § 1 become important. We have to appeal at one point to a theorem* which we have not stated before.

Suppose that $H(z)$ is the function (1 . 2), that $0 < p \leqslant 1$, and that the integral (1 . 3) is bounded. We shall then say that $H(z)$ belongs to the (complex) class L^p. It is still true that $H(re^{i\theta})$ tends almost always to a limit function $h(\theta)$, but $h(\theta)$ is not necessarily integrable.

We shall now require the following theorems.

THEOREM γ†. *If $H(z)$ belongs to L^p, where $p \leqslant 1$, then*

$$c_n = o\{n^{-(p-1)/p}\}, \tag{5 . 1}$$

and the series

$$\Sigma n^{p-2}|c_n|^p, \qquad \Sigma n^{-1/p}|c_n| \tag{5 . 2}$$

are convergent.

THEOREM $\delta^{\dagger}_{\ddagger}$. *If $0 < p \leqslant 1$, $(1-p)/p < \alpha < 1/p$, H belongs to L^p, and*

$$H_\alpha = \sum_1^\infty n^{-\alpha} c_n z^n,$$

then H_α belongs to $L^{p/(1-p\alpha)}$.

The last theorem is an analogue, for $p \leqslant 1$, of Theorem β. Since h is no longer integrable, our former definition of the "fractional integral" is inapplicable.

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* Theorem δ below.

† Hardy and Littlewood, **2** (Theorem 11) and **3** (Theorem 16).

‡ We have not stated this theorem before. A proof will be given in our memoir referred to in § 3 (p. 291, f.n. *).

THEOREM 3. *If $\Sigma a_n z^n$ and $\Sigma b_n z^n$ are functions of the classes L^p and L^q, then the series (2 . 4) is absolutely convergent whenever $0 < p \leqslant q \leqslant 2$, and convergent whenever $0 < p \leqslant 2 < q \leqslant p'$. If $p \leqslant 1$, the last inequality $q \leqslant p'$ is to be omitted.*

We may suppose $p \leqslant 1$, since otherwise there is nothing new to prove. If $q \leqslant 1$, then

$$\Sigma n^{-\mu} |a_n b_n| = \Sigma \{ n^{-1/p} |a_n| . n^{(q-1)/q} |b_n| \}$$

is convergent by Theorem γ. If $1 < q \leqslant 2$ it is sufficient, as in § 3, to prove the convergence of

$$\Sigma (n^{-\mu} |a_n|)^q = \Sigma \{ n^{p-2} |a_n|^p . (n^{(p-1)/p} |a_n|)^{q-p} \}; \tag{3 . 1}$$

and this is again convergent by Theorem γ. Finally, if $2 < q \leqslant p'$, f_μ belongs to $L^{q'}$, by Theorem δ, and the conclusion follows from Riesz's theorem as in § 3.

For the sake of completeness we add

THEOREM 4. *The series (2 . 4) is absolutely convergent whenever $0 < p \leqslant 1 \leqslant q$, f belongs to L^p, and $\Sigma |b_n|^{q'}$ is convergent.*

This has been proved above when $q = 1$. If $q > 1$, the argument above shows that (3 . 1) is convergent, and the conclusion then follows by Hölder's inequality, as in § 3.

Memoirs referred to.

1. G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation: A remarkable trigonometrical series", *Proc. Nat. Acad. of Sciences*, 2 (1916), 583–585 [for a correction, see *ibid.*, 3 (1917), 87].

2. G. H. Hardy and J. E. Littlewood, "Some properties of fractional integrals", *Proc. London Math. Soc.* (2), 24, xxxvii–xli (*Records* for 12 March, 1925).

3. G. H. Hardy and J. E. Littlewood, "New properties of Fourier constants", *Math. Annalen*, 97 (1926), 159–209.

4. F. Hausdorff, "Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen", *Math. Zeitschrift*, 16 (1923), 163–167.

5. M. Riesz, "Sur les séries trigonométriques conjuguées", *Comptes rendus*, 28 April, 1924.

6. H. Weyl, "Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung", *Vierteljahrsschrift d. naturf. Ges. in Zürich*, 62 (1917), 296–302.

7. W. H. and G. C. Young, "On the theorem of Riesz-Fischer", *Quart. J. of Math.*, 44 (1913), 49–88.

SOME IDENTITIES IN THE THEORY OF NUMBERS

By A. OPPENHEIM.

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1 *Introduction and notation.*

If x is a positive integer, let

$$\sigma_k(x) = \sum_{d \mid x} d^k, \tag{1.11}$$

and, if x is not a positive integer, let

$$\sigma_k(x) = 0. \tag{1.12}$$

By $r_p(x)$ I denote the number of integral solutions of the equation

$$x_1^2 + \ldots + x_p^2 = x \quad (p \geqslant 2),$$

so that $r_p(x)$ is the number of lattice-points on the hyper-sphere $\Sigma x_i^2 = x$.

Since*

$$\sigma_0(x) = d(x) = O(x^\epsilon), \quad r_2(x) = r(x) = O(x^\epsilon), \tag{1.13}$$

if $\epsilon > 0$, it is trivial that

$$\sigma_k(x) = O(x^{\frac{1}{2}k_1 + \frac{1}{2}|k_1| + \epsilon}), \quad r_p(x) = O(x^{\frac{1}{2}p - 1 + \epsilon}), \tag{1.14}$$

where k_1 denotes the real part of k†.

From these arithmetic functions we derive the sums

$$\sum_{n \leqslant x} \sigma_k(n), \qquad \sum_{n \leqslant x} r_p(n),$$

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* See, *e.g.*, E. Landau, "Über die Anzahl der Gitterpunkte in gewissen Bereichen", *Göttinger Nachrichten* (1912), 1–84 (30, 64).

† More accurately, if $|k_1| > 1$,

$$\sigma_k(x) = O(x^{\frac{1}{2}|k_1| + \frac{1}{2}k_1}),$$

and, if $|k_1| = 1$

$$\sigma_k(x) = o(x^{\frac{1}{2}|k_1| + \frac{1}{2}k_1} \log x).$$

about which two questions may be asked. The first and more important question is with what degree of accuracy can we approximate to these sums by means of elementary functions of x? The second question is: can we express these sums by means of analytic functions of x, and, if so, what are these expressions?

In this paper I am concerned only with the second question, and the answer to this question is contained in Theorems 1 and 2. Herein, for brevity, I write

$$(1.21) \qquad F_\nu(z) \equiv \cos \tfrac{1}{2}\nu\pi\, J_\nu(z) - \sin \tfrac{1}{2}\nu\pi \left\{ Y_\nu(z) + \frac{2}{\pi} K_\nu(z) \right\},$$

where J_ν, Y_ν, K_ν are the Bessel functions of order ν usually so designated*, and I also write $\Phi_k(x)$ for the sum of the residues of the function

$$z^{-1}\zeta(z)\zeta(z-k)x^z\dagger.$$

1.3. THEOREM 1. *If k is real and x is positive, then*

$$(1.31) \qquad \sum_{n \leqslant x} \sigma_k(n) - \tfrac{1}{2}\sigma_k(x) = \Phi_k(x) + x^{\frac{1}{2}+\frac{1}{2}k} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k}} F_{1+k}\{4\pi\sqrt{(nx)}\},$$

where the series is summable $(R, n, |k| - \frac{1}{2} + \epsilon)$ for any positive ϵ if $|k| \geqslant \frac{1}{2}$, and is convergent if $|k| < \frac{1}{2}$, the summability and convergence being uniform in any closed interval free from integral values of x.

This theorem is not entirely new. The case $k = 0$ is due to Voronoï‡, who proved it so long ago as 1904 by complex analysis. Other proofs by complex analysis have been given by Hardy§, Walfisz‖, and

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* G. N. Watson, *Theory of Bessel functions* (Cambridge, 1922), §§ 3.1 (8), 3.54 (1), 3.7 (6).

† We have

$$\Phi_0(x) = x\log x + (2\gamma - 1)x + \tfrac{1}{4},$$

$$\Phi_{-1}(x) = \tfrac{1}{6}\pi^2 x - \tfrac{1}{2}(\gamma + \log 2\pi + \log x),$$

and, if $k \neq 0, -1$,

$$\Phi_k(x) = x\zeta(1-k) + x^{1+k}\zeta(1+k)/(1+k) - \tfrac{1}{2}\zeta(-k).$$

‡ G. Voronoï, "Sur une fonction transcendante et ses applications à la sommation de quelques séries", *Annales Sci. de l'Ecole Normale* (3), 21 (1904), 207–267 and 459–533 (513).

§ G. H. Hardy, "On Dirichlet's divisor problem", *Proc. London Math. Soc.* (2), 15 (1915), 1–25. See 16–22, and also Hardy's paper in *Quarterly Journal* cited below. Hardy indicates two proofs.

‖ A. Walfisz, "Über die summatorischen Functionen einiger Dirichletscher Reihen", *Inaugural dissertation* (Göttingen, 1922), 1–55.

Cramér*. The only known proof by real analysis is due to Rogosinski†, who uses Pfeiffer's method.

The only other cases previously known are given by $k = \pm 1$, and these, but only with summability $(R, n, 1)$, are due to Walfisz‡, the proofs again depending on complex analysis.

1.4. THEOREM 2. *If p is an integer greater than unity and x is positive, then*

$$(1.41) \qquad \sum_{0 \leqslant n \leqslant x} r_p(n) - \tfrac{1}{2} r_p(x) = \frac{\pi^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)} x^{\frac{1}{2}p} + x^{\frac{1}{2}p} \sum_1^\infty \frac{r_p(n)}{n^{\frac{1}{2}p}} J_{\frac{1}{2}p}\{2\pi\sqrt{(nx)}\},$$

where the series is convergent if $p = 2$ and is summable

$$\{R, n, \tfrac{1}{2}(p-3)+\epsilon\}$$

if $p > 2$, the summability and convergency being uniform in any closed interval free from integral values of x.

This theorem, too, is not entirely new. The case $p = 2$ was first stated by Voronoï§, who, however, gave no proof. The first rigorous proof is due to Hardy‖, who uses complex analysis. Later, Landau¶ gave a proof by real analysis, employing Pfeiffer's method. Quite recently considerably simplified forms of these proofs have been published in a joint paper by Hardy and Landau**. By developing a method used in the latter paper, Hardy†† has obtained yet another proof by real analysis.

The general theorem, but with summability of order $[\frac{1}{2}p - \frac{1}{2}]$, is due to Walfisz‡‡, who, indeed, gives the more general identity corresponding to the sum

$$\Sigma \exp\left(2\pi i \sum_1^p v_j h_j\right),$$

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* H. Cramér, "Über das Teilerproblem von Piltz", *Arkiv for Mat.*, 16 (1922), No. 21, 1–40.

† W. Rogosinski, "Neue Anwendung der Pfeifferschen Methode bei Dirichlets Teilerproblem", *Inaugural dissertation* (Göttingen, 1922), 1–32.

‡ Walfisz, *loc. cit. supra*, 52-53.

§ Voronoï, "Sur le développement, à l'aide des fonctions cylindriques, des sommes doubles $\Sigma f(pm^2 + 2qmn + rn^2)$ où $pm^2 + 2qmn + rn^2$ est une forme à coefficients entiers", *Verhandlungen des dritten internationalen Math.-Kongresses in Heidelberg* (1904), 241–245.

‖ Hardy, "On the representation of a number as the sum of two squares", *Quart. J. of Math.*, 46 (1915), 268–283.

¶ Landau, "Über die Gitterpunkte in einem Kreise (3)", *Göttinger Nachrichten* (1920), 109–134.

** Hardy and Landau, "The lattice points of a circle", *Proc. Royal Soc.* (A), 105 (1924), 244–258.

†† Hardy, "The lattice points of a circle", *Proc. Royal Soc.* (A), 107 (1925), 623–635.

‡‡ Walfisz, *loc. cit. supra*, 54.

where $Q(w) = Q(w_1, \ldots, w_p)$ is a positive definite quadratic form in p variables and with integral coefficients, the numbers h and z are given real numbers, and the summation extends over all integers v such that $0 < Q(v+z) \leqslant x$.

The novelty in Theorem 2 lies in the improvement of the order of summability required by Walfisz by nearly $\frac{1}{2}$ if p is even and exceeds 2, and by nearly 1 if p is odd*.

1.5. It is natural to ask what happens when we differentiate formally the identities of Theorems 1 and 2. The answer to this question is contained in Theorems 3 and 4.

1.6. THEOREM 3†. *If k is real and x is positive and not integral, then the series*

$$(1.61) \qquad 2\pi x^{\frac{1}{2}k} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \left[-\sin \tfrac{1}{2}k\pi \, J_k - \cos \tfrac{1}{2}k\pi \left(Y_k - \frac{2}{\pi} K_k \right) \right],$$

obtained by formal differentiation of the series (1.31), *the argument of the Bessel functions being $4\pi\sqrt{(nx)}$, is summable $(R, n, \lambda > |k| + \frac{1}{2})$ to the sum*‡

$$(1.62) \qquad -\Phi_k'(x).$$

If, however, x is a positive integer, the series is not summable (R, n, λ) for any λ, however large.

1.7. THEOREM 4. *If x is positive and not integral, then the series*

$$(1.71) \qquad \pi x^{\frac{1}{2}p-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{r_p(n)}{n^{\frac{1}{2}p-\frac{1}{2}}} J_{\frac{1}{2}p-1}\{2\pi\sqrt{(nx)}\},$$

obtained by formal differentiation of the series (1.41) *is summable $(R, n, \lambda > \frac{1}{2}p - \frac{1}{2})$ to the sum*

$$(1.72) \qquad -\frac{\pi^{\frac{1}{2}p}}{\Gamma(\frac{1}{2}p)} x^{\frac{1}{2}p-1}.$$

If, however, x is a positive integer, the series is not summable (R, n, λ) for any λ, however large.

The idea of investigating these differentiated identities was suggested to me by Prof. Hardy, to whom I am deeply indebted for much assistance and advice.

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* From a recent paper in the *Math. Zeitschrift*, I gather that Walfisz has now proved Theorem 2 completely, and, moreover, has proved that the order of summability there given is the best possible, *i.e.* that the series (1.41) is not summable $\{R, n, \frac{1}{2}(p-3)\}$.

† This Theorem and Theorem 4 were given incompletely in an abstract of this paper published in *Proc. London Math. Soc.* (*Records* for January, 1925).

‡ We have $\Phi_0'(x) = \log x + 2\gamma$

and, if $k \neq 0$, $\Phi_k'(x) = \zeta(1-k) + x^k \zeta(1+k)$.

It is plain that we can go on differentiating these series, each differentiation raising by unity the order of summability required, and integral values of x again proving to be exceptional.

It is possible also to go in the other direction, that is to say, of integrating the series. Integration, it appears, destroys the non-uniformity of convergence associated with integral values of x. Writing, for brevity,

$$(1.74) \quad F_{\nu,\rho}(z) \equiv \cos \tfrac{1}{2}\nu\pi \, J_{\nu+\rho} - \sin \tfrac{1}{2}\nu\pi \left\{ Y_{\nu+\rho} + \frac{2}{\pi} \cos \rho\pi \, K_{\nu+\rho} \right\},$$

so that $F_{\nu,0}(z) = F_\nu(z)$, we have, for example,

1.8. THEOREM 5. *If x is positive and k is real, then*

$$(1.81) \quad \sum_{n<x} \sigma_k(n)(x-n) = \Phi_{k,1}(x) + \frac{1}{2\pi} x^{1+\frac{1}{2}k} \sum_1^\infty \frac{\sigma_k(n)}{n^{1+\frac{1}{2}k}} F_{1+k,1}\{4\pi\sqrt{(nx)}\},$$

where $\Phi_{k,1}(x)$ denotes the sum of the residues of the function

$$\frac{\zeta(z)\,\zeta(z-k)}{z(z+1)} x^{z+1},$$

and the series is convergent if $|k| < \frac{3}{2}$, but summable

$$(R, n, \lambda > |k| - \tfrac{3}{2})$$

if $|k| \geqslant \frac{3}{2}$, the convergence and summability being uniform in any finite interval $0 < x_1 \leqslant x \leqslant x_2$.

The case $k = 0$ is due to Voronoï*. The case $k = -1$ is due to Wigert†. Both proofs involve complex analysis. A different proof of the latter case was given later by Landau‡, who also uses complex analysis.

There is, of course, an analogous theorem for $\Sigma r_p(n)(x-n)$, but this I do not here write down.

Identities are also obtained for sums of a rather different type, namely for

$$\frac{1}{k} \left\{ x^k \sum_{n<x} \sigma_{-k}(n) - \sum_{n<x} \sigma_k(n) \right\}$$

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* Voronoï, *loc. cit.*, 296, f.n. ‡, 485, 497.

† S. Wigert, "Sur quelques fonctions arithmétiques", *Acta Math.*, 37 (1914), 113–140 (140).

‡ Landau, "Über eine Aufgabe aus der Theorie der quadratischen Formen", *Sitz. der Akad. Wiss. in Wien*, 124 (1915), 1–24 (21–24)

if $k \neq 0$, and (the limit of this when $k \to 0$)

$$\sum_{n<x} d(n) \log \frac{x}{n}.$$

[*Note.*—The complete theorem, of which Theorem 5 is a special case, is given in (8.6); the identity for $\Sigma r_p(n)(x-n)^\rho$ in (8.7); and the identities for the sums above in § 9.]

2. *The method of proof.*

This is, in principle, very simple. The problem is to find an analytic expression for the sum $\sum_{n<x} a_n$, where x is positive and a_n is a function of the positive integer n. We know that, if $c > 0$,

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{ws}}{s}\, ds = 1, \tfrac{1}{2}, 0, \tag{2.1}$$

according as w is positive, zero, or negative. Hence, by classical theorems in the theory of Dirichlet's series*, it follows that

$$\sum_{n\leqslant x}{}' a_n = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{\lambda_x s}}{s} f(s)\, ds, \tag{2.2}$$

where λ_x is a positive function of x, tending steadily to infinity with x, and

$$f(s) = \sum_1^\infty a_n e^{-\lambda_n s}, \tag{2.21}$$

provided that this series is convergent when $\sigma > \sigma_0$, and that c is positive and greater than σ_0.

The formula (2.2) expresses the sum Σa_n as an integral, and the expression of this integral in other terms depends on the nature of the function $f(s)$ defined when $\sigma > \sigma_0$ by the Dirichlet's series (2.21).

In the present paper we do not use the discontinuous factor (2.1), but a generalization of it, which, it turns out, is better suited to our purpose. This factor is

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} H_\mu^{(1)}(isa)\, H_{\mu+1}^{(2)}(isb)\, ds = \begin{cases} 0 & (a > b) \\ (1/\pi b) & (a = b) \\ (2a^\mu/\pi b^{\mu+1}) & (a < b), \end{cases} \tag{2.31}$$

* G. H. Hardy and M. Riesz, *General theory of Dirichlet's series* (Cambridge, 1915), Theorem 13.

where a, b, c are positive and $H_\mu^{(1)}$, $H_\mu^{(2)}$ are Hankel's cylinder functions*. It reduces, when $\mu = -\frac{1}{2}$, to (2 . 1). A similar result is

$$(2\,.\,32)\qquad \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} H_\mu^{(2)}(isa)\, H_{\mu+1}^{(1)}(isb)\, ds = \begin{cases} 0 & (b > a) \\ (-1/\pi b) & (b = a) \\ (-2a^\mu/\pi b^{\mu+1}) & (b < a), \end{cases}$$

which, when $\mu = 0$, reduces to the discontinuous factor used by Hardy and Landau in their joint paper†, in virtue of the formulae

$$(2\,.\,33)\qquad \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} H_\mu^{(1)}(isa)\, H_\nu^{(1)}(isb)\, ds = 0$$

and $H_\mu^{(1)} + H_\mu^{(2)} = 2J_\mu$.

From (2 . 31) and (2 . 33) we derive formally the result

$$(2\,.\,41)\qquad \sum_{n\leqslant x}{}' a_n = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s)\, J'_{\mu+1}(is\lambda_x)\, ds$$

$$(2\,.\,42)\qquad = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s)\, J_{\mu+1}(is\lambda_x)\left(1 - \frac{(-)^m s^{2m}}{T^{2m}}\right)^\lambda ds,$$

where

$$(2\,.\,43)\qquad f(s) = \tfrac{1}{2}\pi\, \lambda_x^{\mu+1} \sum_1^\infty a_n\, \lambda_n^{-\mu}\, H_\mu^{(1)}(is\lambda_n),$$

m is any positive integer, λ any positive number, and λ_x has the meaning previously assigned. That (2 . 42) exists if (2 . 41) does, is easily proved. The solution of the problem now turns on the analytic character of the function $f(s)$ defined by (2 . 43). If $a_n = \sigma_k(n)$, we take $\mu = k$ and $\lambda_x = 4\pi\sqrt{x}$; if $a_n = r_p(n)$, we take $\mu = \frac{1}{2}p - 1$ and $\lambda_x = 2\pi\sqrt{x}$. The corresponding functions $f(s)$ then defined by (2 . 43) if $\sigma > 0$, are regular in the s-plane cut along the negative real axis, except for simple poles at the points $s = \pm i\sqrt{n}$ $(n = 1, 2, \ldots)$. We have, for example, in the case of $r_3(n)$,

$$(2\,.\,44)\qquad \pi i e^{\frac{1}{4}\pi i} \sum_1^\infty \frac{r_3(n)}{n^{\frac{1}{4}}}\, H_{\frac{1}{2}}^{(1)}(2\pi i s\sqrt{n})$$

$$= \frac{A}{s^{\frac{3}{2}}} + Bs^{\frac{1}{2}} + \frac{C}{s^{\frac{1}{2}}} - \frac{1}{\pi s^{\frac{1}{2}}} \sum_1^\infty r_3(n)\left(\frac{1}{n} - \frac{1}{s^2+n}\right),$$

where A, B, and C are certain constants. Taking $m = 1$ in (2 . 42), we find, by means of Cauchy's theorem and (2 . 44), that, for any given

* Watson, *loc. cit.*, 73.

† Hardy and Landau, *loc. cit.*, 248 (Lemma 2).

positive x and any positive λ,

$$(2.46)\qquad \sum_{1\leqslant n\leqslant x}{}' r_3(n)$$

$$= \lim_{T\to\infty}\left[-1+\frac{\pi^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}}+x^{\frac{3}{4}}\sum_{1\leqslant n\leqslant T^2}\frac{r_3(n)}{n^{\frac{3}{4}}}J_{\frac{3}{2}}\{2\pi\sqrt{(nx)}\}\left(1-\frac{n}{T^2}\right)^{\lambda}+O(T^{-\lambda})\right].$$

But this is merely Theorem 2 when $p=3$.

To obtain Theorems 3 and 4 we need a differentiated form of the factors (2.31) and (2.33). These are (the numbers concerned having the meanings already assigned)

$$(2.51)\qquad \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} sH_{\mu}^{(1)}(isa)\,H_{\mu}^{(2)}(isb)\left(1-\frac{(-)^m s^{2m}}{T^{2m}}\right)^{\lambda}ds=0$$

if $a\neq b$, the limit not existing if $a=b$; and

$$(2.52)\qquad \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} sH_{\mu}^{(1)}(isa)\,H_{\nu}^{(1)}(isb)\left(1-\frac{(-)^m s^{2m}}{T^{2m}}\right)^{\lambda}ds=0.$$

The application of these factors to the proofs of Theorems 3 and 4 is now clear. It is merely a question of evaluating the integral

$$(2.54)\qquad \frac{1}{2\pi i}\int_{c-iT}^{c+iT} sf(s)\,J_{\mu}(is\lambda_x)\left(1-\frac{(-)^m s^{2m}}{T^{2m}}\right)^{\lambda}ds;$$

and the calculation of this integral proceeds on exactly the same lines as the calculation of the similar integral (2.42).

It is plain also that these theorems form part of a chain of theorems which may be derived by considering the integral

$$(2.61)\qquad \frac{1}{2\pi i}\int_{c-iT}^{c+iT} s^{1-\rho}H_{\mu}^{(1)}(isa)\,H_{\mu+\rho}^{(2)}(isb)\left(1-\frac{(-)^m s^{2m}}{T^{2m}}\right)^{\lambda}ds,$$

where ρ is any integer or zero.

All these integrals are calculated by means of Cauchy's theorem applied to integrals of analytic functions taken round one or other of the contours C_1 and C_2 defined as follows. C_1 is formed by the line $\sigma=c$ and the arc of the circle $|s|=T$ which is on the right of the line; C_2 is formed by the lines $\sigma=0$ and $\sigma=c$ and the arcs of the circle $|s|=T$ which are intercepted between them. It is supposed, of course, that the contours are indented wherever it may be necessary.

In virtue of the formula*

$$\pi i e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(2i\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \xi^{-2z}\,\Gamma(z-\tfrac{1}{2}\nu)\,\Gamma(z+\tfrac{1}{2}\nu)\,dz, \tag{2.71}$$

which is valid for $\mathbf{R}(\xi) > 0$ and $c > \frac{1}{2}|\mathbf{R}(\nu)|$, the continuation of the function $f(s)$ defined by the series (2.43) is made to depend on the Dirichlet's series

$$\Sigma a_n \lambda_n^{-\mu-2z}. \tag{2.72}$$

If $a_n = \sigma_k(n)$ this series reduces to

$$\zeta(z-\tfrac{1}{2}k)\,\zeta(z+\tfrac{1}{2}k)$$

apart from a numerical factor; if $a_n = r_p(n)$, the series is equivalent to $\zeta_p(z+\frac{1}{2}u)$, where $\zeta_p(z)$ is Epstein's zeta-function†.

In § 3 I consider the functions $f(s)$ defined by (2.43). In §§ 4 and 5 I consider the integrals (2.41) and (2.42). Combining, in § 6, the results of §§ 4 and 5, the proofs of Theorems 1 and 2 are completed. In § 7 *et seq.*, I outline the proofs of Theorems 3, 4, 5, etc. The proof of Voronoï's identity is given in the addendum at the end of the paper. In § 10 I give a list of interesting discontinuous factors obtained in the course of the analysis.

3. *The function* $g_k(s) = \pi i e^{\frac{1}{2}k\pi i}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{1}{2}k}} H_k^{(1)}(4\pi i s\sqrt{n})$.

3.1. I suppose, for simplicity, that k is real and positive. For brevity I write

$$\kappa = [\tfrac{1}{2}+\tfrac{1}{2}k] \tag{3.11}$$

and

$$G(s, z) = G(z) = (2\pi s)^{-2z}\,\Gamma(z-\tfrac{1}{2}k)\,\Gamma(z+\tfrac{1}{2}k)\,\zeta(z-\tfrac{1}{2}k)\,\zeta(z+\tfrac{1}{2}k), \tag{3.12}$$

---

* Watson, *loc. cit.*, 192. The notation has been altered slightly.

† $\zeta_p(z)$ is defined when $\mathbf{R}(z) > \frac{1}{2}p$ by the equation

$$\zeta_p(z) = \sum_{(n_j)=-\infty}^{\infty}{}' (n_1^2 + n_2^2 + \ldots + n_p^2)^{-z} = \sum_1^\infty r_p(n)\,n^{-z}.$$

It satisfies the functional relation

$$\Gamma(z)\,\zeta_p(z)\,\pi^{-z} = \Gamma(\tfrac{1}{2}p-z)\,\zeta_p(\tfrac{1}{2}p-z)\,\pi^{z-\frac{1}{2}p}$$

and is regular for all bounded z except at $z = \frac{1}{2}p$, which is a simple pole with residue $\pi^{\frac{1}{2}p}/\Gamma(\frac{1}{2}p)$; also $\zeta_p(0) = -1$, and, in the strip $x_1 \leqslant \mathbf{R}(z) \leqslant x_2$,

$$\zeta_p(z) = O\{|\mathbf{I}(z)|^A\}$$

where $A = A(x_1, x_2)$.

For $p = 1$, $\zeta_p(z)$ reduces to $2\zeta(2z)$.

where

$$s = \sigma + it, \quad z = x + iy, \tag{3.13}$$

and σ, t, x, y are real.

From the formulae

$$\zeta(z-\tfrac{1}{2}k)\,\zeta(z+\tfrac{1}{2}k) = \Sigma\sigma_k(n)\,n^{-z-\frac{1}{2}k} \quad (x > 1+\tfrac{1}{2}k) \tag{3.14}$$

and*

$$\pi i e^{\frac{1}{2}k\pi i}\,H_k^{(1)}(2is) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s^{-2z}\,\Gamma(z-\tfrac{1}{2}k)\,\Gamma(z+\tfrac{1}{2}k)\,dz, \tag{3.15}$$

valid for $\sigma > 0$ and $c > \frac{1}{2}k$, we deduce that

$$g_k(s) = \pi i e^{\frac{1}{2}k\pi i}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{1}{2}k}}\,H_k^{(1)}(4\pi i s\sqrt{n}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s,z)\,dz, \tag{3.16}$$

provided now that $\sigma > 0$ and $c > 1+\frac{1}{2}k$. From (3.16) it follows, by Cauchy's theorem†, that

$$g_k(s) = \phi_k(s) + \frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} G(s,z)\,dz \tag{3.17}$$

for any positive a [such that $z = -a$ is not a pole of $G(z)$], where $\phi_k(s)$ is the sum of the residues of $G(s, z)$ in the strip $-a < x < c$. We take a such that

$$\tfrac{1}{2}k < a < \operatorname{Min}(\tfrac{1}{2}k+1,\ 2\kappa+1-\tfrac{1}{2}k), \tag{3.18}$$

and, with this value of a, we have for non-integral k

$$\begin{aligned}\phi_k(s) = -\tfrac{1}{2}\Big\{&\frac{\Gamma(k)\,\zeta(k)}{(2\pi s)^k} + \frac{\Gamma(-k)\,\zeta(-k)}{(2\pi s)^{-k}}\Big\} \\ &+ \Big\{\frac{\Gamma(1+k)\,\zeta(1+k)}{(2\pi s)^{2+k}} + \frac{\Gamma(1-k)\,\zeta(1-k)}{(2\pi s)^{2-k}}\Big\} \\ &+ \frac{2}{(2\pi s)^{2+k}}\sum_{r=1}^{\kappa}(-)^{r-1}\,\zeta(2r)\,\zeta(1+k-2r)\,\Gamma(1+k-2r)(2\pi s^2)^{2r}.\end{aligned} \tag{3.21}$$

Of course, if k is an integer, (3.21) has to be modified accordingly, since some of the poles of $G(s, z)$ then coalesce to form double poles.

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* Watson, *loc. cit.*, 192, with a slight alteration in notation.

† We use the results

$$|\,\Gamma(z)\,| = O(|\,y\,|^A\,e^{-\frac{1}{2}\pi|y|}), \quad |\,\zeta(z)\,| = O(|\,y\,|^A),$$

where A is independent of y, uniformly for $-a \leqslant x \leqslant c$.

Explicitly

$$\phi_0(s) = -\tfrac{1}{2}(\gamma+\log s)-2(2\pi s)^{-2}\log(2\pi s), \tag{3.211}$$

$$\phi_1(s) = \frac{\zeta(2)}{(2\pi s)^3}+\frac{1}{4\pi s}\{\gamma+2\log(2\pi s)-\log 2\pi\} + \frac{s\zeta(2)}{2\pi}\left\{\gamma+2\log s+\frac{\zeta'(2)}{\zeta(2)}\right\}; \tag{3.212}$$

if $k = 2\kappa$ $(\kappa = 1, 2, \ldots)$,

$$\phi_k(s) = -\tfrac{1}{2}\frac{\Gamma(2\kappa)\,\zeta(2\kappa)}{(2\pi s)^{2+2\kappa}} - \frac{\zeta(2\kappa+1)\,s^{2\kappa}}{4\cos\kappa\pi} + \frac{\zeta(2\kappa)\,s^{2\kappa-2}}{2\pi^2\cos\kappa\pi}\left\{\log s+\frac{\zeta'(2\kappa)}{\zeta(2\kappa)}\right\} + \frac{2}{(2\pi s)^{2+2\kappa}}\sum_{r=0}^{\kappa-1}(-)^{r+1}\zeta(2r)\,\zeta(1+2\kappa-2r)\,\Gamma(1+2\kappa-2r)(2\pi s^2)^{2r}; \tag{3.213}$$

and, if $k = 2\kappa-1$ $(\kappa = 2, 3, \ldots)$,

$$\phi_k(s) = -\tfrac{1}{2}\frac{\Gamma(2\kappa-1)\,\zeta(2\kappa-1)}{(2\pi s)^{2\kappa-1}} - \frac{\zeta(2\kappa-1)\,s^{2\kappa-3}}{4\pi\cos\kappa\pi} - \frac{\zeta(2\kappa)\,s^{2\kappa-1}}{2\pi\cos\kappa\pi}\left\{\gamma+\log s+\frac{\zeta'(2\kappa)}{\zeta(2\kappa)}\right\} + \frac{2}{(2\pi s)^{2\kappa+1}}\sum_{r=0}^{\kappa-1}(-)^{r+1}\zeta(2\kappa)\,\zeta(2\kappa-2r)\,\Gamma(2\kappa-2r)(2\pi s^2)^{2r}. \tag{3.214}$$

Now, using the functional relation satisfied by the Riemann zeta-function, we have

$$G(s, z) = \tfrac{1}{4}s^{-2z}\frac{\zeta(1-\frac{1}{2}k-z)\,\zeta(1+\frac{1}{2}k-z)}{\cos\frac{1}{2}\pi(z+\frac{1}{2}k)\cos\frac{1}{2}\pi(z-\frac{1}{2}k)} \tag{3.22}$$

and, on the line $z = -a$, where a satisfies (3.18),

$$\zeta(1-\tfrac{1}{2}k-z)\,\zeta(1+\tfrac{1}{2}k-z) = \sum_1^\infty \sigma_k(n)\,n^{z-\frac{1}{2}k-1}, \tag{3.23}$$

the series being absolutely and uniformly convergent, so that

$$\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} G(z)\,dz = \sum_{n=1}^\infty \frac{\sigma_k(n)}{n^{\frac{1}{2}k}}\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty}\frac{\frac{1}{4}s^{-2z}\,n^{z-1}\,dz}{\cos\frac{1}{2}\pi(z+\frac{1}{2}k)\cos\frac{1}{2}\pi(z-\frac{1}{2}k)}, \tag{3.24}$$

the inversion being justified by the existence of the integral

$$\int_{-a-i\infty}^{-a+i\infty}\left|\frac{\frac{1}{4}s^{-2z}\,dz}{\cos\frac{1}{2}\pi(z+\frac{1}{2}k)\cos\frac{1}{2}\pi(z-\frac{1}{2}k)}\right|\sum_{n-1}^\infty|\sigma_k(n)\,n^{z-\frac{1}{2}k-1}|.$$

We are thus led to consider the integral

$$(3.25) \qquad I_k(u) = \frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} \frac{\frac{1}{4}u^{2z-2}\,dz}{\cos\frac{1}{2}\pi(z+\frac{1}{2}k)\cos\frac{1}{2}\pi(z-\frac{1}{2}k)},$$

where a satisfies (3.18) and, for convergence, $|\arg u| < \frac{1}{2}\pi$. Supposing for the moment that $0 < u < 1$, we obtain, by Cauchy's theorem*,

$$(3.26) \qquad I_k(u) = \frac{u^{-k}-u^{-4\kappa+k}}{2\pi\sin\frac{1}{2}k\pi}\,\frac{1}{1-u^2},$$

unless k is an even integer, in which case

$$(3.261) \qquad I_k(u) = -\frac{2}{\pi^2\cos\frac{1}{2}k\pi}\,\frac{u^{-k}\log u}{1-u^2}.$$

The theory of analytic continuation now shows that these formulae, proved when $0 < u < 1$, are valid in the region $|\arg u| < \frac{1}{2}\pi$.

From (3.17), (3.24), and (3.26) we thus derive the formula, valid for $\sigma > 0$,

$$(3.3) \qquad g_k(s) = \phi_k(s) + \sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{1}{2}k}}\left\{\frac{(s/\sqrt{n})^k-(s/\sqrt{n})^{4\kappa-k}}{2\pi\sin\frac{1}{2}k\pi}\right\}\frac{s^2}{s^4-n^2},$$

unless k is an even integer, in which case, from (3.261) instead of (3.26),

$$(3.31) \qquad g_k(s) = \phi_k(s) + \frac{2s^k}{\pi^2\cos\frac{1}{2}k\pi}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^k}\log\left(\frac{s}{\sqrt{n}}\right)\frac{s^2}{s^4-n^2}.$$

The latter formula, in the case $k = 0$, is due to Voronoï†. The first formula takes a much simpler form when k is an odd integer or half an odd integer. Thus, if $k = 2\kappa-1$ where $\kappa = 1, 2, \ldots$, we have

$$(3.32) \qquad g_k(s) = \phi_k(s) + \frac{s^{2\kappa-1}}{2\pi\cos\kappa\pi}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{2\kappa-1}}\left(\frac{1}{n}-\frac{1}{s^2+n}\right),$$

---

* The integrand has on the right of the path simple poles at the points

$$z = -\tfrac{1}{2}k+2r-1, \quad z = \tfrac{1}{2}k-2\kappa+2r-1 \quad (r = 1, 2, \ldots),$$

unless k is an even integer 2κ ($\kappa = 1, 2, \ldots$), when the two sets of simple poles form a single set of double poles at the points $z = -\kappa+2r-1$ ($r = 1, 2, \ldots$).

The condition $0 < u < 1$ is necessary to ensure the convergence of the series of residues.

† *Loc. cit.*, 296, f.n. ‡, 245, 260. The function $\xi(x)$ of Voronoï is $\pi i H_0^{(1)}(2i\sqrt{x})$ in the standard notation of Watson's treatise.

and, if $k = \frac{1}{2}(4\kappa \pm 1)$ where $\kappa = 0, 1, 2, \ldots$,

$$(3.33) \qquad g_k(s) = \phi_k(s) + \frac{s^{2\kappa-\frac{1}{2}}}{2\pi \cos \kappa\pi} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k+\kappa+\frac{1}{4}}} \frac{s^2}{(s^2+n)(s+\sqrt{n})}.$$

We infer, from (3 . 3), or, if k is an even integer, from (3 . 31), that $g_k(s)$ is a branch of an analytic function of s which has a branch-point at the origin and simple poles at the points $s = \pm i\sqrt{n}$ $(n = 1, 2, \ldots)$ and is regular elsewhere. The branch represented when $\sigma > 0$ by the series (3 . 16) is obtained by supposing $\arg s$ to have its principal value and the s-plane to be cut along the negative real axis.

It was assumed during the analysis that k is positive. This restriction may be removed provided that we now define κ (k still being real) by the equation

$$(3.34) \qquad \kappa = [\tfrac{1}{2} + \tfrac{1}{2}|k|]$$

and, on the right-hand side of (3 . 3), etc., replace k wherever it occurs by $|k|$. This follows from the formulae

$$n^{-\frac{1}{2}k}\sigma_k(n) = n^{\frac{1}{2}k}\sigma_{-k}(n)$$

and* $$\pi i e^{\frac{1}{2}k\pi i} H_k^{(1)}(z) = \pi i e^{-\frac{1}{2}k\pi i} H_{-k}^{(1)}(z),$$

which show that $g_k(s)$, defined by (3 . 16), is an even function of k.

3 . 5. *The function $f_q(s)$.* Writing for brevity

$$(3.51) \qquad q = \tfrac{1}{2}p - 1, \quad q' = [q],$$

and

$$(3.52) \qquad A(s, z) = A(z) = \pi s^{-2z}\Gamma(z - \tfrac{1}{2}q)\,\Gamma(z + \tfrac{1}{2}q)\,\zeta_p(z + \tfrac{1}{2}q),$$

we easily see that

$$(3.53) \qquad f_q(s) = \pi i e^{\frac{1}{2}q\pi i} \sum_{n=1}^{\infty} r_p(n)\, n^{-\frac{1}{2}q} H_q^{(1)}(2\pi i s\sqrt{n}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} A(z)\,dz$$

provided that $\sigma > 0$ and $c > 1 + \frac{1}{2}q$. Hence

$$(3.54) \qquad f_q(s) = \psi_q(s) + \frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} A(z)\,dz,$$

* Watson, *loc. cit.*, 74, foot.

where

$$\tfrac{1}{2}q < a < q' - \tfrac{1}{2}q + 1, \tag{3.541}$$

and $\psi_q(s)$ denotes the sum of the residues of $A(s, z)$ at its poles in the strip $-a < x < c$, *i.e.* at the points

$$z = -\tfrac{1}{2}q, \quad \tfrac{1}{2}q - r \quad (r = -1, 0, 1, \ldots q').$$

If p is odd, then $q = q' + \tfrac{1}{2}$, $p = 2q' + 3$, and

$$\psi_q(s) = \frac{\pi^{q+1} s^q}{\Gamma(1+q) \sin q\pi} + \sum_{r=-1}^{q'} \frac{\xi_p(1+r)\, s^{2r-q}}{\pi \cos r\pi}. \tag{3.542}$$

If p is even, so that $q = q'$ and $p = 2q' + 2$, then

$$\psi_q(s) = \frac{(\pi s)^q}{q!\, \cos q\pi} \left\{ 2 \log(\pi s) + \xi_p'(0) + \gamma - \frac{\Gamma'(1+q)}{\Gamma(1+q)} \right\} + \frac{1}{\pi} \sum_{r=-1}^{q-1} (-)^r \xi_p(1+r)\, s^{2r-q}. \tag{3.543}$$

Now, using the functional relation

$$\Gamma(z)\, \xi_p(z)\, \pi^{-z} = \Gamma(\tfrac{1}{2}p - z)\, \xi_p(\tfrac{1}{2}p - z)\, \pi^{z - \frac{1}{2}p},$$

we obtain

$$A(z) = \frac{s^{-2z}\, \xi_p(1 - z + \frac{1}{2}q)}{\sin(z - \frac{1}{2}q)\, \pi} = \frac{s^{-2z}}{\sin(z - \frac{1}{2}q)\, \pi} \sum_{n=1}^{\infty} r_p(n)\, n^{z - 1 - \frac{1}{2}q}, \tag{3.55}$$

the series being absolutely and uniformly convergent on the line $x = -a$. From (3.54), therefore, we have

$$f_q(s) = \psi_q(s) + \sum_{n=1}^{\infty} \frac{r_p(n)}{n^{\frac{1}{2}q - 1}} \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \frac{s^{-2z}\, n^z\, dz}{\sin(z - \frac{1}{2}q)\, \pi}, \tag{3.56}$$

the inversion being plainly legitimate. And an easy application of Cauchy's theorem shows that

$$\frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \frac{u^{2z}}{\sin(z - \frac{1}{2}q)\, \pi}\, dz = -\frac{u^{q - 2q'}}{\pi \cos q'\pi} \frac{1}{1+u^2} \tag{3.561}$$

provided that $|\arg u| < \frac{1}{2}\pi$ and a satisfies (3.541). From (3.56) and (3.561) it follows that

$$f_q(s) = \psi_q(s) - \frac{s^{2q'-q}}{\pi \cos q'\pi} \sum_{n=1}^{\infty} \frac{r_p(n)}{n^{q'}} \left(\frac{1}{n} - \frac{1}{s^2+n} \right) \tag{3.57}$$

provided that $\sigma > 0$. Since the series in (3.57) is uniformly convergent in any closed domain free from the points $s = \pm i\sqrt{n}$ $(n = 1, 2, \ldots)$ we easily deduce that the function $f_q(s)$ defined when $\sigma > 0$ by the series (3.53) is a branch of an analytic function with a branch point at the origin, simple poles at $s = \pm i\sqrt{n}$ $(n = 1, 2, \ldots)$, and regular elsewhere; the branch represented by (3.53) is obtained by giving arg s its principal value and cutting the s-plane along the negative real axis.

4. *The integral* $G(x) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} g_k(s)\, J_{1+k}(4\pi i s\sqrt{x})\left(1-\frac{s^4}{T^4}\right)^\lambda ds.$

4.1. In this section we consider the integral

$$G(x) = G_{k,\lambda}(x, T, c) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} g_k(s)\, J_{1+k}(4\pi i s\sqrt{x})\left(1-\frac{s^4}{T^4}\right)^\lambda ds, \tag{4.11}$$

where

$$\lambda \geqslant 0, \quad T > c > 0, \quad T' = \sqrt{(T^2-c^2)}, \quad x > 0. \tag{4.12}$$

We shall suppose that x is either an integer $m(\geqslant 1)$ or confined to a closed interval I free from integers. We shall finally make T tend to infinity so that, if it is convenient, we may suppose that $T > 2c$. The evaluation of the integral depends on certain lemmas. Write for brevity

$$F = F(\lambda, \mu, \nu, a, b, T) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} H_\mu^{(1)}(ias)\, J_\nu(ibs)\left(1-\frac{s^4}{T^4}\right)^\lambda ds, \tag{4.13}$$

where a and b are positive. We proceed to find the value of F for large values of T in the cases $\nu = \mu \pm 1$.

4.2. LEMMA 1. *If* $\lambda+1 > 0$, $T > 2c > 0$ *and* $\beta > 0$, *then*

$$\omega_\lambda(\beta) = \int_{-\frac{1}{2}\pi+\arcsin(c/T)}^{\frac{1}{2}\pi-\arcsin(c/T)} |\sin 2\theta|^\lambda e^{-\beta T\cos\theta}\, d\theta = O\left\{\frac{e^{-\beta c}}{\beta T^{1+\lambda}}\right\} \tag{4.21}$$

uniformly for $\beta \geqslant \delta > 0$ *and all large* T.

The proof is simple. The change of variable, $\cos\theta = u$, gives

$$\omega_\lambda(\beta) = 2^{\lambda+1}\int_{c/T}^1 u^\lambda (1-u^2)^{\frac{1}{2}\lambda-\frac{1}{2}} e^{-\beta T u}\, du \tag{4.22}$$

$$= 2^{\lambda+1}(I_1+I_2),$$

say, where

$$(4.23)\quad I_2 = \int_{\frac{1}{2}}^{1} u^\lambda (1-u^2)^{\frac{1}{2}\lambda-\frac{1}{2}}\, e^{-\beta T u}\, du < e^{-\frac{1}{2}\beta T} \int_{\frac{1}{2}}^{1} u^\lambda (1-u^2)^{\frac{1}{2}\lambda-\frac{1}{2}}\, du,$$

and

$$(4.24)\quad I_1 = \int_{c/T}^{\frac{1}{2}} u^\lambda (1-u^2)^{\frac{1}{2}\lambda-\frac{1}{2}}\, e^{-\beta T u}\, du$$

$$< \mathrm{Max}\left\{\left(\tfrac{2}{3}\right)^{\frac{1}{2}\lambda-\frac{1}{2}}, \left(1-\frac{c^2}{T^2}\right)^{\frac{1}{2}\lambda-\frac{1}{2}}\right\} \int_{c/T}^{\frac{1}{2}} u^\lambda\, e^{-\beta T u}\, du$$

$$= O(e^{-\beta c}/\beta T^{1+\lambda}).$$

Since I_1 and I_2 are positive, the lemma follows at once from (4.22), (4.23), and (4.24).

4.3. LEMMA 2. *If $a > b > 0$ and T is large, then*

$$(4.31)\quad F(\lambda, \mu, \nu, a, b, T) = O\left\{\frac{e^{-(a-b)c}}{a^{\frac{1}{2}} b^{\frac{1}{2}} (a-b)\, T^{1+\lambda}}\right\},$$

and uniformly for $a \geqq A' > A \geqq b \geqq B > 0$.

By Cauchy's theorem, if C_1 is the contour formed by the line $\sigma = c$ and the arc Γ_1 of the circle $|s| = T$ on the right of this line,

$$\int_{C_1} H_\mu^{(1)}(isa)\, J_\nu(isb) \left(1-\frac{s^4}{T^4}\right)^\lambda ds = 0,$$

C_1 being indented, if necessary, at $s = T$. Since $\lambda+1$ is positive, the contribution of the indentation, if any, tends to zero with the radius of the indentation, so that, Γ_1 being described positively, we obtain

$$(4.32)\quad 2\pi i F = P\int_{\Gamma_1}.$$

Now, if $|z|$ is large, we have*

$$(4.33)\quad H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)} \left\{1+O\left(\frac{1}{|z|}\right)\right\}$$

uniformly for $-\pi < \arg z < 2\pi$, and

$$(4.34)\quad J_\nu(z) = \left(\frac{1}{2\pi z}\right)^{\frac{1}{2}} \left[e^{-i\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)} \left\{1+O\left(\frac{1}{|z|}\right)\right\} + c_1 e^{i\left(z+\frac{1}{2}\nu\pi+\frac{1}{4}\pi\right)} \left\{1+O\left(\frac{1}{|z|}\right)\right\}\right]$$

where $c_1 = 1$ if $|\arg z| < \pi$ and $c_1 = e^{(\nu+\frac{1}{2})\pi i}$ if $|\arg z + \pi| < \pi$.

* Watson, *loc. cit.*, 198, 202.

Hence*, if $\theta = \arg s$,

$$(4.35)\qquad P\int_{\Gamma_1} = O\int_{-\frac{1}{2}\pi+\arcsin(c/T)}^{\frac{1}{2}\pi-\arcsin(c/T)} a^{-\frac{1}{2}} b^{-\frac{1}{2}} |\sin 2\theta|^{\lambda} e^{-(a-b)T\cos\theta}\, d\theta$$

$$= O\{a^{-\frac{1}{2}} b^{-\frac{1}{2}} \omega_\lambda(a-b)\}.$$

The lemma follows from (4.32), (4.35), and Lemma 1.

4.4. LEMMA 3. *If $a = b > 0$ and T is large, then*

$$(4.41)\qquad F(\lambda, \mu, \nu, a, a, T) = \frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{2\pi i a} + O\left(\frac{1}{aT^{1+\lambda}}\right) + O\left(\frac{1}{a^2 T}\right)$$

uniformly for $a \geqslant A > 0$.

Proceeding as in Lemma 2, but now using (4.33) and (4.34) fully, we obtain at once, when $a = b > 0$,

$$(4.42)\qquad F(\lambda, \mu, \nu, a, a, T) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{\pi i a s}\left(1-\frac{s^4}{T^4}\right)^{\lambda}\left\{1+O\left(\frac{1}{aT}\right)\right\} ds$$

$$+ O\int_{\Gamma_1}\left|\left(1-\frac{s^4}{T^4}\right)^{\lambda}\right|\left|\frac{1}{\pi a s}\, e^{-2as}\right| ds$$

$$= \frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{\pi i a} J_1 + J_2 + J_3,$$

say, where

$$(4.43)\qquad J_3 = O\int_{-\frac{1}{2}\pi+\arcsin(c/T)}^{\frac{1}{2}\pi-\arcsin(c/T)} a^{-1} |\sin 2\theta|^{\lambda} e^{-2aT\cos\theta}\, d\theta = O\{a^{-1}\omega_\lambda(2a)\},$$

$$(4.44)\qquad J_2 = O\int_{\Gamma_1}\left|\left(1-\frac{s^4}{T^4}\right)^{\lambda}\right|\frac{|ds|}{a^2 T^2} = O(a^{-2}T^{-1}),$$

and

$$(4.45)\qquad J_1 = \frac{1}{2\pi i}\int_{\Gamma_1}\left(1-\frac{s^4}{T^4}\right)^{\lambda}\frac{ds}{s}.$$

Denote by Γ_2 and Γ_3 the arcs of $|s| = T$ intercepted between $\sigma = 0$ and $\sigma = c$, by Γ_4 the arc on the right of $\sigma = 0$, and by C_3 the contour formed by Γ_4 and its base on $\sigma = 0$, the contour being indented at the

* Here we require only the dominant terms on the right-hand sides of (4.33) and (4.34).

points $s = 0,\ T,\ \pm Ti$. Plainly, by Cauchy's theorem,

$$\int_{C_3} \left(1-\frac{s^4}{T^4}\right)^{\lambda} \frac{ds}{s} = 0. \tag{4.46}$$

To this integral the contributions of the indentations at $s = T,\ \pm Ti$ are, in the limit, zero, since $\lambda+1 > 0$. The indentation at $s = 0$ contributes in the limit $-\pi i$. The sum of the contributions of the two parts of the imaginary axis is plainly zero. Hence, the arcs being described positively,

$$-\pi i+\int_{\Gamma_1}+\int_{\Gamma_2}+\int_{\Gamma_3} = 0. \tag{4.47}$$

But plainly

$$\int_{\Gamma_2} = O\int_{\frac{1}{2}\pi-\arcsin(c/T)}^{\frac{1}{2}\pi} |\sin 2\theta|^{\lambda} d\theta = O(T^{-1-\lambda}), \tag{4.48}$$

and similarly for $\int_{\Gamma_3}$. From (4.45)–(4.48), therefore,

$$J_1 = \tfrac{1}{2}+O(T^{-1-\lambda}). \tag{4.49}$$

The lemma follows from (4.42)–(4.45), (4.49), and Lemma 1.

In Lemmas 2 and 3 we have put no restriction on μ and ν. In Lemma 4, however, we take $\nu = \mu+1$, μ being unrestricted.

4.5. LEMMA 4. *If* $0 < a < b$, $\nu = \mu+1$, *and* T *is large, then*

$$F(\lambda, \mu, \mu+1, a, b, T) = \frac{a^{\mu}}{\pi b^{\mu+1}}+O\left\{\frac{e^{(b-a)c}}{a^{\frac{1}{2}} b^{\frac{1}{2}} T^{1+\lambda}}\right\}+O\left\{\frac{e^{(a-b)c}}{a^{\frac{1}{2}} b^{\frac{1}{2}}(b-a)\, T^{1+\lambda}}\right\}, \tag{4.51}$$

and uniformly for $0 < A \leqq a \leqq A' < B \leqq b$.

Plainly the function

$$Q(s) = \{H_{\mu}^{(1)}(ias)\, J_{1+\mu}(ibs)-H_{1+\mu}^{(1)}(ibs)\, J_{\mu}(ias)\}\left(1-\frac{s^4}{T^4}\right)^{\lambda} \tag{4.52}$$

is regular inside and on the contour C_2 formed by the lines $\sigma = 0$, $\sigma = c$, and the arcs Γ_2 and Γ_3, the contour being indented at $s = \pm Ti$ and $s = 0$. Hence, by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{C_2} Q(s)\, ds = 0. \tag{4.53}$$

Since $\lambda+1 > 0$, the contributions of the indentations at $s = \pm Ti$ tend to zero with the radii of the indentations. Next, by using (4.33)

and (4.34), the contributions of Γ_2 and Γ_3 are each of order

$$(4.54)\quad O\int_{\frac{1}{2}\pi-\arcsin(c/T)}^{\frac{1}{2}\pi} a^{-\frac{1}{2}}b^{-\frac{1}{2}}|\sin 2\theta|^{\lambda}\, e^{|b-a|\,T\cos\theta}\,d\theta = O\left\{\frac{e^{c|b-a|}}{a^{\frac{1}{2}}b^{\frac{1}{2}}T^{1+\lambda}}\right\}.$$

Now*, if $t \geq 0$,

$$4.55)\qquad J_\nu(te^{\pi i}) = e^{\nu\pi i}J_\nu(t),\quad H_\nu^{(1)}(te^{\pi i}) = -e^{-\nu\pi i}H_\nu^{(2)}(t),$$

so that the contribution of $\sigma = 0$ is

$$\frac{1}{2\pi i}\int_T^\delta\left(1-\frac{t^4}{T^4}\right)^\lambda idt\,\{H_\mu^{(2)}(at)J_{1+\mu}(bt)-H_{1+\mu}^{(2)}(bt)J_\mu(at)\}$$

$$+\frac{1}{2\pi i}\int_\delta^T\left(1-\frac{t^4}{T^4}\right)^\lambda(-idt)\,\{H_\mu^{(1)}(at)J_{1+\mu}(bt)-H_{1+\mu}^{(1)}(bt)J_\mu(at)\},$$

and this is zero, since $H_\nu^{(1)}+H_\nu^{(2)} = 2J_\nu$.

To find the contribution of the origin, we suppose for the moment that μ is not an integer. Since†

$$(4.56)\qquad J_{-\nu} = i\sin\nu\pi\, H_\nu^{(1)}+e^{-\nu\pi i}J_\nu,$$

we have

$$(4.57)\quad i\sin\mu\pi\, Q(s) = \left(1-\frac{s^4}{T^4}\right)^\lambda\{J_{-\mu}(ias)J_{\mu+1}(ibs)+J_\mu(ias)J_{-1-\mu}(ibs)\},$$

so that $Q(s)$ has a simple pole at the origin. Hence, making $\delta \to 0$, the contribution of the indentation at the origin is

$$-\pi i\,\frac{1}{2\pi i}\,\frac{1}{i\sin\mu\pi}\,\frac{(\frac{1}{2}ia)^\mu(\frac{1}{2}ib)^{-\mu-1}}{\Gamma(1+\mu)\,\Gamma(-\mu)} = -\frac{a^\mu}{\pi b^{\mu+1}}\,\ddagger.$$

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* Watson, *loc. cit.*, 75.

† Watson, *loc. cit.*, 74.

‡ This is the contribution even if μ is integral. For, if μ is a positive integer,

$$Q(s) = i\left\{Y_\mu(ias)J_{1+\mu}(ibs)-J_\mu(ias)Y_{1+\mu}(ibs)\right\},$$

while, if μ is a negative integer,

$$Q(s) = i\left\{Y_{-\mu-1}(ibs)J_{-\mu}(ias)-J_{-\mu-1}(ibs)Y_{-\mu}(ias)\right\},$$

so that [Watson *loc. cit.*, 62 (3); we use also $Y_n = \pi^{-1}\mathbf{Y}_n$] near $s = 0$,

$$Q(s) = \frac{i}{\pi}\,\frac{(\frac{1}{2}ias)^\mu}{(\frac{1}{2}ibs)^{\mu+1}} + O(1),$$

whether μ is positive or negative. In any case, therefore, the origin contributes, in the limit,

$$-\pi i\,\frac{1}{2\pi i}\,\frac{i}{\pi}\,\frac{(\frac{1}{2}ia)^\mu}{(\frac{1}{2}ib)^{\mu+1}} = -\frac{a^\mu}{\pi b^{\mu+1}}.$$

Finally the contribution of $\sigma = c$ is plainly

$$(4.58) \qquad F(\lambda, \mu, \mu+1, a, b, T) - F(\lambda, \mu+1, \mu, b, a, T),$$

and, after Lemma 2, since $b > a$,

$$(4.59) \quad F(\lambda, \mu+1, \mu, b, a, T) = O\{b^{-\frac{1}{2}} a^{-\frac{1}{2}} (b-a)^{-1} e^{(a-b)c} T^{-1-\lambda}\}.$$

Lemma 4 follows by combining (4.53)–(4.59).

[We have only used the fact that $b > a$ in applying Lemma 2 to $F(\lambda, \mu+1, \mu, b, a, T)$. If, instead, we suppose that $b < a$, and apply Lemma 2 to $F(\lambda, \mu, \mu+1, a, b, T)$, the preceding analysis shows that, if $0 < b < a$,

$$(4.51)' \quad F(\lambda, \mu+1, \mu, b, a, T) = -\frac{a^\mu}{\pi b^{\mu+1}} + O\left\{\frac{e^{(b-a)c}}{a^{\frac{1}{2}} b^{\frac{1}{2}} (a-b) T^{1+\lambda}}\right\} + O\left\{\frac{e^{(a-b)c}}{a^{\frac{1}{2}} b^{\frac{1}{2}} T^{1+\lambda}}\right\},$$

a result parallel to (4.51).

Making T tend to infinity in (4.31), (4.41), and (4.51)′ we see that, if a, b, c are positive, and $\lambda > -1$,

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT'}^{c+iT'} H_\mu^{(1)}(ias)\, J_{\mu+1}(ibs) \left(1 - \frac{s^4}{T^4}\right)^\lambda ds = \begin{cases} 0 & (a > b), \\ (1/2\pi a) & (a = b), \\ (a^\mu/\pi b^{\mu+1}) & (a < b). \end{cases}$$

Using (4.51)′ we obtain, the notation being altered slightly,

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT'}^{c+iT'} H_{\mu+1}^{(1)}(ias)\, J_\mu(ibs) \left(1 - \frac{s^4}{T^4}\right)^\lambda ds = \begin{cases} 0 & (a > b), \\ (-1/2\pi a) & (a = b), \\ (-b^\mu/\pi a^{\mu+1}) & (a < b), \end{cases}$$

the conditions that a, b, c and $\lambda+1$ are positive being retained.]

4.6. LEMMA 5. *If x is a given positive number and T is large, then*

$$(4.61) \qquad G_{k,\lambda}(x) = \frac{ie^{\frac{1}{2}k\pi i}}{4\pi x^{\frac{1}{2}+\frac{1}{2}k}} \sum_{n\leqslant x}{}' \sigma_k(n) + O\left(\frac{1}{T}\right) + O\left(\frac{1}{T^{1+\lambda}}\right),$$

and uniformly in I.

The proof is simple. We substitute in (4.11) for $g_k(s)$ the uniformly convergent series (3.16), which defines it when $\sigma > 0$. The term-by-

term integration being thus permissible, we obtain

$$G_{k,\lambda}(x) = \pi i e^{\frac{1}{2}k\pi i} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} F(\lambda, \mu, 4\pi\sqrt{n}, 4\pi\sqrt{x}, T)$$

$$= \pi i e^{\frac{1}{2}k\pi i} \left(\sum_{n<x} + \sum_{n=x} + \sum_{n>x} \right). \tag{4.62}$$

By Lemma 2,

$$\sum_{n>x} = O\left(\sum_{n>x} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \frac{e^{-4\pi(\sqrt{n}-\sqrt{x})c}}{x^{\frac{1}{4}} n^{\frac{1}{4}} (\sqrt{n}-\sqrt{x})} \frac{1}{T^{1+\lambda}} \right) = O\left(\frac{1}{T^{1+\lambda}} \right) \tag{4.63}$$

for any given positive x, and uniformly in I, since

$$|\sqrt{n}-\sqrt{x}| \geqslant \delta = \delta(I) > 0,$$

if x belongs to I. Next, after Lemma 3,

$$\sum_{n=x} = \frac{\sigma_k(x)}{x^{\frac{1}{2}k}} \left\{ \frac{1}{8\pi^2 x^{\frac{1}{2}}} + O\left(\frac{1}{T^{1+\lambda}} \right) \right\} + O\left(\frac{1}{T} \right); \tag{4.64}$$

and finally, by Lemma 4,

$$\sum_{n<x} = \sum_{n<x} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \left\{ \frac{(4\pi\sqrt{n})^k}{\pi(4\pi\sqrt{x})^{1+k}} + O\left(\frac{e^{4\pi(\sqrt{x}-\sqrt{n})}}{x^{\frac{1}{4}} n^{\frac{1}{4}} T^{1+\lambda}} \right) \right.$$
$$\left. + O\left(\frac{e^{4\pi(\sqrt{n}-\sqrt{x})}}{x^{\frac{1}{4}} n^{\frac{1}{4}} (\sqrt{x}-\sqrt{n}) T^{1+\lambda}} \right) \right\} \tag{4.65}$$

for any given positive x and uniformly in I.

From (4.62)–(4.65) we deduce (4.61), which proves the lemma. It may be observed that the error term $O(1/T)$ can be omitted if x is not integral or if x lies in I.

4.7. LEMMA 5′. *If x is a given positive number and T is large, then*

$$R(x) = R_{q,\lambda}(x) = \frac{1}{2\pi i} \int_{c-iT'}^{c+iT'} f_q(s) J_{1+q}(2\pi i s \sqrt{x}) \left(1 - \frac{s^4}{T^4} \right)^{\lambda} ds \tag{4.71}$$

$$= \frac{i e^{\frac{1}{2}q\pi i}}{2\pi x^{\frac{1}{2}+\frac{1}{2}q}} \sum_{1 \leqslant n \leqslant x}' r_p(n) + O\left(\frac{1}{T} \right) + O\left(\frac{1}{T^{1+\lambda}} \right), \tag{4.72}$$

and uniformly in I.

The proof is similar to that of Lemma 5. The same result is true if, in the integral defining $R(x)$, we replace the factor $(1-s^4T^{-4})^{\lambda}$ by $(1+s^2T^{-2})^{\lambda}$. For it is easy to see that Lemmas 2, 3, 4 still hold when this change is made. As the analysis is much the same—if anything, indeed, rather simpler—it is omitted.

5. *The integrals $G(x)$ and $R(x)$ from another aspect.*

5.1. In the last section we calculated $G(x)$ by replacing $g_k(s)$ in the integral (4.11) which defines $G(x)$ by the series (3.16), which, when $\sigma > 0$, defines $g_k(s)$. We shall now use the analytic continuation (3.3) or, if k is an even integer, (3.31), of $g_k(s)$ in order to evaluate $G(x)$ in other terms.

It is convenient in the analysis which follows to suppose that k is not an integer. It is easy, however, so to modify the analysis as to include this case too*.

We require certain preliminary lemmas.

5.2. LEMMA 6. *If ρ is real, a is positive, and T is large, then*

$$I_1(\rho, \nu, a, T) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} s^\rho H_\nu^{(1)}(ias)\left(1-\frac{s^4}{T^4}\right)^\lambda ds = O(T^{\rho-\frac{1}{2}-\lambda}), \tag{5.21}$$

and uniformly for $a \geqq A > 0$.

The proof is similar to that of Lemma 2 (4.3), and so is omitted.

5.3. LEMMA 7. *If, in addition, $\rho+\nu$ is an odd integer, then*

$$I_2(\rho, \nu, a, T) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} s^\rho J_\nu(ias)\left(1-\frac{s^4}{T^4}\right)^\lambda ds \tag{5.31}$$

$$= \tfrac{1}{2}\eta(a, \rho, \nu)+O(T^{\rho-\frac{1}{2}-\lambda})+O(T^{-4}), \tag{5.32}$$

where

$$\eta(a, \rho, \nu) = \frac{(-)^r(\frac{1}{2}ia)^{-1-\rho}}{\Gamma(1+r)\Gamma(1+\nu+r)}, \tag{5.33}$$

if

$$\rho+\nu = -1-2r \quad (r = 0, 1, \ldots) \tag{5.331}$$

and

$$\eta = 0 \quad (\rho+\nu = 1, 3, \ldots). \tag{5.34}$$

The analysis is similar to the analysis of Lemma 4. By Cauchy's theorem,

$$\int_{C_2} s^\rho J_\nu(ias)\left(1-\frac{s^4}{T^4}\right)^\lambda ds = 0, \tag{5.35}$$

* In the Addendum, the proof of Theorem 1 is given for the case $k = 0$, to show what modifications are required.

where C_2 is the contour of Lemma 4 (3.5). The indentations at $s = \pm Ti$ contribute, in the limit, zero; the arcs Γ_2 and Γ_3 yield $O(T^{\rho-\frac{1}{2}-\lambda})$; the side $\sigma = c$ yields $I_2(\rho, \nu, a, T)$, while the side $\sigma = 0$ yields zero, since $\rho+\nu$ is an odd integer. Finally, if $\rho+\nu+1 > 0$, the contribution of the origin is zero; but, if $\rho+\nu = -1-2r$ $(r = 0, 1, \ldots)$, this contribution is $\frac{1}{2}\eta(a, \rho, \nu)+O(T^{-4})$*.

The proof of Lemma 7 is thus completed. The error term $O(T^{-4})$ may be omitted if $\rho+\nu \geqslant -3$.

5.4. LEMMA 8. *If $\rho-\nu$ is an odd integer, then*

$$I_2(\rho, \nu, a, T) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} s^\rho J_\nu(ias)\left(1-\frac{s^4}{T^4}\right)^\lambda ds \tag{5.41}$$

$$= \tfrac{1}{2}e^{\nu\pi i}\eta(a, \rho, -\nu)+O(T^{\rho-\frac{1}{2}-\lambda})+O(T^{-4}), \tag{5.42}$$

where $\eta(a, \rho, -\nu)$ is defined by (5.33) *and* (5.34).

For, from (4.56),

$$I_2(\rho, \nu, a, T) = -i\sin\nu\pi\, I_1(\rho, -\nu, a, T)+e^{\nu\pi i} I_2(\rho, -\nu, a, T),$$

and the lemma follows from Lemmas 6 and 7.

5.5. We apply Lemmas 7 and 8 to the calculation of

$$I_{k,\lambda}(x) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} \phi_k(s) J_{1+k}(4\pi is\sqrt{x})\left(1-\frac{s^4}{T^4}\right)^\lambda ds. \tag{5.41}$$

LEMMA 9. *If x is positive and T is large, then*

$$I_{k,\lambda}(x) = \frac{ie^{\frac{1}{2}k\pi i}}{4\pi x^{\frac{1}{2}+\frac{1}{2}k}}\Phi_k(x)+O(T^{|k|-\frac{1}{2}-\lambda}), \tag{5.52}$$

where

$$\Phi_k(x) = x\zeta(1-k)+x^{1+k}\frac{\zeta(1+k)}{1+k}-\tfrac{1}{2}\zeta(-k). \tag{5.53}$$

For, from (3.21) and the remark on page 307,

$$\phi_k(s) = -\tfrac{1}{2}\frac{\Gamma(k)\,\zeta(k)}{(2\pi s)^k}-\tfrac{1}{2}\frac{\Gamma(-k)\,\zeta(-k)}{(2\pi s)^{-k}}+\frac{\Gamma(1+k)\,\zeta(1+k)}{(2\pi s)^{2+k}}$$

$$+\frac{\Gamma(1-k)\,\zeta(1-k)}{(2\pi s)^{2-k}}+\frac{2}{(2\pi s)^{2+k}}\sum_{r=1}^{\kappa} A_r s^{4r},$$

* The integrand is an *odd* function of s; and

$$\int_\gamma s^{-1-2r}(1-s^4T^{-4})^\lambda\, ds = 0 \quad (r = 1, 2, \ldots),$$

where γ is the semi-circle $|s| = \delta$, $|\arg s| \leqslant \frac{1}{2}\pi$.

where the A_r are constants, so that

$$I_{k,\lambda}(x) = -\tfrac{1}{2}\Gamma(k)\,\xi(k)\,(2\pi)^{-k} I_2(-k,\,1+k) - \tfrac{1}{2}\Gamma(-k)\,\xi(-k)(2\pi)^{k} I_2(k,\,1+k)$$

$$+\frac{\Gamma(1+k)\,\xi(1+k)}{(2\pi)^{2+k}} I_2(-2-k,\,1+k)$$

$$+\frac{\Gamma(1-k)\,\xi(1-k)}{(2\pi)^{-2-k}} I_2(k-2,\,1+k)$$

$$+\sum_{r=1}^{\kappa} B_r I_2(4r-2-|k|,\,1+k),$$

where the B_r are constants, and the remaining arguments of the functions I_2 are $4\pi\sqrt{x}$ and T.

The application of Lemmas 7 and 8 gives, after a slight reduction,

$$I_{k,\lambda}(x) = \frac{ie^{\frac{1}{2}k\pi i}}{4\pi x^{\frac{1}{2}+\frac{1}{2}k}}\left\{x\xi(1-k)+x^{1+k}\,\frac{\xi(1+k)}{1+k}-\tfrac{1}{2}\xi(-k)\right\}$$

$$+O(T^{|k|-\frac{1}{2}-\lambda})+O(T^{4\kappa-2-|k|-\frac{1}{2}-\lambda}),$$

and this proves the lemma, since $\kappa=[\frac{1}{2}+\frac{1}{2}|k|]$, so that

$$4\kappa-2-|k|\leqslant|k|.$$

5.6. In what follows it is necessary in one case slightly to restrict T, and, throughout, to take $c<1$. When $\lambda=0$ we shall suppose that $T^2=N+\frac{1}{2}$, where N is a positive integer, which we shall finally make to tend to infinity. If $\lambda>0$, we do not restrict T; we shall make T in this case tend to infinity continuously.

LEMMA 10. *If $\lambda\geqslant 0$, ρ is real, $a>0$, and n is a positive integer, then for all large T,*

$$U_1(\rho,\nu,n,a,T)=\frac{1}{2\pi i}\int_{c-iT'}^{c+iT'}\frac{s^\rho}{s^4-n^2}\,H_\nu^{(1)}(ias)\left(1-\frac{s^4}{T^4}\right)^\lambda ds \tag{5.61}$$

$$=\xi_\lambda(\rho,\nu,n,a,T)+\Lambda_\lambda(n,T,\rho), \tag{5.62}$$

where

$$\xi_\lambda=-\tfrac{1}{4}n^{\frac{1}{2}\rho-\frac{3}{2}}H_\nu^{(1)}(ia\sqrt{n})\left(1-\frac{n^2}{T^4}\right)^\lambda \quad (n<T^2) \tag{5.63}$$

and

$$\xi_\lambda=0 \quad (n\geqslant T^2), \tag{5.631}$$

and

$$(5.64)\quad \Lambda_0 = O\left(\frac{T^{\rho-\frac{1}{2}}}{|T^4-n^2|}\right), \qquad \Lambda_\lambda = O\left[\mathrm{Min}\left\{\frac{T^{\rho-\frac{1}{2}-\lambda}}{|T^4-n^2|}, \frac{T^{\rho-\frac{1}{2}-\lambda}}{Tn}\right\}\right],$$

if $\lambda > 0$.

The proof is similar to that of Lemma 2, the same contour C_1 being used. We apply Cauchy's theorem to the integral round C_1 of the integrand of (5 . 61). If we use the inequality

$$|T^4 e^{4i\theta} - n^2| \geqslant \mathrm{Max}\,(|T^4-n^2|,\ 2T^2 n \sin 2\theta),$$

valid for all real θ, n, and T, it follows without difficulty, by employing (4 . 33) and Lemma 1, that the contribution of the circular part Γ_1 of C_1 is $\Lambda_\lambda(n, T, \rho)$, the convention about λ and T being borne in mind.

If $\lambda = 0$, the point $s = \sqrt{n}$ is a simple pole of the integrand, and yields the term ξ_λ. If $\lambda > 0$, $s = \sqrt{n}$ is still a simple pole yielding ξ_λ provided that $n \neq T^2$. If $n = T^2$, the pole $s = \sqrt{n}$ coalesces with the branch-point $s = T$; the contribution is still zero since $(\lambda-1)+1$ is positive, *i.e.* it is ξ_λ. The lemma follows at once.

[*Note.* The restriction that n is integral is merely for convenience.]

5 . 7. LEMMA 11. *If to the conditions of Lemma* 10 *we add that* $\rho+\nu$ *is an odd positive integer, then*

$$(5.71)\quad U_2(\rho, \nu, a, T) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} \frac{s^\rho}{s^4-n^2} J_\nu(ias)\left(1-\frac{s^4}{T^4}\right)^\lambda ds$$

$$(5.72)\quad = \xi_\lambda(\rho, \nu, n, a, T) + \Lambda_\lambda(n, T, \rho),$$

where

$$(5.73)\quad \xi_\lambda = -\tfrac{1}{4} e^{\frac{1}{2}\nu\pi i} \sin \tfrac{1}{2}(\rho+\nu)\pi \,.\, n^{\frac{1}{2}\rho-\frac{3}{4}} J_\nu(a\sqrt{n})\left(1-\frac{n^2}{T^4}\right)^\lambda,$$

if $n < T^2$, *and*

$$(5.74)\quad \xi_\lambda = 0,$$

if $n \geqslant T^2$.

The proof is similar to that of Lemma 7. By Cauchy's theorem the integral round C_2 of the integrand of (5 . 71) is zero. Now C_2 has on the side $\sigma = 0$ the simples poles $s = \pm i\sqrt{n}$ of the integrand, which together contribute $-\zeta_\lambda$ when $n < T^2$ or $n > T^2$. When $n = T^2$, since now $\lambda > 0$ (by our convention), the contribution is zero and is given by $-\xi_\lambda$. The side $\sigma = 0$ itself contributes zero, since $\rho+\nu$ is an odd integer,

while the contribution of the arcs Γ_2 and Γ_3 is $\Lambda_\lambda(n, T, \rho)$. The origin, being a regular point, contributes zero; and, lastly, the contribution of $\sigma = c$ is plainly U_2. The lemma is therefore proved.

Using (4 . 56) we deduce, from Lemmas 10 and 11,

LEMMA 12. *If to the conditions of Lemma* 10 *we add that* $\rho - \nu$ *is an odd positive integer, then*

$$(5\,.\,75)\quad U_2(\rho, \nu, n, a, T) = -i \sin \nu\pi\, \xi_\lambda(\rho, -\nu) + e^{\nu\pi i}\zeta_\lambda(\rho, -\nu) + \Lambda_\lambda(n, T, \rho),$$

the remaining arguments of ξ *and* ζ *being* n, a, *and* T.

5 . 8. We proceed to calculate $G(x)$. From (3 . 3), (4 . 11), (5 . 51), and (5 . 71) we deduce that

$$(5\,.\,81)\quad G(x) = I_{k,\lambda}(x) + \frac{1}{2\pi \sin \frac{1}{2}|k|\pi} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \times \left\{ \frac{U_2(2+|k|,\ 1+k)}{n^{\frac{1}{2}|k|}} - \frac{U_2(4\kappa+2-|k|,\ 1+k)}{n^{2\kappa-\frac{1}{2}|k|}} \right\},$$

where the remaining arguments of U_2 are n, $4\pi\sqrt{x}$ and T, the term-by-term integration being legitimate owing to the finiteness of the path of integration and the uniform convergence thereon of the series (3 . 3).

Now, from Lemmas 10, 11, and 12, we see that, whether k is positive or negative,

$$(5\,.\,82)\quad \frac{1}{2\pi \sin \frac{1}{2}|k|\pi} \left\{ \frac{U_2(2+|k|, 1+k)}{n^{\frac{1}{2}|k|}} - \frac{U_2(4\kappa+2-|k|, 1+k)}{n^{2\kappa-\frac{1}{2}|k|}} \right\}$$

$$= \frac{-ie^{\frac{1}{2}k\pi i}}{8\pi\sqrt{n} \sin \frac{1}{2}k\pi} \{J_{1+k}(a) + J_{-1-k}(a) + e^{-\frac{1}{2}k\pi i} \sin k\pi H^{(1)}_{-1-k}(ia)\} \left(1 - \frac{n^2}{T^4}\right)^\lambda$$

$$+ \frac{1}{2\pi \sin \frac{1}{2}|k|\pi} \left\{ \frac{\Lambda_\lambda(n, T, 2+|k|)}{n^{\frac{1}{2}|k|}} - \frac{\Lambda_\lambda(n, T, 4\kappa+2-|k|)}{n^{2\kappa-\frac{1}{2}|k|}} \right\},$$

where $a = 4\pi\sqrt{(xn)}$, provided that $n < T^2$; if $n \geqslant T^2$, the first term on the right has to be omitted. But*

$$J_\nu(z) + J_{-\nu}(z) - ie^{-\frac{1}{2}\nu\pi i} H^{(1)}_{-\nu}(iz)$$

$$= 2 \cos \tfrac{1}{2}\nu\pi \left[J_\nu \cos \tfrac{1}{2}\nu\pi - \left\{ Y_\nu + \frac{2}{\pi} K_\nu \right\} \sin \tfrac{1}{2}\nu\pi \right]$$

$$(5\,.\,83)\qquad = 2 \cos \tfrac{1}{2}\nu\pi F_\nu(z),$$

* Watson, *loc. cit.*, 74, 78.

from the definition (1.21), so that

$$(5.84)\quad G(x) = I_{k,\lambda}(x) + \frac{ie^{\frac{1}{2}k\pi i}}{4\pi} \sum_{n<T^2} \frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k}} F_{1+k}(a)\left(1-\frac{n^2}{T^4}\right)^{\lambda}$$

$$+ \frac{1}{2\pi \sin \frac{1}{2}|k|\pi} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \left\{ \frac{\Lambda_\lambda(n, T, 2+|k|)}{n^{\frac{1}{2}|k|}} \right.$$

$$\left. - \frac{\Lambda_\lambda(n, T, 4\kappa+2|k|)}{n^{2\kappa-\frac{1}{2}|k|}} \right\},$$

where $a = 4\pi\sqrt{(nx)}$.

In order to deal with the last sum we require

LEMMA 13. *If* $-1 \leqslant b < 1$, *then*

$$(5.85)\quad \sum_{n=1}^{\infty} n^b \operatorname{Min}\left\{ \frac{1}{|T^4-n^2|}, \frac{1}{nT} \right\} = O(T^{2b-2+\epsilon})$$

for all large T *and any positive* ϵ.

Writing $N = [T^2]$, we have

$$(5.86)\quad \sum_{1}^{\infty} = \sum_{n \leqslant \frac{1}{2}N} + \sum_{\frac{1}{2}N}^{N-1} + \sum_{N}^{N+1} + \sum_{N+2}^{2N} + \sum_{2N+1}^{\infty} = S_1+S_2+S_3+S_4+S_5,$$

say. It is evident that, in S_1,

$$T^4-n^2 > AT^4,$$

where A denotes generally positive constants, so that

$$(5.861)\quad S_1 < A \sum_{1}^{\frac{1}{2}N} T^{-4}n^b < AT^{-4}N^{b+1+\epsilon} < AT^{2b-2+\epsilon},$$

where $\epsilon > 0$; in S_2,

$$T^2+n > AT^2, \quad AT^2 < n < AT^2,$$

so that

$$(5.862)\quad S_2 < A \sum_{\frac{1}{2}N}^{N-1} \frac{T^{2b}}{T^2(T^2-n)} < A \sum_{1}^{N} \frac{T^{2b-2}}{m} < AT^{2b-2+\epsilon};$$

similarly

$$(5.863)\quad S_4 < AT^{2b-2+\epsilon},$$

while

$$(5.864)\quad S_3 \leqslant \frac{N^{b-1}}{T} + \frac{(N+1)^{b-1}}{T} < AT^{2b-3}$$

and

$$S_5 \leqslant \sum_{m=1}^{\infty} \frac{(2N+m)^b}{(2N+m-T^2)(2N+m+T^2)} < A \sum_{m=1}^{\infty} (N+m)^{b-2}$$

$$< AT^{2b-2}. \tag{5.865}$$

From (5.86) and (5.861)–(5.865), (5.85) follows immediately, since the sums involved are positive. The constant implied in the O is, of course, a function of b and ϵ.

In precisely the same way we may prove that, if $T^2 = N+\frac{1}{2}$ and $-1 \leqslant b < 1$, then

$$\sum_{1}^{\infty} \frac{n^b}{|T^4-n^2|} = O(T^{2b-2+\epsilon}) \tag{5.87}$$

for each positive ϵ.

Since

$$\sigma_k(n)\, n^{-\frac{1}{2}k} = O(n^{\frac{1}{2}|k|+\epsilon}),$$

for each positive ϵ, we see, from (5.64), that, if $\rho = |k|+2$ or $4\kappa-|k|+2$,

$$\sum_{1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \frac{\Lambda_\lambda(n, T, \rho)}{n^{\frac{1}{2}\rho-1}} = O \sum_{1}^{\infty} \frac{T^{\rho-\frac{1}{2}-\lambda}}{n^{\frac{1}{2}\rho-1-\frac{1}{2}k|-\epsilon}} \operatorname{Min}\left\{\frac{1}{|T^4-n^2|}, \frac{1}{Tn}\right\}$$

$$= O\{T^{\rho-\frac{1}{2}-\lambda+2+|k|+2\epsilon-\rho+\epsilon'}\} = O\{T^{|k|-\frac{1}{2}+\epsilon-\lambda}\} \tag{5.88}$$

by (5.85), since* $b = 1+\frac{1}{2}|k|+\epsilon-\frac{1}{2}\rho$ satisfies the conditions of Lemma 13 if ϵ is sufficiently small.

This result holds also for $\lambda = 0$, but instead of (5.85) we must use (5.87).

From Lemma 9, (5.81) and (5.88) we deduce that

$$G(x) = \frac{ie^{\frac{1}{2}k\pi i}}{4\pi}\left\{\frac{\Phi_k(x)}{x^{\frac{1}{2}+\frac{1}{2}k}} + \sum_{1 \leqslant n < T^2} \frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k}} F_{1+k}\{4\pi\sqrt{(nx)}\}\left(1-\frac{n^2}{T^4}\right)^\lambda\right\} \tag{5.89}$$

$$+ O(T^{|k|-\frac{1}{2}+\epsilon-\lambda}),$$

where $x > 0$, $\lambda \geqslant 0$, and T is large (with the condition that, if $\lambda = 0$, $T^2 = N+\frac{1}{2}$, where N is a positive integer). Further, (5.89) holds uniformly with respect to x in any finite interval $0 < x_1 \leqslant x \leqslant x_2$, as is easily seen from the analysis.

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* If $\rho = |k|+2$, then $b = \epsilon$; if $\rho = 4\kappa-|k|+2$, then $b = |k|+\epsilon-2\kappa$. Since $\kappa = [\frac{1}{2}+\frac{1}{2}|k|]$, we have $2\kappa \leqslant 1+|k| < 2\kappa+1$, so that $b \geqslant -1$, and $b < 1$ if ϵ is small enough.

5 . 9. *The corresponding result for* $R(x)$. In exactly the same way we can prove that

$$R(x) = \frac{ie^{\frac{1}{2}q\pi i}}{2\pi}\left\{\frac{\pi^{q+1}x^{\frac{1}{2}+\frac{1}{2}q}}{\Gamma(q+2)} - \frac{1}{x^{\frac{1}{2}+\frac{1}{2}q}} + \sum_{1\leqslant n<T^2}\frac{r_p(n)}{n^{\frac{1}{2}+\frac{1}{2}q}}J_{1+q}\{2\pi\sqrt{(nx)}\}\left(1-\frac{n^2}{T^4}\right)^{\lambda}\right\}$$

$$+O(T^{q-\frac{1}{2}+\epsilon-\lambda}) \qquad (5\,.\,91)$$

with the same convention as to T and λ. Here it is possible to replace the factor $\{1-(n^2/T^4)\}^{\lambda}$ by $\{1-(n/T^2)\}^{\lambda}$.

I omit the proof since the analysis is practically identical with that used in proving (5 . 89).

6. *The proofs of Theorem* 1 *and Theorem* 2.

6 . 1. *Proof of Theorem* 1. From Lemma 5 (4 . 61) and (5 . 89) we deduce that, if $|k|<\frac{1}{2}$,

$$\sum_{n\leqslant x}{}' \sigma_k(n) = \Phi_k(x) + x^{\frac{1}{2}+\frac{1}{2}k}\sum_{n=1}^{N}\frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k}}F_{1+k}\{4\pi\sqrt{(nx)}\} + O(N^{\frac{1}{2}|k|-\frac{1}{4}+\epsilon}), \qquad (6\,.\,11)$$

and, if $|k|\geqslant\frac{1}{2}$, $\lambda>0$,

$$\sum_{n\leqslant x}{}' \sigma_k(n) = \Phi_k(x) + x^{\frac{1}{2}+\frac{1}{2}k}\sum_{1\leqslant n<T^2}\frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k}}F_{1+k}\{4\pi\sqrt{(nx)}\}\left(1-\frac{n^2}{T^4}\right)^{\lambda}$$

$$+O(T^{-1})+O(T^{-1-\lambda})+O(T^{|k|-\frac{1}{2}+\epsilon-\lambda}), \qquad (6\,.\,12)$$

each formula being valid for any given $x>0$ and uniformly in any closed interval I free from integers; ϵ is any positive number.

Since $|k|-\frac{1}{2}+\epsilon<0$ if $|k|<\frac{1}{2}$, for small enough ϵ, we deduce, by making $N\to\infty$ in (6 . 11), the first part of Theorem 1.

Making $T\to\infty$ continuously in (6 . 12), we deduce, if $\lambda>|k|-\frac{1}{2}$, so that $\lambda\geqslant|k|-\frac{1}{2}+\epsilon$ for sufficiently small ϵ, the second part of Theorem 1, except that we have proved the series to be summable $(R, n^2, \lambda\geqslant|k|-\frac{1}{2})$. But it is known* that summability (R, n^2, λ) implies summability (R, n, λ). And this completes the proof of Theorem 1 in the case when k is not integral. The modifications required when k is an odd integer are trivial; those required when k is an even integer are a little more difficult.

* Hardy and Riesz, *loc. cit.*, 33.

6.2. *Proof of Theorem* 2. From (4.71) and (5.91) we infer that

$$(6.21)\qquad \sum_{0\leqq n\leqq x}{}' r_2(n) = \pi x + x^{\frac{1}{2}} \sum_{n=1}^{N} \frac{r_2(n)}{n^{\frac{1}{2}}} J_1\{2\pi\sqrt{(nx)}\} + O(N^{-\frac{1}{2}+\epsilon}),$$

and, if $p \geqq 3$, $\lambda > 0$,

$$(6.22)\qquad \sum_{0\leqq n\leqq x}{}' r_p(n) = \frac{\pi^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}\, x^{\frac{1}{2}p} + x^{\frac{1}{4}p} \sum_{1\leqq n<T^2} \frac{r_p(n)}{n^{\frac{1}{4}p}} J_{\frac{1}{2}p}\{2\pi\sqrt{(nx)}\}\left(1-\frac{n^2}{T^4}\right)^{\lambda} + O(T^{-1}) + O(T^{-1-\lambda}) + O(T^{\frac{1}{2}p-3+\epsilon-\lambda}),$$

each formula being valid for any given $x > 0$ and uniformly in any closed interval I free from integers; ϵ is any positive number.

Taking $\epsilon < \frac{1}{4}$, we deduce, from (6.21), the first part of Theorem 2 by making $N \to \infty$. The second part follows, from (6.22), by taking $\lambda > \frac{1}{2}p - \frac{3}{2}$ and then $\epsilon < \lambda - \frac{1}{2}p + \frac{3}{2}$ and making $T \to \infty$ continuously. This proves Theorem 2, but with summability $(R,\ n^2,\ \lambda > \frac{1}{2}p - \frac{3}{2})$. Appealing, as before, to known theorems on summability, the proof of Theorem 2 is completed.

[We can prove the summability $(R,\ n,\ \lambda > \frac{1}{2}p - \frac{3}{2})$ directly by replacing the factor $\{1-(s^4/T^4)\}^{\lambda}$ in the definition of $R(x)$ by $\{1+(s^2/T^2)\}^{\lambda}$ and modifying the analysis correspondingly. The analysis, in fact, becomes rather simpler.]

7. *The proof of Theorem* 3 *and Theorem* 4, *and generalizations.*

7.1. It is plain that the results contained in the Lemmas 6–13 of § 5 enable us to calculate the integral

$$(7.11)\qquad G_{k,\rho,\lambda}(x) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} s^{-\rho} g_k(s)\, J_{k+\rho+1}(4\pi i s\sqrt{x})\left(1-\frac{s^4}{T^4}\right)^{\lambda} ds$$

for any integral value of ρ. We have already considered the case $\rho = 0$. The case $\rho = -1$ furnishes us with a proof of Theorem 3. We obtain, in fact,

$$(7.12)\qquad G_{k,-1,\lambda}(x) = \frac{e^{\frac{1}{2}k\pi i}}{8\pi^2}\{x^{\frac{1}{2}k}\zeta(1+k) + x^{-\frac{1}{2}k}\zeta(1-k)\} + \frac{e^{\frac{1}{2}k\pi i}}{4\pi} \sum_{n<T^2} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} F_{k+1,-1}\{4\pi\sqrt{(nx)}\}\left(1-\frac{n^2}{T^4}\right)^{\lambda} + O(T^{|k|+\frac{1}{2}-\lambda+\epsilon})$$

for $\lambda > 0$, $x > 0$ and large T, where

$$(7.13) \quad F_{\nu,\rho}(z) = \cos \tfrac{1}{2}\nu\pi J_{\nu+\rho} - \sin \tfrac{1}{2}\nu\pi \left\{ Y_{\nu+\rho} + \frac{2}{\pi} \cos \rho\pi K_{\nu+\rho} \right\}.$$

Now substituting for $g_k(s)$ the defining series (3.16) and integrating term by term, which is permissible by uniform convergence in the finite range of integration, we obtain

$$(7.14) \quad G_{k,\rho,\lambda}(x) = i\pi e^{\frac{1}{2}k\pi i} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} F(\lambda, k, k+\rho+1, \rho, 4\pi\sqrt{n}, 4\pi\sqrt{x}, T),$$

where

$$(7.15) \quad F(\lambda, \mu, \nu, a, a, b, T) = \frac{1}{2\pi i} \int_{c-iT'}^{c+iT'} s^{-a} H_{\mu}^{(1)}(ias) J_{\nu}(ibs) \left(1 - \frac{s^4}{T^4}\right)^{\lambda} ds,$$

a, b being positive and a real. This integral, in the case $a = 0$ and $\mu - \nu = \pm 1$, has been discussed in Lemmas 2, 3, and 4 of § 4. By a similar piece of analysis we may evaluate the integral in the case $\nu = \mu + \rho'$, $a = \rho$ where ρ and ρ' are integers and $\rho' - \rho$ is an odd integer. It is on this integral that the problem of finding the sum (R, n^2, λ), for some λ, of the series

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}\rho}} F_{k+1,\rho}\{4\pi\sqrt{(nx)}\}$$

depends.

7.2. LEMMA 2′. *If $\lambda + 1 > 0$, a is real and $a > b > 0$, then, for all large T,*

$$(7.21) \quad F(\lambda, \mu, \nu, a, a, b, T) = O\{a^{-\frac{1}{2}} b^{-\frac{1}{2}} (a-b)^{-1} T^{-a-\lambda-1} e^{-(a-b)c}\}$$

uniformly for $a \geqq A' > A \geqq b \geqq B > 0$; and, if $a \geqq b$,

$$(7.22) \quad F = O\{T^{-a} e^{-(a-b)c} a^{-\frac{1}{2}} b^{-\frac{1}{2}}\}$$

uniformly for $a \geqq b > B > 0$.

Proceeding as in Lemma 2 (4.3) we find that

$$(7.23) \quad F = O \int_{-\frac{1}{2}\pi + \arcsin(c/T)}^{\frac{1}{2}\pi - \arcsin(c/T)} T^{-a} a^{-\frac{1}{2}} b^{-\frac{1}{2}} |\sin 2\theta|^{\lambda} e^{-(a-b)T\cos\theta} d\theta.$$

From (7.23) and Lemma 1 (4.21), (7.21) follows immediately.

To obtain (7 . 22) we observe that

$$(a-b)\,T\cos\theta \geqslant (a-b)\,c \quad \{|\theta| \leqslant \tfrac{1}{2}\pi - \arcsin(c/T)\},$$

so that, from (7 . 23),

$$F = O\int_{-\frac{1}{2}\pi+\arcsin(c/T)}^{\frac{1}{2}\pi-\arcsin(c/T)} T^{-\alpha}a^{-\frac{1}{2}}b^{-\frac{1}{2}}e^{-(a-b)c}|\sin 2\theta|^{\lambda}\,d\theta,$$

whence (7 . 22) follows at once, since

$$\int_{-\frac{1}{2}\pi+\arcsin(c/T)}^{\frac{1}{2}\pi-\arcsin(c/T)} |\sin 2\theta|^{\lambda}d\theta \leqslant 2^{\lambda}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}|\sin 2\theta|^{\lambda}d\theta = O(1).$$

7 . 3. LEMMA 3′. *If* $\lambda+1>0$, a *is real and* $a=b(>0)$, *then, for all large* T,

(7 . 31) $F(\lambda, \mu, \nu, a, a, a, T)$

$$= \frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{2\pi i a T^{a}}\,\frac{\Gamma(1+\lambda)\cos\frac{1}{4}a\pi}{\Gamma(1+\frac{1}{4}a)\,\Gamma(1+\lambda-\frac{1}{4}a)}$$

$$+\frac{(\nu^2-\mu^2)\,e^{\frac{1}{2}(\nu-\mu)\pi i}}{4\pi i a^2 T^{a+1}}\,\frac{\Gamma(1+\lambda)\cos\frac{1}{4}(a+1)\,\pi}{\Gamma(1+\frac{1}{4}+\frac{1}{4}a)\,\Gamma(1+\lambda-\frac{1}{4}-\frac{1}{4}a)}$$

$$+O\left(\frac{1}{aT^{a+\lambda}}\right)+O\left(\frac{1}{a^3T^{a+2}}\right)$$

uniformly for $a \geqslant A > 0$.

We obtain (7 . 31) just as we obtained (4 . 41). We require, however, an extra term in the asymptotic expansions of $H_{\nu}^{(1)}(z)$ and $J_{\nu}(z)$. These expansions are*

(7 . 32) $$H_{\nu}^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}\left\{1-\frac{\nu^2-\frac{1}{4}}{2iz}+O\left(\frac{1}{|z|^2}\right)\right\}$$

and

(7 . 33) $$J_{\nu}(z) = \left(\frac{1}{2\pi z}\right)^{\frac{1}{2}} e^{-i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}\left\{1+\frac{\nu^2-\frac{1}{4}}{2iz}+O\left(\frac{1}{|z|^2}\right)\right\}$$

$$+c_1 e^{i(z+\frac{1}{2}\nu\pi+\frac{1}{4}\pi)}\left\{1+O\left(\frac{1}{|z|}\right)\right\},$$

valid under the same conditions as (4 . 33) and (4 . 34).

* Watson, *loc. cit.*, 198, 202.

Proceeding as in Lemma 3 (4 . 4) we obtain

$$F(\lambda, \mu, \nu, \alpha, a, a, T)$$

$$= \frac{1}{2\pi i} P\int_{\Gamma_1} \frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{\pi i a s^{1+\alpha}} \left\{1+\frac{\mu^2-\nu^2}{2as}+O\left(\frac{1}{a^2T^2}\right)\right\}\left(1-\frac{s^4}{T^4}\right)^\lambda ds$$

$$+O\int_{\Gamma_1}\left|\left(1-\frac{s^4}{T^4}\right)^\lambda\right|\left|\frac{1}{\pi a s^{1+\alpha}}e^{-2as}\right||ds|$$

$$(7\,.\,34) \qquad = \frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{\pi i a}\left\{J_1(\alpha)+\frac{\mu^2-\nu^2}{2a}J_1(\alpha+1)\right\}+J_2+J_3,$$

say, where

$$(7\,.\,35) \qquad J_3 = O\int_{-\frac{1}{2}\pi+\arcsin(c/T)}^{\frac{1}{2}\pi-\arcsin c/T} a^{-1}T^{-\alpha}|\sin 2\theta|^\lambda e^{-2aT\cos\theta}d\theta$$

$$= O\{a^{-1}T^{-\alpha}\omega_\lambda(2a)\},$$

$$(7\,.\,36) \qquad J_2 = O\int_{\Gamma_1}\left|\frac{e^{\frac{1}{2}(\nu-\mu)\pi i}}{\pi i a s^{1+\alpha}}\left(1-\frac{s^4}{T^4}\right)^\lambda ds\right| O\left(\frac{1}{a^2T^2}\right) = O\left(\frac{1}{a^3T^{\alpha+2}}\right),$$

and

$$(7\,.\,37) \qquad J_1(\alpha) = \frac{1}{2\pi i}\int_{\Gamma_1} s^{-1-\alpha}\left(1-\frac{s^4}{T^4}\right)^\lambda ds$$

$$= \frac{1}{2\pi i}\int_{\Gamma_4} - \frac{1}{2\pi i}\int_{\Gamma_2} - \frac{1}{2\pi i}\int_{\Gamma_3},$$

the arcs being those of (4 . 4), and each being described positively. Plainly

$$(7\,.\,38) \qquad \frac{1}{2\pi i}\int_{\Gamma_2} = O\int_{\frac{1}{2}\pi-\arcsin(c/T)}^{\frac{1}{2}\pi} T^{-\alpha}|\sin 2\theta|\,d\theta = O(T^{-\alpha-\lambda-1}),$$

and similarly for $\int_{\Gamma_3}$. As for $\int_{\Gamma_4}$, assuming for the moment that α is negative, we can flatten Γ_4 into the imaginary axis from $-Ti$ to Ti without altering the value of the integral. [This follows by Cauchy's theorem: the indentations at $\pm Ti$ yield, in the limit, zero, since $\lambda+1>0$; so, too, does the indentation at $s=0$, since $\alpha<0$.] Hence

$$\int_{\Gamma_4} = \int_T^0 e^{\frac{1}{2}\alpha\pi i}\left(1-\frac{t^4}{T^4}\right)^\lambda\frac{dt}{t^{1+\alpha}}+\int_0^T e^{-\frac{1}{2}\alpha\pi i}\left(1-\frac{t^4}{T^4}\right)^\lambda\frac{dt}{t^{1+\alpha}}$$

$$= -2i\sin\tfrac{1}{2}\alpha\pi\int_0^T t^{-1-\alpha}\left(1-\frac{t^4}{T^4}\right)^\lambda dt$$

$$(7\,.\,39) \qquad = \frac{-i\sin\frac{1}{2}\alpha\pi}{2T^\alpha}\frac{\Gamma(-\frac{1}{4}\alpha)\,\Gamma(\lambda+1)}{\Gamma(1+\lambda-\frac{1}{4}\alpha)} = \frac{\pi i}{T^\alpha}\frac{\Gamma(1+\lambda)\cos\frac{1}{4}\alpha\pi}{\Gamma(1+\frac{1}{4}\alpha)\,\Gamma(1+\lambda-\frac{1}{4}\alpha)}.$$

This result, proved for $a<0$, is now seen, by analytic continuation, to be true for all a. From (7 . 34)–(7 . 39) we deduce (7 . 31). And this is Lemma 3′.

[*Note.* The coefficient of T^{-1-a} has been calculated because the coefficient of T^{-a} vanishes when a is a negative even integer. In this case, supposing $\mu \neq \nu$, the coefficient of T^{-1-a} does not vanish. In any case, therefore, if a is a negative integer,

$$|F(\lambda, \mu, \nu, a, a, T)| \to \infty$$

when $T \to \infty$, for any $\lambda > 0$.

This result has to be proved when we wish to generalize, by further differentiations, the second part of Theorem 3. The retention of the term in T^{-1-a} is thus necessary.]

7 . 4. LEMMA 4′. *If $\lambda+1>0$, ρ is an integer, and T is large, then*

$$(7 . 41)\quad F(\lambda, \mu, \mu+\rho+1, \rho, a, b, T)-F(\lambda, \mu+\rho+1, \mu, \rho, b, a, T)$$
$$= u_\rho(a, b, \mu)+v_\rho(a, b, \mu, T)+O(e^{|b-a|c}a^{-\frac{1}{2}}b^{-\frac{1}{2}}T^{-\rho-1-\lambda}),$$

*where**

$$(7 . 42)\qquad u_\rho(a, b, \mu) = \frac{e^{\frac{1}{2}\rho\pi i}}{2^\rho \pi}\,\frac{a^\mu}{b^{\mu+\rho+1}}\,\frac{(b^2-a^2)^\rho}{\Gamma(1+\rho)},$$

and, if $\lambda = 0$ or if $\rho \leqslant 1$,

$$(7 . 43)\qquad v_\rho(a, b, \mu, T) = 0,$$

but, if $\lambda \neq 0$ and $\rho \geqslant 2$,

$$(7 . 44)\qquad |v_\rho| < K(\lambda, \mu, \rho)\, a^\mu b^{-\mu-\rho-1}\{\mathrm{Max}(a, b)\}^{2\rho-4}T^{-4},$$

uniformly for $a \geqslant A > 0$ and $b \geqslant B > 0$.

Suppose, for simplicity, that μ is not integral. Then, writing for brevity

$$(7 . 51)\qquad Q(s) = H^{(1)}_\mu(ias)J_{\mu+\rho+1}(ibs)-H^{(1)}_{\mu+\rho+1}(ibs)J_\mu(ias),$$

the formula† $\qquad i \sin \mu\pi H^{(1)}_\mu = J_{-\mu}-e^{-\mu\pi i}J_\mu$

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* Plainly $u_\rho = 0$ if $\rho = -1, -2, \ldots$.

† Watson, *loc. cit.*, 74.

gives immediately

$$(7.52) \qquad i \sin \mu\pi Q(s) = J_{-\mu}(ias) J_{\mu+\rho+1}(ibs) + \cos \rho\pi J_{\mu}(ias) J_{-\mu-\rho-1}(ibs).$$

It follows at once that the function

$$P(s) = s^{-\rho} Q(s) \left(1 - \frac{s^4}{T^4}\right)^{\lambda}$$

is analytic inside and on the contour C formed by the lines $\sigma = \pm c$ and the arcs Γ_5 and Γ_6 of the circle $|s| = T$ intercepted between them, these arcs being indented at $s = \pm Ti$. Also it is plain that $P(s)$ is an *odd* function of s. If ρ is non-negative, $P(s)$ is regular throughout C save for a pole of order $2\rho+1$ at the origin; if ρ is negative, $P(s)$ is regular throughout C.

Cauchy's theorem applied to $\int_C P(s)ds$ gives us, therefore,

$$(7.54) \qquad \frac{1}{2\pi i} \int_{c-iT'}^{c+iT'} + \frac{1}{2\pi i} \int_{\Gamma_5} = 0$$

if ρ is negative, and, if ρ is non-negative,

$$(7.55) \qquad \frac{1}{2\pi i} \int_{c-iT'}^{c+iT'} + \frac{1}{2\pi i} \int_{\Gamma_5} = \tfrac{1}{2} S,$$

where S is the residue (since $\rho \geqslant 0$) of $P(s)$ at the origin.

We proceed to calculate S. Since

$$J_{\nu}(iz) = 2^{-\nu} e^{\frac{1}{2}\nu\pi i} z^{\nu} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{2m}}{m!\, \Gamma(\nu+1+m)},$$

we may write

$$(7.56) \qquad s^{-\rho} J_{\mu}(ias) J_{-\mu-\rho-1}(ibs) = \frac{e^{-\frac{1}{2}(\rho+1)\pi i} a^{\mu} s^{-2\rho-1}}{2^{-(\rho+1)} b^{\mu+\rho+1}} \sum_{l=0}^{\infty} c_l s^{2l},$$

where

$$(7.561) \qquad c_l = c_l(a, b) = \frac{1}{4^l} \sum_{m+n=l} \frac{b^{2m} a^{2n}}{m!\, n!\, \Gamma(-\mu-\rho+m)\, \Gamma(\mu+1+m)}.$$

Also

$$(7.57) \qquad \left(1 - \frac{s^4}{T^4}\right)^{\lambda} = \sum_{r=0}^{\infty} (-)^r \binom{\lambda}{r} \frac{s^{4r}}{T^{4r}}.$$

Hence the residue at $s = 0$, being the coefficient of $1/s$ in the expansion

in powers of s of $P(s)$, is equal to

$$\frac{\cos\rho\pi}{i\sin\mu\pi}\,\frac{e^{-\frac{1}{2}(\rho+1)\pi i}a^{\mu}}{2^{-(\rho+1)}b^{\mu+\rho+1}}$$

times the coefficient of $s^{2\rho}$ in the expansion of

$$\sum_{l=0}^{\infty} c_l s^{2l} \sum_{r=0}^{\infty} (-)^r \binom{\lambda}{r} s^{4r} T^{-4r},$$

and this coefficient is clearly

$$d_\rho = \sum_{m=0}^{\frac{1}{2}\rho} (-)^m \binom{\lambda}{m} c_{\rho-2m} T^{-4m}. \tag{7.581}$$

Now, if $m+n=\rho$,

$$\Gamma(-\mu-\rho+m)\Gamma(\mu+1+n) = \pi \operatorname{cosec}(\mu+1+n)\pi = (-)^{n+1}\pi \operatorname{cosec}\mu\pi,$$

so that, from (7.561),

$$c_\rho = \frac{1}{4^\rho}\sum_{n=0}^{\rho}\frac{(-)^{n+1}\sin\mu\pi}{\pi}\,\frac{a^{2n}b^{2\rho-2n}}{n!\,(\rho-n)!} \tag{7.582}$$

$$= -\frac{\sin\mu\pi}{4^\rho\pi}\,\frac{(b^2-a^2)^\rho}{\Gamma(1+\rho)}.$$

Again, from (7.561),

$$|c_l| < K(\mu,\rho)\operatorname{Max}(a^{2l}, b^{2l}) \quad (0 \leqslant l \leqslant \rho-1). \tag{7.583}$$

Hence

$$S = 2\bar{u}_\rho(a, b, \mu)+2\bar{v}_\rho(a, b, \mu, T) \tag{7.59}$$

where

$$\bar{u}_\rho = \tfrac{1}{2}\frac{\cos\rho\pi}{i\sin\mu\pi}\,\frac{e^{-\frac{1}{2}(\rho+1)\pi i}a^{\mu}}{2^{-\rho-1}b^{\mu+\rho+1}}\,c_\rho = u_\rho(a, b, \mu), \tag{7.591}$$

and, if $\lambda = 0$ or $\rho \leqslant 1$*,

$$\bar{v}_\rho(a, b, \mu, T) \equiv 0, \tag{7.592}$$

* From (7.581) since $\binom{\lambda}{m} = 0$ if $\lambda = 0$, $m \geqslant 1$; and, if $\rho \leqslant 1$, $\frac{1}{2}\rho < 1$; so that in either case d_ρ reduces to c_ρ.

but, if $\lambda \neq 0$ and $\rho \geqslant 2$,

$$\bar{v}_\rho = \tfrac{1}{2} \frac{\cos \rho\pi}{i \sin \mu\pi} \frac{e^{-\frac{1}{2}(\rho+1)\pi i} a^\mu}{2^{-\rho-1} b^{\mu+\rho+1}} \sum_{m=1}^{\frac{1}{2}\rho} (-1)^m \binom{\lambda}{m} c_{\rho-2m} T^{-4m}$$

from (7 . 581), so that, by (7 . 583),

$$|\bar{v}_\rho| < K(\mu, \rho)\, a^\mu\, b^{-\mu-\rho-1} \sum_{m=1}^{\frac{1}{2}\rho} \binom{\lambda}{m} T^{-4m} \{\mathrm{Max}\,(a, b)\}^{2\rho-4m}$$

$$(7.593) \qquad < K(\lambda, \mu, \rho)\, a^\mu\, b^{-\mu-\rho-1} \{\mathrm{Max}\,(a, b)\}^{2\rho-4} T^{-4}.$$

Summing up the results of (7 . 54)–(7 . 593) we see that

$$(7.594) \qquad \frac{1}{2\pi i} \int_{c-iT'}^{c+iT'} P(s)\, ds + \frac{1}{2\pi i} \int_{\Gamma_3} P(s)\, ds = u_\rho + v_\rho.$$

But plainly

$$(7.595) \qquad \frac{1}{2\pi i} \int_{c-iT'}^{c+iT'} P(s)\, ds$$

$$= F(\lambda, \mu, \mu+\rho+1, \rho, a, b, T) - F(\lambda, \mu+\rho+1, \mu, \rho, a, b, T)$$

and

$$\int_{\Gamma_3} = O \int_{\frac{1}{2}\pi + \arcsin(c/T)}^{\frac{1}{2}\pi - \arcsin(c/T)} a^{-\frac{1}{2}} b^{-\frac{1}{2}} T^{-\rho} |\sin 2\theta|^\lambda e^{|a-b|T\cos\theta}\, d\theta$$

$$(7.596) \qquad = O(a^{-\frac{1}{2}} b^{-\frac{1}{2}} e^{|a-b|c} T^{-\rho-1-\lambda}).$$

The lemma follows from (7 . 594)–(7 . 596).

The restriction in the proof that μ is not integral can be removed. It is not worth while to give the analysis.

7 . 6. From (7 . 14), (7 . 15), and Lemmas 2′, 3′, and 4′ we obtain, in the case $\rho = -1$, when x is positive and not integral,

$$(7.61) \qquad G_{k,-1,\lambda}(x) = O(T^{-\lambda}),$$

and, when x is a positive integer,

$$(7.62) \qquad G_{k,-1,\lambda}(x) = T \frac{\Gamma(1+\lambda)\cos\frac{1}{4}\pi}{\Gamma(\frac{3}{4})\,\Gamma(1+\frac{1}{4}+\lambda)} \frac{e^{\frac{1}{2}k\pi i}}{8\pi x^{\frac{1}{2}}} \frac{\sigma_k(x)}{x^{\frac{1}{2}k}} + O(1).$$

From (7 . 12) and (7 . 61) we deduce, by taking $\lambda > |k| + \frac{1}{2}$ and making $T \to \infty$, that, for any positive non-integral x,

$$(7.63) \qquad \lim_{T\to\infty} 2\pi x^{\frac{1}{2}k} \sum_{n<T^2} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} F_{k+1,-1}\{4\pi\sqrt{(nx)}\} \left(1 - \frac{n^2}{T^4}\right)^\lambda \quad (\lambda > |k| + \tfrac{1}{2})$$

exists and is equal to

$$-\{x^k\zeta(1+k)+\zeta(1-k)\} = -\Phi_k'(x), \tag{7.64}$$

and, moreover, uniformly in any closed interval free from integers.

On the other hand, if x is a positive integer, we see, from (7.12) and (7.62), that, when $T\to\infty$,

$$\Phi_k'(x)+2\pi x^{\frac{1}{2}k}\sum_{n<T^2}\frac{\sigma_k(n)}{n^{\frac{1}{2}k}}F_{k+1,-1}\{4\pi\sqrt{(nx)}\}\left(1-\frac{n^2}{T^4}\right)^{\lambda} \tag{7.65}$$

$$= T\frac{\Gamma(1+\lambda)\cos\frac{1}{4}\pi}{\Gamma(\frac{3}{4})\Gamma(1+\frac{1}{4}+\lambda)}\frac{\pi\sigma_k(x)}{x^{\frac{1}{2}}}+O(1)+O(T^{|k|+\frac{1}{2}-\lambda}).$$

It follows at once that the series

$$2\pi x^{\frac{1}{2}k}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{1}{2}k}}F_{k+1,-1}\{4\pi\sqrt{(nx)}\}$$

is summable $(R, n^2, \lambda>|k|+\frac{1}{2})$ to the sum

$$-\Phi_k'(x)$$

for any positive non-integral x, and is not summable (R, n^2, λ) for any λ, however large, if x is a positive integer.

But this is Theorem 3, since summability (R, n^2, λ) and (R, n, λ) are equivalent*.

7.7. *Theorem* 4. The analysis, being in all respects similar to that employed in proving Theorem 3, is omitted.

7.8. *Extension of Theorem* 3. The preceding analysis gives us the theorem:

The series obtained by differentiating term by term r times the series (1.31) *of Theorem* 1 *is summable* $(R, n, \lambda>|k|-\frac{1}{2}+r)$ *to the sum* $-\Phi_k^{(r)}(x)$ *if x is positive and not integral; and is not summable* (R, n, λ) *for any* λ, *however large, if x is a positive integer. The summability is uniform in any closed interval free from integers.*

The case $r=1$ constitutes Theorem 3. We can, of course, extend Theorem 4 in the same way.

* Hardy and Riesz, *loc. cit.*, 33, f.n. †.

It is also possible, of course, to go in the other direction. The general result is that *the series obtained by integrating term by term r times the series of Theorem* 1 *is uniformly convergent if* $r > |k| - \frac{1}{2}$, *and uniformly summable* $(R, n, \lambda > |k| - r - \frac{1}{2})$ *if* $r \leqslant |k| - \frac{1}{2}$, *in any finite interval* $0 < x_1 \leqslant x \leqslant x_2$.

Here $r = 1, 2, \ldots$. The constants introduced by the integration are adjusted so that the new series is

$$Ax^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}r} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}r}} F_{k+1,r}\{4\pi\sqrt{(nx)}\}.$$

For, using (7 . 13) and the simple properties of Bessel functions*, we have

$$\frac{d}{dx}\left[x^{\frac{1}{2}\nu+\frac{1}{2}\rho} F_{\nu,\rho}\{4\pi\sqrt{(nx)}\}\right] = 2\pi\sqrt{n}\, x^{\frac{1}{2}\nu+\frac{1}{2}\rho-\frac{1}{2}} F_{\nu,\rho-1}\{4\pi\sqrt{(nx)}\}.$$

It is not so easy, however, to determine the sum of this series. In order to obtain the sum of the series, it is necessary to complete Lemma 11 (5 . 7) by considering the case when $\rho+\nu$ is a *negative* odd integer. This is done in

LEMMA 11′. *If, in Lemma* 11 (5 . 7), *we suppose that* $\rho+\nu$ *is a negative odd integer,* $-1-2r$, *say, where* $r = 0, 1, \ldots$, *then*

$$U_2 = \xi_\lambda + \Lambda_\lambda - \frac{(\frac{1}{2}ia)^\nu}{2n^2} \sum_{h=0}^{\frac{1}{2}r} (-)^h \binom{\lambda}{h} \frac{a_{r-2h}(\nu)}{T^{4h}},$$

where

$$a_f(\nu) = a_f = \sum_{m=0}^{\frac{1}{2}f} \frac{(\frac{1}{2}a)^{2f-4m} n^{-2m}}{(f-2m)!\,\Gamma(1+\nu+f-2m)},$$

and ξ_λ, Λ_λ *have the meanings assigned in Lemma* 11.

The proof is similar to that of Lemma 4′, and so may be omitted.

By means of Lemma 10 and Lemma 11′ we can complete Lemma 12. We obtain, in fact,

LEMMA 12′. *If* $\rho-\nu$ *is a negative odd integer,* $-1-2r$ *say, then*

$$U_2 = -i \sin\nu\pi\, \xi_\lambda(\rho, -\nu) + e^{\nu\pi i}\, \xi_\lambda(\rho, -\nu)$$
$$+ \Lambda_\lambda(n, T, \rho) - e^{\nu\pi i} \frac{(\frac{1}{2}ia)^{-\nu}}{2n^2} \sum_{0}^{\frac{1}{2}r} (-)^h \binom{\lambda}{h} \frac{a_{r-2h}(-\nu)}{T^{4h}},$$

the remaining arguments of ξ *and* ζ *being* n, a, *and* T.

* Watson, *loc. cit.*, 3 . 2 (5), 3 . 56, and 3 . 71 (5).

8. *Generalizations, continued.*

8.1. We apply the preceding lemmas to calculate $G_{k,\rho,\lambda}(x)$ when ρ is a positive integer. From Lemma 2′ (7.22), Lemma 4′ (7.4), and (7.14), it follows that, when ρ is a positive integer, $x>0$ and $\lambda \geqq 0$,

$$(8.11)\quad G_{k,\rho,\lambda}(x) = \frac{(2\pi)^{\rho-1}\, i e^{\frac{1}{2}(k+\rho)\pi i}}{2x^{\frac{1}{2}k+\frac{1}{2}\rho+\frac{1}{2}}\,\rho!} \sum_{n<x} \sigma_k(n)(x-n)^{\rho} + O(T^{-\rho}) + O(T^{-4})$$

for all large T, and uniformly in any finite interval $0<x_1\leqq x\leqq x_2$.

On the other hand, using (3.3) and integrating term by term (permissible by uniform convergence over the finite range of integration), we obtain

$$(8.21)\quad G_{k,\rho,\lambda}(x) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} s^{-\rho}\,\phi_k(s)\,J_{k+\rho+1}\{4\pi i s\sqrt{x}\}\left(1-\frac{s^4}{T^4}\right)^{\lambda} ds$$

$$+\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{1}{2}k}}\left\{\frac{U_2(|k|+2-\rho,\ k+\rho+1)}{n^{\frac{1}{2}|k|}}\right.$$

$$\left.-\frac{U_2(4\kappa-|k|+2-\rho,\ k+\rho+1)}{n^{2\kappa-\frac{1}{2}|k|}}\right\}\frac{1}{2\pi\sin\frac{1}{2}k\pi},$$

the remaining arguments of U_2 being n, $4\pi\sqrt{x}$, and T.

Case (I). $k>0$. Suppose, first, that $k>0$. Then, by Lemmas 11 and 12′, we obtain, after some reduction,

$$(8.31)\quad n^{-\frac{1}{2}k}U_2(k+2-\rho,\ k+\rho+1) - n^{\frac{1}{2}k-2\kappa}U_2(4\kappa-k+2-\rho,\ k+\rho+1)$$

$$= \tfrac{1}{2}i\sin\tfrac{1}{2}k\pi e^{\frac{1}{2}(k+\rho)\pi i}\, n^{-\frac{1}{2}\rho-\frac{1}{2}}\left(1-\frac{n^2}{T^4}\right)^{\lambda} F_{k+1,\rho}(a\sqrt{n})$$

$$+n^{-\frac{1}{2}k}\Lambda_{\lambda}(n,\ T,\ k+2-\rho) - n^{\frac{1}{2}k-2\kappa}\Lambda_{\lambda}(n,\ T,\ 4\kappa-k+2-\rho)$$

$$-\frac{2^{k+\rho}\, i e^{\frac{1}{2}(k+\rho)\pi i}}{a^{k+\rho+1}\, n^{\frac{1}{2}k+2}}\sum_{h=0}^{\frac{1}{2}(\rho-1)}(-)^h\binom{\lambda}{h}\frac{\alpha_{\rho-1-2h}(-k-\rho-1)}{T^{4h}},$$

if $n<T^2$; if $n\geqq T^2$, the first term on the right has to be omitted [$a=4\pi\sqrt{x}$]. Now from Lemma 11′ above

$$(8.32)\qquad \alpha_f(\nu) = \sum_{m=0}^{\frac{1}{2}f}\frac{(\frac{1}{2}a)^{2f-4m}\, n^{-2m}}{(f-2m)!\,\Gamma(1+\nu+f-2m)},$$

so that

$$(8.33)\qquad |\alpha_f(-k-\rho-1)| < A(k,\rho)\,a^{2\rho}\quad (f\leqq\rho)$$

for all n. Hence

$$\left| \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \frac{1}{n^{\frac{1}{2}k+2}} \sum_{h=1}^{\frac{1}{2}(\rho-1)} (-)^h \binom{\lambda}{h} \frac{a_{\rho-1-2h}(-k-\rho-1)}{T^{4h}} \right|$$

$$< A(k, \rho, \lambda)\, a^{2\rho}\, T^{-4} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{k+2}}$$

$$= A(k, \rho, \lambda)\, a^{2\rho}\, \xi(2)\, \xi(k+2)\, T^{-4}, \tag{8.34}$$

and

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \frac{1}{n^{\frac{1}{2}k+2}} a_{\rho-1}(-k-\rho-1)$$

$$= \sum_{m=0}^{\frac{1}{2}\rho-\frac{1}{2}} \frac{(\frac{1}{2}a)^{2\rho-2-4m}}{(\rho-1-2m)!\, \Gamma(-1-k-2m)} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{k+2}} \frac{1}{n^{2m}}$$

$$= \frac{1}{\pi} \sin k\pi (\tfrac{1}{2}a)^{2\rho-2} \sum_{m=0}^{\frac{1}{2}\rho-\frac{1}{2}} \frac{\Gamma(k+2+2m)\, \xi(k+2+2m)\, \xi(2+2m)}{(\rho-1-2m)!\, (\frac{1}{2}a)^{4m}}. \tag{8.35}$$

Next, by (5.88),

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \left\{ \frac{\Lambda_\lambda(n, T, k+2-\rho)}{n^{\frac{1}{2}k}} - \frac{\Lambda_\lambda(n, T, 4\kappa-k+2-\rho)}{n^{2\kappa-\frac{1}{2}k}} \right\}$$

$$= O(T^{|k|-\frac{1}{2}-\rho+\epsilon-\lambda}). \tag{8.36}$$

The equations (8.31)–(8.36) enable us to deal with the second term on the right of (8.21). In order to deal with the first term, we use Lemmas 6, 7, and 8 [(5.2), (5.3), and (5.4)].

Proceeding as in Lemma 9 (5.5), we find that

$$I_{k,\rho,\lambda}(x) = \frac{1}{2\pi i} \int_{c-iT'}^{c+iT'} s^{-\rho}\, \phi_k(s)\, J_{k+\rho+1}\{4\pi i s\sqrt{x}\} \left(1-\frac{s^4}{T^4}\right)^{\lambda} ds \tag{8.41}$$

$$= -\tfrac{1}{2} \frac{\Gamma(k)\, \xi(k)}{(2\pi)^k}\, I_2(-\rho-k,\ k+\rho+1)$$

$$-\tfrac{1}{2} \frac{\Gamma(-k)\, \xi(-k)}{(2\pi)^{-k}}\, I_2(-\rho+k,\ k+\rho+1)$$

$$+ \frac{\Gamma(1+k)\, \xi(1+k)}{(2\pi)^{2+k}}\, I_2(-\rho-k-2,\ k+\rho+1)$$

$$+ \frac{\Gamma(1-k)\, \xi(1-k)}{(2\pi)^{2-k}}\, I_2(-\rho+k-2,\ k+\rho+1)$$

$$+ \sum_{r=1}^{\kappa} B_r I_2(-\rho+4r-2-k,\ k+\rho+1),$$

where the numbers B_r are constants, and the remaining arguments of the functions I_2 are $4\pi\sqrt{x}$ and T. Applying Lemmas 7 and 8, we obtain

$$(8.42)\quad I_{k,\rho,\lambda}(x) = \frac{(2\pi)^{\rho-1} i e^{\frac{1}{2}(k+\rho)\pi i}}{2x^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}\rho}} \times \left\{ \frac{\Gamma(1+k)\,\zeta(1+k)}{\Gamma(2+k+\rho)}\, x^{1+k+\rho} + \zeta(1-k)\,\frac{x^{\rho+1}}{(\rho+1)!} - \tfrac{1}{2}\zeta(-k)\,\frac{x^{\rho}}{\rho!} \right\} + O(T^{k-\rho-\frac{1}{2}-\lambda}) + O(T^{-4}).$$

Using (8.31)–(8.36) and (8.42) we obtain, from (8.21),

$$(8.51)\quad G_{k,\rho,\lambda}(x) = \frac{(2\pi)^{\rho-1} i e^{\frac{1}{2}(k+\rho)\pi i}}{2x^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}\rho}} \times \left\{ \frac{\Gamma(1+k)\,\zeta(1+k)}{\Gamma(2+k+\rho)}\, x^{1+k+\rho} + \zeta(1-k)\,\frac{x^{\rho+1}}{(\rho+1)!} - \tfrac{1}{2}\zeta(-k)\,\frac{x^{\rho}}{\rho!} \right\} + \frac{i e^{\frac{1}{2}(k+\rho)\pi i}}{4\pi} \sum_{n<T^2} \frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}\rho}}\, F_{k+1,\rho}\{4\pi(nx)\} \left(1-\frac{n^2}{T^4}\right)^{\lambda} - \frac{2ie^{\frac{1}{2}(k+\rho)\pi i}\cos\frac{1}{2}k\pi}{(2\pi)^{k+5-\rho}\, x^{\frac{1}{2}k-\frac{1}{2}\rho+\frac{3}{2}}} \sum_{m=0}^{\frac{1}{2}\rho-\frac{1}{2}} \frac{\Gamma(k+2+2m)\,\zeta(k+2+2m)\,\zeta(2+2m)}{(\rho-1-2m)!\,(2\pi\sqrt{x})^{4m}} + O(T^{k-\frac{1}{2}-\rho+\epsilon-\lambda}) + O(T^{-4}).$$

From (8.11) and (8.51) we deduce that, if $k>0$,

$$(8.52)\quad \frac{1}{\rho!} \sum_{n<x} \sigma_k(n)(x-n)^{\rho} = \frac{\Gamma(1+k)\,\zeta(1+k)}{\Gamma(2+k+\rho)}\, x^{1+k+\rho} + \zeta(1-k)\,\frac{(\rho+1)!}{x^{\rho+1}} - \tfrac{1}{2}\zeta(-k)\,\frac{x^{\rho}}{\rho!} + \frac{1}{2\pi^2}\, x^{\rho-1} \sum_{m=0}^{\frac{1}{2}\rho-\frac{1}{2}} \frac{(-)^m\,\zeta(-1-k-2m)\,\zeta(2+2m)}{(\rho-1-2m)!\,(2\pi x)^{2m}} + \frac{x^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}\rho}}{(2\pi)^{\rho}} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k+\rho}}\, F_{k+1,\rho}\{4\pi\sqrt{(nx)}\},$$

where the series is uniformly convergent if $\rho > k-\frac{1}{2}$, *and is uniformly summable* $(R, n^2, \lambda > k-\frac{1}{2}-\rho)$ *if* $\rho \leqslant k-\frac{1}{2}$, *in any finite interval* $0 < x_1 \leqslant x \leqslant x_2$.

This constitutes the generalization of Theorem 5*, to which it reduces when $\rho = 1$, for the case when $k > 0$.

It is to be observed that the first two lines on the right of (8.52) form the sum of the residues of the function

$$\frac{\xi(z)\,\xi(z-k)}{z(z+1)\ldots(z+\rho)}\,x^{k+\rho}.$$

Denoting this sum by $\Phi_{k,\rho}(x)$, we have

$$\Phi_{k,\rho}(x) = \frac{\Gamma(1+k)\,\xi(1+k)}{\Gamma(2+k+\rho)}\,x^{1+k+\rho} + \xi(1-k)\,\frac{x^{\rho+1}}{(\rho+1)!} - \tfrac{1}{2}\xi(-k)\,\frac{x^{\rho}}{\rho!}$$

$$- x^{\rho-1}\sum_{m=0}^{\frac{1}{2}\rho-\frac{1}{2}} \frac{\xi(-1-2m)\,\xi(-1-2m-k)}{(2m+1)!\,(\rho-1-2m)!\,x^{2m}}.$$

[The Σ of course may be omitted when k is an odd integer.]

(8.6) Case (II) $k < 0$. In this case we have

$$(8.61)\quad n^{-\frac{1}{2}|k|}U_2(|k|+2-\rho,\ k+\rho+1)$$

$$-n^{\frac{1}{2}|k|-2\kappa}U_2(4\kappa-|k|+2-\rho,\ k+\rho+1)$$

$$= n^{\frac{1}{2}k}U_2(2-\rho-k,\ k+\rho+1) - n^{-\frac{1}{2}k-2\kappa}U_2(4\kappa+k+2-\rho,\ k+\rho+1)$$

$$= -\tfrac{1}{2}i\,\sin\tfrac{1}{2}k\pi e^{\frac{1}{2}(k+\rho)\pi i}n^{-\frac{1}{2}\rho-\frac{3}{2}}\left(1-\frac{n^2}{T^4}\right)^{\lambda}F_{k+1,\rho}\{4\pi\sqrt{(nx)}\}$$

$$+n^{\frac{1}{2}k}\Lambda_\lambda(n,\ T,\ 2-\rho-k) - n^{-\frac{1}{2}k-2\kappa}\,\Lambda_\lambda(n,\ T,\ 4\kappa+k+2-\rho)$$

$$-\frac{(2\pi\sqrt{x})^{-k-\rho-1}ie^{\frac{1}{2}(k+\rho)\pi i}}{2n^{2+\frac{1}{2}k+2\pi}}\sum_{h=0}^{\frac{1}{2}r}(-)^h\binom{\lambda}{h}\frac{a_{r-2h}(-k-\rho-1,4\pi\sqrt{x})}{T^{4h}},$$

where $r = \rho-1-2\kappa$, if $n < T^2$; if $n \geqslant T^2$ the first term on the right has to be omitted. If r is negative, the last term is to be interpreted to be zero.

Now, if $r \geqslant 0$,

$$a_r(-k-\rho-1,\ 4\pi\sqrt{x}) = \sum_{m=0}^{\frac{1}{2}\nu}\frac{(2\pi\sqrt{x})^{2r-4m}n^{-2m}}{(r-2m)!\,\Gamma(-k-1-2\kappa-2m)},$$

* For summability (R, n^2, λ) is equivalent to summability (R, n, λ).

so that

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}k}} \frac{1}{n^{2+\frac{1}{2}k+2\kappa}} \alpha_r(-k-\rho-1,\ 4\pi\sqrt{x})$$

$$= \sum_{m=0}^{\frac{1}{2}r} \frac{(2\pi\sqrt{x})^{2r-4m}}{(r-2m)!\,\Gamma(-k-1-2\kappa-2m)} \sum_{n=0}^{\infty} \frac{\sigma_k(n)}{n^{2+2\kappa+k+2m}}$$

$$= \sum_{m=0}^{\frac{1}{2}r} \frac{(2\pi\sqrt{x})^{2r-4m}\,\zeta(2+2\kappa+k+2m)\,\zeta(2+2\kappa+2m)}{(r-2m)!\,\Gamma(-k-1-2\kappa-2m)}$$

$$(8.612) \quad = (2\pi)^{1+k+2\rho} \sin\tfrac{1}{2}k\pi \sum_{m=0}^{\frac{1}{2}r} \frac{\zeta(-k-1-2\kappa-2m)\,\zeta(-1-2\kappa-2m)}{\Gamma(\rho-2\kappa-2m)\,\Gamma(+2+2\kappa+2m)}$$

$$\times x^{\rho-1-2\kappa-2m}.$$

We have still to deal with $I_{k,\rho,\lambda}(x)$. In the present case we have

$$I_{k,\rho,\lambda}(x) = -\tfrac{1}{2} \frac{\Gamma(k)\,\zeta(k)}{(2\pi)^k} I_2(-\rho-k,\ k+\rho+1)$$

$$-\tfrac{1}{2} \frac{\Gamma(-k)\,\zeta(-k)}{(2\pi)^{-k}} I_2(-\rho+k,\ k+\rho+1)$$

$$+ \frac{\Gamma(1+k)\,\zeta(1+k)}{(2\pi)^{2+k}} I_2(-\rho-k-2,\ k+\rho+1)$$

$$+ \frac{\Gamma(1-k)\,\zeta(1-k)}{(2\pi)^{2-k}} I_2(-\rho+k-2,\ k+\rho+1)$$

$$+ \sum_{r=1}^{\kappa} B_r I_2(-\rho+4r-2+k,\ k+\rho+1),$$

where

$$B_r = (-)^{r-1} \frac{2}{(2\pi)^{2-k}} \zeta(2r)\,\zeta(1-k-2r)\,\Gamma(1-k-2r)(2\pi)^{2r},$$

whence, by (5.3) and (5.4),

$$(8.614) \quad I_{k,\rho,\lambda}(x)$$

$$= \frac{(2\pi)^{\rho-1} i e^{\frac{1}{2}(k+\rho)\pi i}}{2x^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}\rho}} \left\{ \frac{\Gamma(1+k)\,\zeta(1+k)}{\Gamma(2+k+\rho)} x^{1+k+\rho} + \zeta(1-k) \frac{x^{\rho+1}}{(\rho+1)!} - \tfrac{1}{2}\zeta(-k) \frac{x^{\rho}}{\rho!} \right.$$

$$\left. - \sum_{r\geqslant 1}^{r\leqslant \mathrm{Min}(\kappa,\,\frac{1}{2}\rho+\frac{1}{2})} \frac{\zeta(1-2r)\,\zeta(1-k-2r)}{\Gamma(2r)\,\Gamma(2+\rho-2r)} x^{\rho+1-2r} \right\}$$

$$+ O(T^{-k-\rho-\frac{1}{2}-\lambda}) + O(T^{-4}).$$

From (8.11), (8.21), (8.61), etc., we deduce that, *when* $k<0$, $\rho\geqq 1$, *and* $x>0$,

$$(8.62)\quad \frac{1}{\rho!}\sum_{n<x}\sigma_k(n)(x-n)^\rho$$

$$=\frac{\Gamma(1+k)\,\zeta(1+k)}{\Gamma(2+k+\rho)}x^{1+k+\rho}+\zeta(1-k)\frac{x^{\rho+1}}{(\rho+1)!}-\tfrac{1}{2}\zeta(-k)\frac{x^\rho}{\rho!}$$

$$-\sum_{r\geqq 1}^{r\leqq \mathrm{Min}(\kappa,\,\frac{1}{2}\rho+\frac{1}{2})}\frac{\zeta(1-2r)\,\zeta(1-k-2r)}{\Gamma(2r)\,\Gamma(2+\rho-2r)}x^{\rho+1-2r}$$

$$-\sum_{r\geqq 0}^{r\leqq \frac{1}{2}\rho-\frac{1}{2}-\kappa}\frac{\zeta(-1-k-2\kappa-2r)\,\zeta(-1-2\kappa-2r)}{\Gamma(\rho-2\kappa-2r)\,\Gamma(2+2\kappa+2r)}x^{\rho-1-2\kappa-2r}$$

$$+\frac{x^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}\rho}}{(2\pi)^\rho}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}\rho}}F_{k+1,\,\rho}\{4\pi\sqrt{(nx)}\},$$

where the series is uniformly convergent if $\rho>|k|-\frac{1}{2}$, *and uniformly summable* $(R,\ n^2,\ \lambda>|k|-\frac{1}{2}-\rho)$ *if* $\rho\leqq|k|-\frac{1}{2}$, *in any finite interval* $0<x_1\leqq x\leqq x_2$.

This completes the generalization of Theorem 5.

Observing that the second and third lines on the right of (8.62) form

$$-\sum_{r=1}^{\frac{1}{2}\rho+\frac{1}{2}}\frac{\zeta(1-2r)\,\zeta(1-k-2r)}{\Gamma(2r)\,\Gamma(2+\rho-2r)}x^{\rho+1-2r}$$

(since $$\sum_{r\geqq 1}^{r\leqq \mathrm{Min}(m,\,n)}u_r+\sum_{r\geqq m+1}^{r\leqq n}u_r\equiv\sum_{r=1}^{n}u_r;$$

in the third line we put $r+\kappa=m$), we see that the first three lines on the right together make

$$\Phi_{k,\,\rho}(x),$$

the sum of the residues of

$$\frac{\zeta(z)\,\zeta(z-k)}{z(z+1)\ldots(z+\rho)}x^{z+\rho}$$

at its poles.

Thus we can combine (8.52) and (8.62) into a single theorem, the generalization of Theorem 5.

THEOREM 5′. *If ρ is a positive integer, k is real, and x is positive, then*

$$\frac{1}{\rho!}\sum_{n<x}\sigma_k(n)(x-n)^\rho = \Phi_{k,\rho}(x)+\frac{x^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}\rho}}{(2\pi)^\rho}\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k+\frac{1}{4}\rho}}F_{k+1,\rho}\{4\pi\sqrt{(nx)}\},$$

where the series is uniformly convergent if $\rho>|k|-\frac{1}{2}$ and uniformly summable $(R, n, \lambda>|k|-\frac{1}{2}-\rho)$ if $\rho\leqslant|k|-\frac{1}{2}$, uniformly in any finite interval $0<x_1\leqslant x\leqslant x_2$.

This theorem we have proved when k is not integral. The proof can be modified so as to include this case, too.

8.7. Naturally it is possible in the same way to obtain integrated forms of the identity of Theorem 2. The analysis for this is, in fact, rather simpler than the preceding analysis.

THEOREM 2′. *If p is a positive integer $\geqslant 2$, ρ a positive integer $\geqslant 1$, and $x>0$, then*

$$\frac{1}{\rho!}\sum_{0\leqslant n<x} r_p(n)(x-n)^\rho$$
$$=\frac{\pi^{\frac{1}{2}p}}{\Gamma(1+\rho+\frac{1}{2}p)}x^{\rho+\frac{1}{2}p}+\frac{x^{\frac{1}{2}\rho+\frac{1}{4}p}}{\pi^\rho}\sum_{n=1}^{\infty}\frac{r_p(n)}{n^{\frac{1}{2}\rho+\frac{1}{4}p}}J_{\rho+\frac{1}{2}p}\{2\pi\sqrt{(nx)}\},$$

where the series is convergent if $\rho>\frac{1}{2}p-\frac{3}{2}$ and summable

$$(R, n, \lambda>\tfrac{1}{2}p-\tfrac{3}{2}-\rho) \quad \text{if} \quad \rho\leqslant\tfrac{1}{2}p-\tfrac{3}{2},$$

in each case uniformly in any finite interval $0<x_1\leqslant x\leqslant x_2$.

The case $p=2$ is due to Hardy*. The case $\rho\geqslant[\frac{1}{2}p]+1$ is due to Landau†. Since $[\frac{1}{2}p]+1>\frac{1}{2}p-\frac{3}{2}$, this is the simpler part of the theorem.

It should be added that the method of this paper enables us to attack the more general sum

$$\Sigma\exp\left(2\pi i\sum_1^n v_jh_j\right)$$

---

* Hardy, *loc. cit.*, footnote ‖ to Theorem 2 (1.4).

† Landau, "Über die Gitterpunkte in einen mehrdimensionalen Ellipsoid", *Berliner Sitzungsberichte*, 31 (1915), 458–476 (472).

defined at the end of (1 . 4). The identities so obtained are of much the same form as those in Theorems 2, 4, and 2′.

9. *An identity for*

$$\frac{1}{k}\left\{x^k \sum_{n<x} \sigma_{-k}(n) - \sum_{n<x} \sigma_k(n)\right\}$$

if $k \neq 0$, *and for the limit of this when* $k \to 0$, *namely*

$$\sum_{n<x} d(n) \log \frac{x}{n}.$$

We may suppose $k \geqslant 0$. The identities in question are obtained by the consideration of $F(\lambda, \mu, \mu, 1, a, b, T)$ [for which see (7 . 15)].

It is hardly necessary to go through the analysis in detail. The necessary formulae are as follow :

If $\lambda \geqslant 0$ *and* T *is large, then*

$$F = O(T^{-1} a^{-\frac{1}{2}} b^{-\frac{1}{2}} e^{-(a-b)c}) \tag{9 . 11}$$

uniformly for $a \geqslant b > B > 0$, *and*

$$F = \frac{1}{2\mu\pi i}\left\{\left(\frac{b}{a}\right)^{\mu} - \left(\frac{a}{b}\right)^{\mu}\right\} + O(T^{-1} a^{-\frac{1}{2}} b^{-\frac{1}{2}} e^{(b-a)c}) \tag{9 . 12}$$

if $0 < a \leqslant b$*.

Hence we deduce that, uniformly for x in the interval

$$0 < x_1 \leqslant x \leqslant x_2,$$

$$\frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} s^{-1} g_k(s) J_k(4\pi i s\sqrt{x})\left(1 - \frac{s^4}{T^4}\right)^{\lambda} ds$$

$$= \frac{e^{\frac{1}{2}k\pi i}}{2x^{\frac{1}{2}k}k}\left\{x^k \sum_{n<x} \sigma_{-k}(n) - \sum_{n<x} \sigma_k(n)\right\} + O\left(\frac{1}{T}\right). \tag{9 . 2}$$

Employing the lemmas and method of § 5, we find that

$$k^{-1}\left\{x^k \sum_{n<x} \sigma_{-k}(n) - \sum_{n<x} \sigma_k(n)\right\} = \Psi_k(x) + x^{\frac{1}{2}+\frac{1}{2}k} \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{1}{2}+\frac{1}{2}k}} V_k\{4\pi\sqrt{(nx)}\}, \tag{9 . 3}$$

* If $\mu = 0$, the term $\mu^{-1}\{(b/a)^{\mu} - (a/b)^{\mu}\}$ is to be replaced by its limit when $\mu \to 0$, *i.e.* by $2 \log (b/a)$.

where $\Psi_k(x)$ denotes the sum of the residues of the function

$$\frac{\zeta(z)\,\zeta(z-k)}{z(z-k)}\,x^z \tag{9.31}$$

and

$$V_k(z) = \frac{1}{k}\left\{F_{1-k}(z) - F_{1+k}(z)\right\}, \tag{9.32}$$

and the series is convergent if $k < \frac{3}{2}$ and summable $(R, n, \lambda > k-\frac{3}{2})$ if $k \geqslant \frac{3}{2}$, the convergence and summability being uniform in any finite interval $0 < x_1 \leqslant x \leqslant x_2$.

In particular, when $k = 0$, we have, uniformly in $0 < x_1 \leqslant x \leqslant x_2$,

$$\sum_{n<x} d(n) \log\frac{x}{n} = x\log x - x + (2\gamma-1)x + \tfrac{1}{4}\log x + \tfrac{1}{2}\log 2\pi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{d(n)}{n}\left\{Y_0\left\{4\pi\sqrt{(nx)}\right\} + \frac{2}{\pi}K_0\left\{4\pi\sqrt{(nx)}\right\}\right\}. \tag{9.4}$$

Actually this series is absolutely convergent.

Similarly we can obtain identities for

$$\sum_{n<x}\left(x^{\frac{1}{2}p-1} - n^{\frac{1}{2}p-1}\right) r_p(n) \quad (p \geqslant 3)$$

and

$$\sum_{n<x} r(n)\log\frac{x}{n}.$$

Generally, identities of different types may be obtained by considering the values of the integral

$$F(\lambda, \mu, \nu, \alpha, a, b, T) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} s^{-\alpha} H_\mu^{(1)}(ias) J_\nu(ibs)\left(1-\frac{s^4}{T^4}\right)^\lambda ds$$

for various values of the parameters μ, ν, α, when a and b are positive.

The preceding identities have been obtained by evaluating this integral in the cases

$$\alpha = \rho, \quad \nu = \mu+\rho+1,$$

where ρ is integral, and

$$\alpha = 1, \quad \nu = \mu.$$

We can also evaluate it when

$$\alpha = \rho, \quad \nu = \mu + \rho' + 1,$$

where ρ, ρ' are integers and $\rho - \rho'$ is even.

10. *Some infinite integrals.*

The integrals considered in the preceding analysis give rise, when we make $T \to \infty$, to certain interesting infinite integrals.

Thus we have, if a, b, c are positive,

$$(10.1) \qquad \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} H_\mu^{(1)}(isa)\, H_\nu^{(1)}(isb)\, ds = 0,$$

$$(10.2) \qquad \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} H_\mu^{(1)}(isa)\, H_{\mu+1}^{(2)}(isb)\, ds = \begin{cases} 0 & (a > b), \\ 1/(\pi b) & (a = b), \\ 2a^\mu/(\pi b^{\mu+1}) & (a < b), \end{cases}$$

$$(10.21) \qquad \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} H_\mu^{(2)}(isa)\, H_{\mu+1}^{(1)}(isb)\, ds = \begin{cases} 0 & (b > a), \\ -1/(\pi b) & (b = a), \\ -2a^\mu/(\pi b^{\mu+1}) & (b < a); \end{cases}$$

if ρ is a positive integer,

$$(10.3) \qquad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-\rho} H_\mu^{(1)}(ias)\, H_{\mu+\rho+1}^{(2)}(ibs)\, ds = \begin{cases} 0 & (a \geqslant b), \\ \dfrac{e^{\frac{1}{2}\rho\pi i}\, a^\mu\, (b^2 - a^2)^\rho}{2^{\rho-1}\, \pi b^{\mu+\rho+1}\, \rho!} & (a \leqslant b), \end{cases}$$

and

$$(10.31) \qquad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-\rho} H_\mu^{(2)}(ias)\, H_{\mu+\rho+1}^{(1)}(ibs)\, ds = \begin{cases} 0 & (b \geqslant a), \\ -\dfrac{e^{\frac{1}{2}\rho\pi i}\, a^\mu\, (b^2 - a^2)^\rho}{2^{\rho-1}\, \pi b^{\mu+\rho+1}\, \rho!} & (b \leqslant a). \end{cases}$$

Also, if $\mu \neq 0$,

$$(10.4) \qquad \int_{c-i\infty}^{c+i\infty} H_\mu^{(1)}(ias)\, H_\mu^{(2)}(ibs)\, \frac{ds}{s} = \begin{cases} 0 & (a \geqslant b), \\ \dfrac{1}{2\mu} \left\{ \left(\dfrac{b}{a}\right)^\mu - \left(\dfrac{a}{b}\right)^\mu \right\} & (a \leqslant b), \end{cases}$$

and, if $\mu = 0$,

$$(10.41)\quad \int_{c-i\infty}^{c+i\infty} H_0^{(1)}(ias)\, H_0^{(2)}(ibs)\, \frac{ds}{s} = \begin{cases} 0 & (a \geqslant b), \\ 2\log\left(\frac{b}{a}\right) & (a \leqslant b). \end{cases}$$

Other integrals can be obtained from the results of Lemmas 6, 7, etc.

But of the multiplication of examples there is no end.

ADDENDUM.

The proof for the case $k = 0$.

A.1. In order to show how the difficulties in the proof which arise when k is integral can be overcome, I give here the proof for the case $k = 0$.

In this case*

$$(A\,1.1)\quad g(s) = g_0(s) = \pi i \sum_{n=1}^{\infty} d(n)\, H_0^{(1)}(4\pi i s\sqrt{n})$$

$$= -\tfrac{1}{2}\gamma - \tfrac{1}{2}\log s - \frac{2\log(2\pi s)}{(2\pi s)^2}$$

$$+ \frac{1}{\pi^2} \sum_{n=1}^{\infty} d(n) \log\left(\frac{s}{\sqrt{n}}\right)\left\{\frac{1}{s^2-n} + \frac{1}{s^2+n}\right\}.$$

Let x be positive. Consider the integral

$$(A\,1.2)\qquad \mathbf{G}(x) = \frac{1}{2\pi i}\int_{C_2} g(s)\, J_1(4\pi i s\sqrt{x})\, ds,$$

where C_2 is the contour formed by the lines $\sigma = 0$, $\sigma = c > 0$, and the arcs Γ_2, Γ_3 of $|s| = T$, where $T^2 = N + \frac{1}{2}$ (N being a positive integer), which are intercepted between them. The side $\sigma = 0$ is indented, of course, at the points

$$s = 0, \quad s = \pm i\sqrt{n} \quad (n = 1, 2, \ldots, N).$$

By Cauchy's theorem

$$(A\,1.21)\qquad \mathbf{G}(x) = 0.$$

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* Writing, as is usual, $d(n) = \sigma_0(n)$

The contribution of $\sigma = c$ is

$$G(x) = G_{0,0}(x) = \frac{1}{2\pi i}\int_{c-iT'}^{c+iT'} g(s)\, J_1(4\pi i s\sqrt{x})\, ds. \tag{A 1.22}$$

The contribution of the indentation at $s = \pm i\sqrt{n}$ $(1 \leqslant n \leqslant N)$, is, in the limit, when the radii of these indentations tend to zero,

$$\frac{1}{2\pi i}(-\pi i)\frac{1}{\pi^2}\, d(n)\,\frac{\pi}{2}\, i\,\frac{1}{2i\sqrt{n}}\left[-J_1\{4\pi\sqrt{(nx)}\}\right]$$

$$+\frac{1}{2\pi i}(-\pi i)\frac{1}{\pi^2}\, d(n)\left(-\frac{\pi}{2}\, i\right)\frac{1}{-2i\sqrt{n}}\left[J_1\{4\pi\sqrt{(nx)}\}\right] \equiv 0.$$

The contribution of the side $\sigma = 0$ is

$$\frac{1}{2\pi i}\, P\int_T^{\delta} g(it)\, J_1(-4\pi t\sqrt{x})\, i dt + \frac{1}{2\pi i}\, P\int_{\delta}^{T} g(-it)\, J_1(4\pi t\sqrt{x})\,.\,-i dt.$$

Now, if $t > 0$,

$$g(it) = -\tfrac{1}{2}\gamma - \tfrac{1}{2}(\log t + \tfrac{1}{2}\pi i) + \frac{2\{\log(2\pi t) + \frac{1}{2}\pi i\}}{(2\pi t)^2}$$

$$+\frac{1}{\pi^2}\sum_{n=1}^{\infty} d(n)\left(\log\frac{t}{\sqrt{n}} + \frac{\pi}{2}\, i\right)\left(\frac{1}{n-t^2} - \frac{1}{n+t^2}\right),$$

$$g(-it) = -\tfrac{1}{2}\gamma - \tfrac{1}{2}(\log t - \tfrac{1}{2}\pi i) + \frac{2\{\log(2\pi t) - \frac{1}{2}\pi i\}}{(2\pi t)^2}$$

$$+\frac{1}{\pi^2}\sum_{n=1}^{\infty} d(n)\left(\log\frac{t}{\sqrt{n}} - \frac{\pi}{2}\, i\right)\left(\frac{1}{n-t^2} - \frac{1}{n+t^2}\right),$$

so that the contribution of the imaginary axis is

$$-\frac{i}{2\pi i}\, P\int_{\delta}^{T}\{g(-it) - g(it)\}\, J_1(4\pi t\sqrt{x})\, dt$$

$$= \frac{1}{2\pi i}\, P\int_{\delta}^{T} h(it)\, J_1(4\pi t\sqrt{x})\, dt, \tag{A 1.23}$$

where

$$h(s) = \tfrac{1}{2}\pi + \frac{2\pi}{(2\pi s)^2} - \frac{1}{\pi}\sum_{n=1}^{\infty} d(n)\left(\frac{1}{s^2-n} + \frac{1}{s^2+n}\right). \tag{A 1.24}$$

The contribution of the indentation at $s = 0$, formed by the arc C_δ of $|s| = \delta$, where δ is small, to the right of $\sigma = 0$, is

$$\frac{1}{2\pi i}\int_{C_\delta} g(s)\, J_1(4\pi i s\sqrt{x})\, ds, \tag{A 1.25}$$

C_δ being described negatively. We shall deal with this presently. From (A1 . 21), (A1 . 22), and (A1 . 23) we see that

$$G(x) = -\frac{1}{2\pi i}\int_{\Gamma_2} - \frac{1}{2\pi i}\int_{\Gamma_3} + \frac{1}{2\pi i}\int_{C_\delta} \tag{A 1 . 26}$$

$$-\frac{1}{2\pi i} P\int_\delta^T h(it)\, J_1(4\pi t\sqrt{x})\, ds,$$

the arcs Γ_2, Γ_3, and C_δ being described positively. [The integrand for each of these contours is $g(s)\, J_1(4\pi i s\sqrt{x})$.]

The last integral on the right is easily evaluated by considering the integral

$$\frac{1}{2\pi i}\int_C h(s)\, H_i^{(1)}(4\pi i s\sqrt{x})\, ds, \tag{A 1 . 31}$$

where C is formed by $\sigma = 0$ and the arc Γ_4 of $|s| = T$ to the right of $\sigma = 0$, and is indented at $s = 0$, $s = \pm i\sqrt{n}$ $(n = 1, \ldots, N)$, the poles of $h(s)$ on the contour. Plainly the integrand is analytic on and inside C, save for simples poles at the points $s = \sqrt{n}$ $(n = 1, \ldots, N)$, the residue at $s = \sqrt{n}$ being

$$-\frac{1}{\pi}\, d(n)\, \frac{1}{2\sqrt{n}}\, H_1^{(1)}\{4\pi i\sqrt{(nx)}\} = \frac{1}{\pi^2}\, \frac{d(n)}{\sqrt{n}}\, K_1\{4\pi\sqrt{(nx)}\}.$$

By Cauchy's theorem, therefore,

$$\frac{1}{2\pi i}\int_C = \frac{1}{\pi^2}\sum_{n=1}^{N} \frac{d(n)}{\sqrt{n}}\, K_1\{4\pi\sqrt{(nx)}\}. \tag{A 1 . 32}$$

The contribution of $s = \pm i\sqrt{n}$ $(n = 1, 2, \ldots, N)$ taken together is, in the limit,

$$\frac{1}{2\pi i} \cdot -\pi i \cdot -\frac{1}{\pi}\, d(n)\, \frac{1}{2i\sqrt{n}}\, H_1^{(1)}\{-4\pi\sqrt{(nx)}\}$$

$$+\frac{1}{2\pi i} \cdot -\pi i \cdot -\frac{1}{\pi}\, d(n)\, \frac{1}{-2i\sqrt{n}} \cdot -H_1^{(1)}\{4\pi\sqrt{(nx)}\}$$

$$= \frac{d(n)}{4\pi i\sqrt{n}}\left[H_1^{(2)}\{4\pi\sqrt{(nx)} - H_1^{(1)}\{4\pi\sqrt{(nx)}\}\right]$$

$$= -\frac{1}{2\pi}\, \frac{d(n)}{\sqrt{n}}\, Y_1\{4\pi\sqrt{(nx)}\}. \tag{A 1 . 33}$$

Next, the contribution of $\sigma = 0$ is

$$\text{(A 1 . 34)} \quad \frac{1}{2\pi i} P\int_{T}^{\delta} h(it)\, H_1^{(1)}(-4\pi t\sqrt{x})\, i\,dt + \frac{1}{2\pi i} P\int_{\delta}^{T} h(-it)\, H_1^{(1)}(4\pi t\sqrt{x})\,.\,-i\,dt = -\frac{1}{\pi} P\int_{\delta}^{T} h(it)\, J_1(4\pi t\sqrt{x})\, dt.$$

We deduce that

$$\text{(A 1 . 35)} \quad \frac{1}{\pi} P\int_{\delta}^{T} h(it)\, J_1(4\pi t\sqrt{x})\, dt = \frac{1}{2\pi i}\int_{\Gamma_4} - \frac{1}{2\pi i}\int_{C_\delta} - \frac{1}{2\pi}\sum_{n=1}^{N}\frac{d(n)}{\sqrt{n}} \times\left[Y_1\{4\pi\sqrt{(nx)}\} + \frac{2}{\pi} K_1\{4\pi\sqrt{(nx)}\}\right],$$

Γ_4 and C_δ being described positively, the integrand being that of (A1 . 31).

From (A 1 . 26) and (A 1 . 35) we derive the result

$$\text{(A 1 . 4)} \quad G(x) = -\frac{i}{4\pi}\sum_{n=1}^{N}\frac{d(n)}{\sqrt{n}}\left[Y_1\{4\pi\sqrt{(nx)}\} + \frac{2}{\pi} K_1\{4\pi\sqrt{(nx)}\}\right] + J_1 + J_2 + J_3 + J_4,$$

where

$$\text{(A 1 . 41)} \quad J_2 = -\frac{1}{2\pi i}\int_{\Gamma_2} g(s)\, J_1(4\pi i s\sqrt{x})\, ds,$$

$$\text{(A 1 . 42)} \quad J_3 = -\frac{1}{2\pi i}\int_{\Gamma_3} g(s)\, J_1(4\pi i s\sqrt{x})\, ds,$$

$$\text{(A 1 . 43)} \quad J_4 = \frac{1}{4\pi}\int_{\Gamma_4} h(s)\, H_1^{(1)}(4\pi i s\sqrt{x})\, ds,$$

and

$$\text{(A 1 . 44)} \quad J_1 = \frac{1}{2\pi i}\int_{C}\left\{g(s)\, J_1(4\pi i s\sqrt{x}) + \frac{1}{2i} h(s)\, H_1^{(1)}(4\pi i s\sqrt{x})\right\} ds.$$

We can now make $\delta \to 0$. For we have*

$$\text{(A 1 . 45)} \quad \pi Y_1(z) = 2\{\gamma + \log(\tfrac{1}{2}z)\}\, J_1(z) - \frac{2}{z} - \sum_{n=0}^{\infty}\frac{(-)^n(\tfrac{1}{2}z)^{1+2n}}{n!\,(n+1)!}\{a_n + a_{n+1}\},$$

* Watson, *loc. cit.*, 62 (3), 64 (2).

where $$a_0 = 0, \quad a_n = 1+\tfrac{1}{2}+\ldots+\frac{1}{n} \quad (n = 1, 2, \ldots),$$

so that

$$g(s)\,J_1(4\pi is\sqrt{x})+\frac{1}{2i}\,h(s)\,H_1^{(1)}(4\pi is\sqrt{x})$$

$$=\Big[-\tfrac{1}{2}\gamma-\tfrac{1}{2}\log s-\frac{2\log(2\pi s)}{(2\pi s)^2}$$

$$+\frac{2}{\pi^2}\sum_1^\infty d(n)\log\frac{s}{\sqrt{n}}\left\{\frac{1}{s^2-n}+\frac{1}{s^2+n}\right\}\Big]\,J_1(4\pi is\sqrt{x})$$

$$+\frac{1}{2i}\left[\tfrac{1}{2}\pi+\frac{2\pi}{(2\pi s)^2}-\frac{1}{\pi}\sum_1^\infty d(n)\left\{\frac{1}{s^2-n}+\frac{1}{s^2+n}\right\}\right]J_1(4\pi is\sqrt{x})$$

$$+\frac{1}{2\pi}\left[\tfrac{1}{2}\pi+\frac{2\pi}{(2\pi s)^2}-\frac{1}{\pi}\sum_1^\infty d(n)\left\{\frac{1}{s^2-n}+\frac{1}{s^2+n}\right\}\right]$$

$$\times\left[2\left\{\gamma+\tfrac{1}{2}\pi i+\tfrac{1}{2}\log x+\log(2\pi s)\right\}J_1(4\pi is\sqrt{x})-\frac{1}{2\pi is\sqrt{x}}\right.$$

$$\left.-\sum_{n=0}^\infty\frac{i(2\pi s\sqrt{x})^{1+2n}}{n!\,(n+1)!}\,(a_n+a_{n+1})\right]$$

$$=-\frac{1}{8\pi^3 i\sqrt{x}}\,\frac{1}{s^3}+\frac{i}{2\pi\sqrt{x}}\left\{x\log x+(2\gamma-1)x+\tfrac{1}{4}\right\}\frac{1}{s}+O\left(\log\frac{1}{\delta}\right)$$

near $s = 0$. Hence

$$\lim_{\delta\to 0} J_1 = \frac{i}{4\pi\sqrt{x}}\left\{x\log x+(2\gamma-1)x+\tfrac{1}{4}\right\}. \tag{A 1.46}$$

To deal with J_2, J_3, and J_4 we require a lemma which has been proved in the main part of this paper [Lemma 13]

If $-1 \leqslant b < 1$ and $T^2 = N+\frac{1}{2}$, where N is a positive integer, then for all large N,

$$\sum_{n=1}^\infty \frac{n^b}{|T^4-n^2|} = O(N^{b-1+\epsilon}) \tag{A 1.5}$$

for each positive ϵ.

Hence, for all real θ,

$$\text{(A 1.51)} \qquad \Sigma \left| \frac{n^b}{(T^4 e^{4i\theta} - n^2)} \right| < K_{b,\epsilon} N^{b-1+\epsilon},$$

since

$$| T^4 e^{4i\theta} - n^2 | \geqslant | T^4 - n^2 |,$$

so that, on the circle $|s| = T$, we have

$$\text{(A 1.52)} \qquad g(s) = O(N^\epsilon), \quad h(s) = O(N^\epsilon) \quad (\epsilon > 0),$$

since

$$d(n) = O(n^\epsilon) \quad \text{and} \quad |\log s| = O(N^\epsilon).$$

Now on Γ_2 and Γ_3 (the arcs of $|s| = T$ between $\sigma = 0$ and $\sigma = c$) we have, uniformly on $\theta = \arg s$,

$$\text{(A 1.53)} \qquad J_1(4\pi i s) = O(T^{-\frac{1}{2}}) \quad \left(\tfrac{1}{2}\pi - \arcsin \frac{c}{T} \leqslant |\theta| \leqslant \tfrac{1}{2}\pi\right),$$

and on Γ_4 $(-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi)$

$$\text{(A 1.54)} \qquad H_1^{(1)}(4\pi i s\sqrt{x}) = O(T^{-\frac{1}{2}} e^{-4\pi T\sqrt{x}\cos\theta}),$$

both uniformly for x in $0 < x_1 \leqslant x \leqslant x_2$. Hence

$$\text{(A 1.55)} \qquad J_2, J_3 = O \int_{\frac{1}{2}\pi - \arcsin(c/T)}^{\frac{1}{2}\pi} N^\epsilon T^{\frac{1}{2}} d\theta = O(N^{-\frac{1}{4}+\epsilon})$$

and

$$\text{(A 1.56)} \qquad J_4 = O \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} N^\epsilon T^{\frac{1}{2}} e^{-4\pi T\sqrt{x}\cos\theta} d\theta = O(N^{-\frac{1}{4}+\epsilon})$$

as in the proof of Jordan's lemma.

From (A .1 . 4)–(A 1 . 56) we derive

$$\text{(A 1.6)} \quad G(x) = \frac{i}{4\pi\sqrt{x}} \left\{ x \log x + (2\gamma - 1)x + \tfrac{1}{4} - \sqrt{x} \sum_1^N \frac{d(n)}{\sqrt{n}} \left(Y_1 + \frac{2}{\pi} K_1 \right) \right\} + O(N^{-\frac{1}{4}+\epsilon})$$

for large integral N, and uniformly for x in $0 < x_1 \leqslant x \leqslant x_2 < \infty$.

Now, by definition,

$$g(s) = \pi i \sum_{n=1}^{\infty} d(n) H_0^{(1)}(4\pi i s\sqrt{n})$$

so that, since this series is uniformly convergent for $\sigma = c$ and $|t| \leqslant T'$,

$$(\text{A } 1.7) \quad G(x) = \sum_{n=1}^{\infty} d(n) \frac{\pi i}{2\pi i} \int_{c-iT'}^{c+iT'} H_0^{(1)}(4\pi i s\sqrt{n})\, J_1(4\pi i s\sqrt{x})\, ds$$

$$= \frac{i}{4\pi\sqrt{x}} \sum_{n \leqslant x}' d(n) + O\left(\frac{1}{T}\right)$$

by Lemma 2 (4 . 3), Lemma 3 (4 . 4), and Lemma 4 (4 . 5); and this is true for any $x > 0$, and uniformly if x lies in a closed interval free from integers.

From (A 1 . 6) and (A 1 . 7) we see at once that

$$(\text{A } 1.8) \quad \sum_{n \leqslant x}' d(n) = x \log x + (2\gamma - 1)x + \tfrac{1}{4} - \sqrt{x} \sum_{n=1}^{N} \frac{d(n)}{\sqrt{n}}$$

$$\times \left[Y_1\{4\pi\sqrt{(nx)}\} + \frac{2}{\pi} K_1\{4\pi\sqrt{(nx)}\} \right] + O(N^{-\frac{1}{2}+\epsilon})$$

for any positive x and uniformly in any closed interval free from integers, for large integral N and each positive ϵ. Taking $\epsilon < \frac{1}{4}$ and making $N \to \infty$, we obtain Voronoï's identity.

ON CUBIC SURFACES, MILNE QUADRICS, AND OTHER ENVELOPES

By A. L. DIXON.

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The present paper is the outcome of an attempt to find a proof of Milne's theorem*, viz. : *The* 24 *tangent planes to a cubic surface at the* 24 *parabolic points of a double six touch the same quadric* U_2, which should be applicable to the ordinary form, $UVW = U'V'W'$, of the equation to a cubic.

By reciprocating with respect to the Schur quadric of the double six, I determined the equation to Milne's quadric, and was led on to other results regarding the three associated double sixes, which can be formed out of eighteen lines on the cubic, each line belonging to two double sixes.

One theorem I have obtained is :

The tangent planes to a cubic surface along its curve of intersection with the Schur quadric of a double six all touch the cubic envelope which contains the double six.

Milne proves that his quadrics are contact-quadrics of the harmonic envelope T† (that is, that the two envelopes touch wherever they meet), and another theorem is :

The eighteen lines of three associated double sixes meet the parabolic curve in thirty-six points, and the tangent planes to the cubic surface at all these thirty-six points touch the same cubic envelope, which also touches the common tangent planes of the harmonic envelope T *and each of the three Milne quadrics.*

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* (M), p. 9.

† (M), p. 10.

If we consider a pencil of cubic surfaces through nine given lines, the equation to any one being $UVW = kU'V'W'$, the other eighteen lines on each surface form a triad of double sixes (each line occurring twice), and determine a triad of Schur quadrics, and also a triad of quadrics each through six of the lines. Then it follows from the formulae I have obtained (§§ 8, 9) that :

The locus of the parabolic curves of the surfaces of the pencil is a quartic surface, which is also

(i) *the locus of the curve of intersection of any pair of a triad of Schur quadrics, and*

(ii) *The envelope of the quadrics through six of the lines.*

References are made to

(M) W. P. Milne, "The harmonic and equiharmonic envelopes of a cubic surface", *Journal London Math. Soc.*, 1 (1926), 7.

(D) A. L. Dixon, "Cubic surfaces, Schur quadrics, &c.", *Journal London Math Soc.*, 1 (1926), 170.

1. *Notation.*

As the number of symbols used is large, I have thought it best to collect them all in the first paragraph.

The equation to the cubic surface is

$$C \equiv UVW - U'V'W' = 0.$$

The identical relations between the six planes* are taken to be

$$L_r \equiv A_r U + B_r V + C_r W \equiv A'_r U' + B'_r V' + C'_r W' \quad (r = 1, 2, 3),$$

$$L_1 + L_2 + L_3 \equiv 0,$$

and we have the following identities

$$D_{rs} \equiv A_s B_r C_r + B_s C_r A_r + C_s A_r B_r - A'_s B'_r C'_r - B'_s C'_r A'_r - C'_s A'_r B'_r,$$

$$\theta = D_{12} = D_{23} = D_{31} = -D_{21} = -D_{32} = -D_{13}.$$

$$P_r \equiv A_r B_r C_r = A'_r B'_r C'_r,$$

$$(lA_r + mA_s)(lB_r + mB_s)(lC_r + mC_s) - (lA'_r + mA'_s)(lB'_r + mB'_s)(lC'_r + mC'_s)$$
$$\equiv D_{rs} lm(l - m).$$

* (D), pp. 170, 172.

The equations of the three quadrics, each through half of a double six, are

$$S_1(U) \equiv B_1C_1VW+C_1A_1WU+A_1B_1UV$$
$$-B_1'C_1'V'W'-C_1'A_1'W'U'-A_1'B_1'U'V' = 0,$$
$$S_2(U) \equiv B_2C_2VW+C_2A_2WU+A_2B_2UV$$
$$-B_2'C_2'V'W'-C_2'A_2'W'U'-A_2'B_2'U'V' = 0,$$
$$S_3(U) \equiv B_3C_3VW+C_3A_3WU+A_3B_3UV$$
$$-B_3'C_3'V'W'-C_3'A_3'W'U'-A_3'B_3'U'V' = 0$$

or
$$-2S_r(U) \equiv \Sigma A_r^2U^2-\Sigma A_r'^2U'^2 = 0,$$

and the equation of the Schur quadric to that double six which lies half in S_1 and half in S_2 is

$$2T_3(U) \equiv \Sigma(B_1C_2+B_2C_1)VW-\Sigma(B_1'C_2'+B_2'C_1')V'W'$$
$$\equiv -\Sigma A_1A_2U^2+\Sigma A_1'A_2'U'^2$$
$$\equiv S_3(U)-S_1(U)-S_2(U) = 0,$$

so that*

$$S_1+T_2+T_3 \equiv 0, \quad S_2+T_3+T_1 \equiv 0, \quad S_3+T_1+T_2 \equiv 0.$$

The method of this paper is to consider a plane as, at the same time, the polar plane of (U, V, W, U', V', W') with respect to the cubic surface C, and the polar plane of (u, v, w, u', v', w') with respect to the Schur quadric T_3.

The transformation formulae of this (8, 1)-point correspondence are written down, and the geometrical relations, between the lines and surfaces considered, enable us to reduce them to simpler and more suggestive forms. I write

$$S_r \equiv S_r(U), \quad T_r \equiv T_r(U), \quad \text{and} \quad \sigma_r \equiv S_r(u).$$

I shall treat equations in (u, v, w, u', v', w') as representing the envelope of the polar plane of u with respect to T_3, so that, since S_1 and S_2 are polar

* (D), p. 171. I have made a slight change in the notation; S_r and T_r of the former paper have been changed to $-2S_r$ and $-2T_r$.

reciprocals of one another with respect to T_3, I write alternatively (taking $\Sigma_r = 0$ as the tangential equation of $S_r = 0$),

$$\Sigma_1 \equiv \theta P_1 \sigma_2, \quad \Sigma_2 \equiv \theta P_2 \sigma_1, \quad \Sigma_3 \equiv -\theta(P_1\sigma_2 + P_2\sigma_1),$$

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = 0,$$

which agrees with the relation $P_1 a_1^2 + P_2 a_2^2 + P_3 a_3^2 = 0$ found in (D), page 173.

The three Milne quadrics, which touch the tangent planes to the cubic C at the twenty-four parabolic points on a double six, I denote by ϕ_1, ϕ_2, ϕ_3, so that ϕ_3 is the U_2 of Milne's paper.

The three cubic envelopes which pass through the lines of the three double sixes, respectively, I call Γ_1, Γ_2, Γ_3 (Γ_3 is Milne's U_3)*, and I take $\Gamma_3 \equiv \theta^2 P_1 P_2 (uvw - u'v'w')$, since the lines of the double six are conjugate in pairs with respect to the Schur quadric T_3, and so Γ_3 and C are reciprocals with respect to T_3.

The cubic envelope which touches the tangent planes to the cubic C at the thirty-six parabolic points on the three double sixes I call Γ.

S and T have their usual meanings as contravariants of C, whose tangential equation is $T^2 - 64S^3 = 0$. $S = 0$ and $T = 0$ are the envelopes of planes which cut the cubic surface in cubic curves, whose invariant cross ratio is equianharmonic and harmonic respectively†.

Finally there is the point X, whose coordinates are

$$U = \Sigma B_r C_r, \quad V = \Sigma C_r A_r, \quad W = \Sigma A_r B_r,$$

$$U' = \Sigma B'_r C'_r, \quad V' = \Sigma C'_r A'_r, \quad W' = \Sigma A'_r B'_r \quad (r = 1, 2, 3).$$

Taking its tangential equation to be $\xi = 0$, I write

$$-2\xi \equiv A_1 A_2 (\Sigma BC)\, u + B_1 B_2 (\Sigma CA)\, v + C_1 C_2 (\Sigma AB)\, w$$

$$- A'_1 A'_2 (\Sigma B'C')\, u' - B'_1 B'_2 (\Sigma C'A')\, v' - C'_1 C'_2 (\Sigma A'B')\, w'$$

$$\equiv \Sigma A_1 A_2 B_3 C_3 u - \Sigma A'_1 A'_2 B'_3 C'_3 u'$$

$$\equiv \Sigma A_1 A_2 (B_1 C_2 + B_2 C_1)\, u - \Sigma A'_1 A'_2 (B_1 C'_2 + B'_2 C'_1)\, u',$$

where, for example, ΣBC means $B_1 C_1 + B_2 C_2 + B_3 C_3$ and $\Sigma B'C'$ means $B'_1 C'_1 + B'_2 C'_2 + B'_3 C'_3$.

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* (M), p. 10.

† Cf. (M), p. 8.

The properties of the point X are :

(i) Its polar quadric with respect to C is $S_1+S_2+S_3=0$.

(ii) Four common tangent planes to S_1, S_2, S_3, ϕ_1, ϕ_2, ϕ_3 meet at X, and the six points of contact in each plane are collinear. These four planes also touch the equianharmonic envelope $S=0$.

(iii) The tangent cone from X to the cubic envelope Γ is common to Γ, Γ_1, Γ_2, Γ_3.

(iv) The polar plane of X with respect to C meets the Hessian and the quartic surface $T_2T_3+T_3T_1+T_1T_2=0$ in the same curve. This plane curve and the parabolic curve make up the complete intersection of the two quartic surfaces.

2. Take any plane and let (U, V, W, U', V', W') be one of its poles with respect to the cubic C, and (u, v, w, u', v', w') its pole with respect to the quadric T_3. The equations to the two polar planes must be equivalent, so that we have a set of six relations

$$\kappa VW = A_1A_2u+lA_1+mA_2, \ldots,$$

wherein κ, l, and m are undetermined multipliers.

Since $(B_1C_1+B_2C_2+B_3C_3, \ldots)$ are the coordinates of a point, we get*

$$\kappa\Sigma(B_1C_1+B_2C_2+B_3C_3)\,VW-\kappa\Sigma(B_1'C_1'+B_2'C_2'+B_3'C_3')V'W'$$
$$=\Sigma A_1A_2B_3C_3u-\Sigma A_1'A_2'B_3'C_3'w',$$

the other terms disappearing on account of the fundamental identities, so that we have $\kappa(S_1+S_2+S_3)=-2\xi$, or $\xi=\kappa(T_1+T_2+T_3)$.

Also we have

$$\kappa S_1 = m(\Sigma A_2B_1C_1-\Sigma A_2'B_1'C_1') = mD_{12} = m\theta;$$
$$\kappa S_2 = l(\Sigma A_1B_2C_2-\Sigma A_1'B_2'C_2') = lD_{21} = -l\theta.$$

Again, taking the sum of three terms only, for the moment,

$$\kappa^2 UVW(\Sigma A_1U) = \Sigma A_1(B_1B_2v+lB_1+mB_2)(C_1C_2w+lC_1+mC_2)$$
$$= A_1B_1C_1(\Sigma B_2C_2vw)+lA_1B_1C_1\Sigma(B_2v+C_2w)$$
$$+m\Sigma A_1A_2(B_1C_2+B_2C_1)\,u$$
$$+m^2\Sigma A_1B_2C_2+3l^2A_1B_1C_1+lm\Sigma(A_1B_1C_2+A_1C_1B_2),$$

* The summations are for cyclical interchange of A, B, C.

so that, taking the two sets of three terms together,

$$\kappa^2 UVW(\Sigma A_1 U)-\kappa^2 U'V'W'(\Sigma A_1' U') = P_1\sigma_2-2m\xi-\theta m^2+2\theta lm,$$

or $$\theta P_1\sigma_2 = \kappa^2 S_1(S_2-S_3)+\kappa^2\theta C L_1,$$

and similarly

$$\theta P_2\sigma_1 = \kappa^2 S_2(S_3-S_1)+\kappa^2\theta C L_2,$$

so that $$-\theta(P_2\sigma_1+P_1\sigma_2) = \kappa^2 S_3(S_1-S_2)+\kappa^2\theta C L_3.$$

Lastly

$$\kappa^3(U^2V^2W^2-U'^2V'^2W'^2)$$

$$= (A_1A_2u+lA_1+mA_2)(B_1B_2v+lB_1+mB_2)(C_1C_2w+lC_1+mC_2)$$

$$-(A_1'A_2'u'+lA_1'+mA_2')(B_1'B_2'v'+lB_1'+mB_2')(C_1'C_2'w'+lC_1'+mC_2')$$

$$= P_1P_2(uvw-u'v'w')+lP_1\sigma_2+mP_2\sigma_1-2lm\xi+\theta lm(l-m),$$

so that

$$\Gamma_3 \equiv \theta^2 P_1P_2(uvw-u'v'w')$$

$$= \kappa^3 S_1S_2(S_1+S_2-S_3)+\kappa^3\theta^2 C(UVW+U'V'W')-\kappa^3\theta C(L_2S_1-L_1S_2)$$

$$= -2\kappa^3 S_1S_2T_3+\kappa^3\theta^2 C(UVW+U'V'W')-\kappa^3\theta C(L_2S_1-L_1S_2).$$

3. If we denote the quadric envelope

$$\xi^2+\theta P_1\sigma_2-\theta P_2\sigma_1 = 0$$

by $\phi_3 = 0$, we shall find, in § 4, that $\phi_3 = 0$ is a Milne quadric, and using the transformation of § 2, we may write

$$\phi_3 \equiv \xi^2+\theta P_1\sigma_2-\theta P_2\sigma_1 = \tfrac{1}{4}\kappa^2(S_1+S_2-S_3)^2+2\kappa^2 S_1S_2+\kappa^2\theta C(L_1-L_2)$$

so that $$\phi_3 \equiv \xi^2+\Sigma_1-\Sigma_2 = \kappa^2 T_3^2+2\kappa^2 S_1S_2+\kappa^2\theta C(L_1-L_2)$$

and $$\phi_1 \equiv \xi^2+\Sigma_2-\Sigma_3 = \kappa^2 T_1^2+2\kappa^2 S_2S_3+\kappa^2\theta C(L_2-L_3),$$

$$\phi_2 \equiv \xi^2+\Sigma_3-\Sigma_1 = \kappa^2 T_2^2+2\kappa^2 S_3S_1+\kappa^2\theta C(L_3-L_1),$$

so that $$\phi_1+\phi_2+\phi_3 \equiv 3\xi^2 \quad \text{and} \quad \phi_1-\phi_2 = -3\Sigma_3,$$

that is to say, from X can be drawn four tangent planes to touch the six quadrics ϕ_1, ϕ_2, ϕ_3, S_1, S_2, S_3, and the six points of contact are collinear in each plane.

Lastly, for the cubic envelope which touches the tangent planes at the parabolic points on the eighteen lines, we take

$$3\Gamma \equiv 3\Gamma_3+2\xi\phi_3 \equiv 3\Gamma_1+2\xi\phi_1 \equiv 3\Gamma_2+2\xi\phi_2$$

$$= -2\kappa^3(T_1T_2T_3+2S_1S_2S_3)+3\kappa^2\theta^2C(UVW+U'V'W')$$

$$-\kappa^3\theta C\{(L_2-L_3)S_1+(L_3-L_1)S_2+(L_1-L_2)S_3\}.$$

Thus the tangent cone from X to Γ is also the tangent cone to $\Gamma_1, \Gamma_2, \Gamma_3$.

4. If we consider any one of these transformations in § 3, *e.g.*

$$\phi_3 \equiv \xi^2+\Sigma_1-\Sigma_2 = \xi^2+\theta P_1\sigma_2-\theta P_2\sigma_1 = \kappa^2T_3^2+2\kappa^2S_1S_2+\kappa^2\theta C(L_1-L_2),$$

we see that $\xi^2+\Sigma_1-\Sigma_2=0$ is the tangential equation to ϕ_3, and

$$\xi^2+\theta P_1\sigma_2-\theta P_2\sigma_1=0$$

is the point equation of its polar reciprocal with respect to T_3, while $T_3^2+2S_1S_2+\theta C(L_1-L_2)=0$ is the locus of the eight poles, with respect to C, of the tangent planes to ϕ_3.

If now in these equations we put $C=0$, we shall get relations which are satisfied by the points of contact on C of the common tangent planes of C and the envelope (ϕ or Γ), and these will, in general, be enough to determine the envelope. Thus, on putting $C=0$, we have

$$\phi_3 = \kappa^2(T_3^2+2S_1S_2),$$

and the locus of the point of contact on C of the common tangent planes of ϕ_3 and C is $T_3^2+2S_1S_2=0$. But the parabolic points on that double six, which has six lines in S_1 and six in S_2, are given by $C=0$, $T_3=0$, $S_1=0$, or $S_2=0$*, so that ϕ_3 is Milne's quadric which touches the tangent planes to the cubic at these twenty-four points.

Also, since

$$3\Sigma_3 = \phi_2-\phi_1 = 3\kappa^2S_3(S_1-S_2) \quad \text{if} \quad C=0,$$

we see that not only does S_3 go through half a double six (which is the interpretation of the factor S_3), but also that the other common tangent planes to S_3 (*i.e.* Σ_3) and C touch C along its curve of intersection with S_1-S_2.

* Cf. the identity of § 8. $H-CE=4(T_3^2-S_1S_2)$.

5. We have also, putting $C = 0$,

$$3\Gamma = -2\kappa^3(T_1T_2T_3+2S_1S_2S_3) = -2\kappa^3\{3T_1T_2T_3-2(\Sigma T)(\Sigma T_1T_2)\},$$

so that Γ is the cubic envelope determined by the thirty-six tangent planes to the cubic at the parabolic points on the eighteen lines, since these points are given by $S_r = 0$, $T_t = 0$.

Again, putting $C = 0$, we have $\Gamma_3 = -2\kappa^3 S_1S_2T_3$, so that Γ_3 is the cubic envelope through the double six lying in S_1, S_2, as is otherwise evident, and, moreover, it touches the tangent planes drawn to the cubic C at all points of its curve of intersection with T_3.

6. Again, with $C = 0$, we find that

$$\phi_2\phi_3+\phi_3\phi_1+\phi_1\phi_2 = -3\kappa^4\{(\Sigma T_1T_2)^2+6(\Sigma T)T_1T_2T_3-4(\Sigma T)^2(\Sigma T_1T_2)\},$$

so that $$9\xi\Gamma-\Sigma\phi_1\phi_2 = 3\kappa^4(\Sigma T_1T_2)^2 = 3\kappa^4(T_3^2-S_1S_2)^2.$$

Therefore the locus of the poles with respect to C of the tangent planes to the quartic envelope $9\xi\Gamma-\Sigma\phi_1\phi_2 = 0$ passes through the intersection of $C = 0$ and $(T_3^2-S_1S_2)^2 = 0$, and since a point given by $T_3 = 0$, $S_1 = 0$, $C = 0$ is a parabolic point on C, at which the two asymptotic lines coincide with a generator of S_1, we see, by considering a point on the locus in the neighbourhood of this parabolic point, and taking its polar plane with respect to C, that the quartic envelope $9\xi\Gamma-\Sigma\phi_1\phi_2 = 0$ touches C at the thirty-six parabolic points on the eighteen lines, and so* is identified with the equianharmonic envelope $S = 0$. In fact, we may write

$$3S \equiv 9\xi\Gamma-\Sigma\phi_1\phi_2 \quad \text{and} \quad S = \kappa^4(\Sigma T_1T_2)^2 \quad \text{when} \quad C = 0.$$

Since $S = 0$ is inscribed in the parabolic developable, it follows that $T_3^2-S_1S_2 = 0$ must pass through the parabolic curve.

I will give in § 8 a direct proof that $T_3^2-S_1S_2 = 0$ passes through the intersection of C and its Hessian H.

We also find that, with $C = 0$,

$$\phi_1\phi_2\phi_3 = 2\kappa^6(\Sigma T_1T_2)^3+3\kappa^6\{2(\Sigma T)(\Sigma T_1T_2)-3T_1T_2T_3\}^2,$$

and therefore

$$27\Gamma^2-4\phi_1\phi_2\phi_3 = -8\kappa^6(\Sigma T_1T_2)^3 = -8\kappa^6(T_3^2-S_1S_2)^3$$

* (M), p. 8.

so that $27\Gamma^2-4\phi_1\phi_2\phi_3=0$ is identified with the harmonic envelope $T=0$, since it is a sextic envelope touching C along the parabolic curve, and we write

$$T\equiv 27\Gamma^2-4\phi_1\phi_2\phi_3.$$

We have then, with $C=0$,

$$T=-8\kappa^6(\Sigma T_1T_2)^3,\quad S=\kappa^4(\Sigma T_1T_2)^2,$$

so that we get $T^2-64S^3=0$ for the tangential equation to the cubic, as it should be. Thus the tangent planes along the curves of contact of T with ϕ_1, ϕ_2, ϕ_3 all touch Γ.

7. Milne's researches led him to a series of envelopes U_2, U_3, U_4*, Γ_6, U_8 in his notation. The curves of contact, on the cubic surface, of the common tangent planes of these envelopes and the cubic are given in the following table, in which H stands for the Hessian, which intersects C in the parabolic curve.

$$U_2,\quad [T_3^2+2S_1S_2];\quad U_4\equiv S-U_2^2,\quad [S_1S_2(2T_3^2+S_1S_2)];$$

$$U_3,\quad [T_3S_1S_2];\quad \Gamma_6\equiv T-8SU_2,\qquad [H^2S_1S_2];$$

$$U_8\equiv U_2\Gamma_6-4SU_4\equiv TU_2-4SU_2^2-4S^2,\qquad [H^2S_1^2S_2^2].$$

The series may be continued; for example,

$$U_{12}\equiv T^2-24STU_2+192S^2U_2^2-8TU_2^3,\qquad [H^3S_1^3S_2^3]$$

is a duodecimic envelope which touches the cubic surface along the parabolic curve and along the twelve lines of the double six. Similarly the duodecimic envelope which touches the cubic surface along the parabolic curve and contains all the eighteen lines is, in the notation of this paper,

$$9T\Gamma^2-96S^2\Gamma\xi+4ST\xi^2=0\quad [H^3S_1^2S_2^2S_3^2].$$

8. *Proof of the identity*

$$H-CE=4(T_1^2-S_1S_2)=-4(T_2T_3+T_3T_1+T_1T_2)$$
$$=S_1^2+S_2^2+S_3^2-2S_2S_3-2S_3S_1-2S_1S_2,$$

where $E=0$ is the polar plane of X with respect to C.

* (M), p. 10. In my notation $U_4=S-\phi_3^2=3(\xi\Gamma_3+\Sigma_1\Sigma_2)$.

Since the Hessian comes into this formula, it is convenient to change the notation, and write C as the sum of six cubes, ΣX^3.

If we put $A_1 = B_1 = C_1 = A_1' = B_1' = C_1' = 1$, and

$$U = X_2+X_3, \qquad V = X_3+X_1, \qquad W = X_1+X_2,$$

$$-U' = X_5+X_6, \quad -V' = X_6+X_4, \quad -W' = X_4+X_5,$$

and also $\quad r_1 = B_2+C_2, \quad r_2 = C_2+A_2, \quad r_3 = A_2+B_2,$

$$r_4 = B_2'+C_2', \quad r_5 = C_2'+A_2', \quad r_6 = A_2'+B_2',$$

we get $\qquad \Sigma X \equiv 0, \quad \Sigma rX \equiv 0,$

$$S_1 \equiv \Sigma VW - \Sigma V'W' \equiv X_2X_3+X_3X_1+X_1X_2-X_5X_6-X_6X_4-X_4X_5$$

$$\equiv -\tfrac{1}{2}(X_1^2+X_2^2+X_3^2-X_4^2-X_5^2-X_6^2),$$

$$2T_3 \equiv \Sigma(B_2+C_2)VW - \Sigma(B_2'+C_2')V'W' \equiv \Sigma r_1X_2X_3 - \Sigma r_4X_5X_6.$$

I also write

$$D = X_1+X_2+X_3 = -(X_4+X_5+X_6),$$

$$S' \equiv r_1^2X_2X_3+r_2^2X_3X_1+r_3^2X_1X_2-r_4^2X_5X_6-r_5^2X_6X_4-r_6^2X_4X_5.$$

Then for the cubic we have

$$3C \equiv 3(X_2+X_3)(X_3+X_1)(X_1+X_2)+3(X_5+X_6)(X_6+X_4)(X_4+X_5)$$

$$\equiv -\{X_1^3+X_2^3+X_3^3+X_4^3+X_5^3+X_6^3\}$$

$$= 3DS_1 - 3X_1X_2X_3 - 3X_4X_5X_6,$$

since

$$X_1^3+X_2^3+X_3^3-3X_1X_2X_3 = D(X_1^2+X_2^2+X_3^2-X_2X_3-X_3X_1-X_1X_2)$$

and also for the Hessian

$$H \equiv \Sigma(r_1-r_2)^2X_3X_4X_5X_6 \quad \text{(fifteen terms)}$$

$$= X_1X_2X_3X_4X_5X_6\left\{\left(\Sigma\frac{1}{X}\right)\left(\Sigma\frac{r^2}{X}\right)-\left(\Sigma\frac{r}{X}\right)^2\right\}.*$$

Since $\qquad \Sigma\dfrac{1}{X} = \dfrac{X_2X_3+X_3X_1+X_1X_2}{X_1X_2X_3} + \dfrac{X_5X_6+X_6X_4+X_4X_5}{X_4X_5X_6}$

$$= \frac{S_1}{X_1X_2X_3} \quad \text{if} \quad X_1X_2X_3+X_4X_5X_6 = 0,$$

* A. L. Dixon, "On cubic surfaces", *Proc. London Math. Soc.* (2), 7 (1909), 391.

and there are similar relations for $\Sigma r/X$, $\Sigma r^2/X$, we see that $H-4T_3^2+S_1S'$ vanishes when $X_1X_2X_3=-X_4X_5X_6$, so that we get at once

$$H-4T_3^2+S_1S'=(X_1X_2X_3+X_4X_5X_6)\{\Sigma(r_2-r_3)^2X_1+\Sigma(r_5-r_6)^2X_4\},$$

by picking out the coefficient of $X_1X_2X_3X_4$.

We may write this

$$H-4T_3^2+S_1S'=-(DS_1-C)E,$$

or in the desired form $\quad H-CE=4(T_3^2-S_1S_2)$,

if we put $\quad 4S_2=S'+DE\,; \quad -E=\Sigma(r_2-r_3)^2X_1+\Sigma(r_5-r_6)^2X_4.$

It is not difficult to verify that

$$S'+DE=\Sigma r_1^2X_2X_3-\Sigma r_4^2X_5X_6-\Sigma X_1\,.\,\Sigma(r_2-r_3)^2X_1+\Sigma X_4\,.\,\Sigma(r_5-r_6)^2X_4$$
$$+(\Sigma r_1X_1)^2-(\Sigma r_4X_4)^2$$
$$\equiv 4[\Sigma B_2C_2VW-\Sigma B_2'C_2'V'W'].$$

and also, since

$$(A_1C_1+A_2C_2+A_3C_3)(A_1B_1+A_2B_2+A_3B_3)+(A_2B_1-B_2A_1)(A_2C_1-C_2A_1)$$
$$=2A_1P_1+2A_2P_2+2A_3P_3,$$

that

$$-\{\Sigma(r_2-r_3)^2X_1+\Sigma(r_5-r_6)^2X_4\}\equiv\Sigma(\Sigma AC)(\Sigma AB)U-\Sigma(\Sigma A'C')(\Sigma A'B')U'$$

when A_1, B_1, C_1, A_1', B_1', C_1' are put equal to unity, so that

$$E=-\Sigma(r_2-r_3)^2X_1-\Sigma(r_5-r_6)^2X_4=0$$

is the polar plane of X with respect to C.

9. The quartic surface, $T_2T_3+T_3T_1+T_1T_2=0$, is the locus of the parabolic curves of the pencil $UVW=kU'V'W'$.

If we consider any surface of the pencil $UVW=kU'V'W'$, and, keeping A_1, B_1, ..., A_2, B_2, ... unchanged, denote by λ_1, λ_2, λ_3 the roots of the equation

$$k(A_1+\lambda A_2)(B_1+\lambda B_2)(C_1+\lambda C_2)=(A_1'+\lambda A_2')(B_1'+\lambda B_2')(C_1'+\lambda C_2'),$$

we see at once that, for this surface, the quadrics corresponding to S_1

and T_3 are given by the equations

$$(B_1+\lambda_1 B_2)(C_1+\lambda_1 C_2)VW+(C_1+\lambda_1 C_2)(A_1+\lambda_1 A_2)WU$$
$$+(A_1+\lambda_1 A_2)(B_1+\lambda_1 B_2)UV$$
$$=(B_1'+\lambda_1 B_2')(C_1'+\lambda_1 C_2')V'W'+(C_1'+\lambda_1 C_2')(A_1'+\lambda_1 A_2')W'U'$$
$$+(A_1'+\lambda A_2')(B_1'+\lambda B_2')U'V'$$

and

$$\Sigma\{(B_1+\lambda_1 B_2)(C_1+\lambda_2 C_2)+(B_1+\lambda_2 B_2)(C_1+\lambda_1 C_2)\}VW$$
$$=\Sigma\{(B_1'+\lambda_1 B_2')(C_1'+\lambda_2 C_2')+(B_1'+\lambda_2 B_2')(C_1'+\lambda_1 C_2')\}V'W',$$

respectively, so that the three quadrics, each through half of a double six, are, for any value of k,

$$S_1+2\lambda_1 T_3+\lambda_1^2 S_2=0,\quad S_1+2\lambda_2 T_3+\lambda_2^2 S_2=0,\quad S_1+2\lambda_3 T_3+\lambda_3^2 S_2=0,$$

and the three Schur quadrics are

$$S_1+(\lambda_2+\lambda_3)T_3+\lambda_2\lambda_3 S_2=0,\quad S_1+(\lambda_3+\lambda_1)T_3+\lambda_3\lambda_1 S_2=0,$$
$$S_1+(\lambda_1+\lambda_2)T_3+\lambda_1\lambda_2 S_2=0.$$

Since

$$(S_1+2\lambda_1 T_3+\lambda_1^2 S_2)(S_1+2\lambda_2 T_3+\lambda_2^2 S_2)-\{S_1+(\lambda_1+\lambda_2)T_3+\lambda_1\lambda_2 S_2\}^2$$
$$=(\lambda_1-\lambda_2)^2(S_1S_2-T_3^2),$$

it follows that the quartic surface given by

$$S_1S_2=T_3^2\quad\text{or}\quad T_2T_3+T_3T_1+T_1T_2=0$$

is the same for all cubic surfaces belonging to the pencil, so that the last theorem stated in the introduction is proved, since it follows from the identity $H-CE=4(T_3^2-S_1S_2)$ that the intersection of a cubic and its Hessian lies on the surface $S_1S_2=T_3^2$.

WAVES IN CANALS AND BASINS

By B. M. SEN.

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Preface.

The problem of wave-motion in canals and basins has been solved in only a few cases. The known cases of horizontal canals are cylinders with

(*a*) a rectangular section,

(*b*) section consisting of two straight lines inclined at 45° to the vertical†,

(*c*) section consisting of two straight lines inclined at 60° to the vertical‡.

For basins, the only known cases are basins with a rectangular or circular cylindrical boundary and a plane bottom§

The usual method aims at guessing a solution of the equation of continuity, which also satisfies the condition at the free surface and the fixed boundaries. In the present paper, a more general method has been used. A solution involving an arbitrary function has been found for the equation which holds at the free surface and the equation of

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* Two papers entitled "Waves in a horizontal cylindrical canal" and "Periodic motion in basins" have been united into a single paper.

† Kelland, *Trans. Royal Soc. Edinburgh*, 14 (1839), 497–546.

‡ Macdonald, *Proc. London Math. Soc.* (1), 25 (1894), 101–111.

§ Lamb, *Hydrodynamics*, fifth edition, 416.

continuity. Particular values have been given to this function, and the corresponding boundaries determined. It seems that the number of cases which can be solved in finite terms is extremely limited.

Part I.

Waves in horizontal cylindrical canals.

1. The problem of deep water waves in a horizontal cylindrical canal may be formulated analytically as follows. Taking the upward vertical as the z-axis, the x-axis horizontal parallel to the generators of the cylinder, the y-axis horizontal perpendicular to the generators, and the origin at the undisturbed free surface, the equation of the section may be written in the form

$$y = \chi(z). \tag{1}$$

If ϕ be the velocity-potential, it satisfies the equation of continuity

$$\nabla^2 \phi = 0 \tag{2}$$

within the mass of the liquid, the equation

$$\frac{\partial \phi}{\partial n} = 0 \tag{3}$$

at the boundary $y = \chi(z)$,

and

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \tag{4}$$

at the free surface $z = 0$.

If we take $\phi = P(y, z) \cos \kappa x \, e^{i\sigma t}$,

these reduce to

$$\frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = \kappa^2 P, \tag{5}$$

$$\frac{\partial P}{\partial y} - \frac{\partial P}{\partial z} \frac{d\chi}{dz} = 0 \quad \text{when} \quad y = \chi(x), \tag{6}$$

and

$$\sigma^2 P = g \frac{\partial P}{\partial z} \quad \text{when} \quad z = 0. \tag{7}$$

The problem is to find the function P for any prescribed section.

2. It is easily verified that

$$P = \left(1+\frac{\sigma^2}{g}z\right) f(y)+z^2F(y, z) \tag{8}$$

is a solution of (7), $f(y)$ and $F(y, z)$ being arbitrary holomorphic functions of their respective variables. We assume that $F(y, z)$ can be expanded in the form

$$F(y, z) = f_0(y)+zf_1(y)+z^2f_2(y)+\ldots. \tag{9}$$

The validity of the expansion will have to be tested for each individual case.

The equation (5) then gives

$$\left.\begin{aligned}
2\,.\,1\,f_0(y) &= \left(\kappa^2-\frac{\partial^2}{\partial y^2}\right) f(y)\\
3\,.\,2\,f_1(y) &= \frac{\sigma^2}{g}\left(\kappa^2-\frac{\partial^2}{\partial y^2}\right) f(y)\\
4\,.\,3\,f_2(y) &= \left(\kappa^2-\frac{\partial^2}{\partial y^2}\right) f_0(y)\\
5\,.\,4\,f_3(y) &= \left(\kappa^2-\frac{\partial^2}{\partial y^2}\right) f_1(y)\\
6\,.\,5\,f_4(y) &= \left(\kappa^2-\frac{\partial^2}{\partial y^2}\right) f_2(y)\\
\ldots \quad &\ldots \quad \ldots
\end{aligned}\right\}. \tag{10}$$

Putting

$$D^2 = \kappa^2-\frac{\partial^2}{\partial y^2},$$

we have

$$\begin{aligned}
P &= \left(1+\frac{\sigma^2}{g}z\right) f(y)+\frac{z^2}{2!}D^2f(y)+\frac{\sigma^2}{g}\frac{z^3}{3!}D^2f(y)\\
&\qquad +\frac{z^4}{4!}D^4f(y)+\frac{\sigma^2}{g}\frac{z^5}{5!}D^4f(y)+\ldots\\
&= \left\{\cosh zD+\frac{\sigma^2}{gD}\sinh zD\right\} f(y).
\end{aligned} \tag{11}$$

This is a functional equation for the determination of P. Since it is an even function of D, the question of the sign of D does not arise. The arbitrary function $f(y)$ for any given section will be determined by (6).

Or, we may find the value of P corresponding to any chosen value of $f(y)$, and then proceed to find the appropriate boundary for which the solution holds.

3. If we put $$f(y) = \exp \kappa m y,$$

then $D^2 = \kappa^2 n^2$, where $m^2 + n^2 = 1$. We have, therefore,

$$P = \left\{\cosh \kappa n z + \frac{\sigma^2}{g\kappa n} \sinh \kappa n z\right\} e^{\kappa m y}, \tag{12}$$

which leads to the solutions of Kelland and Macdonald.

It is easily verified that the appropriate boundary for Kelland's solution

$$P = \cosh \frac{\kappa y}{\sqrt{2}} \cosh \frac{\kappa z}{\sqrt{2}}$$

is any curve given by the equation

$$\frac{dy}{dz} = \frac{\tanh (\kappa y/\sqrt{2})}{\tanh (\kappa z/\sqrt{2})}$$

so that $$\sinh \frac{\kappa y}{\sqrt{2}} = A \sinh \frac{\kappa z}{\sqrt{2}}, \tag{13}$$

the origin being taken, for the moment, at a depth h below the free surface.

The lines $$y = \pm z$$

are obtained as particular solutions when $A = \pm 1$.

4. Putting $$f(y) = y,$$

we see that the corresponding value of P is given by the equation

$$P = y \left\{\cosh \kappa z + \frac{\sigma^2}{g\kappa} \sinh \kappa z\right\}. \tag{14}$$

Substituting in (6), we have

$$\kappa y \frac{dy}{dz} = \frac{\cosh \kappa z + (\sigma^2/g\kappa) \sinh \kappa z}{\sinh \kappa z + (\sigma^2/g\kappa) \cosh \kappa z}, \tag{15}$$

which is to be satisfied when

$$y = \chi(z).$$

We get $$Ae^{\frac{1}{2}\kappa^2 y^2} = \sinh \kappa z + \frac{\sigma^2}{g\kappa} \cosh \kappa z, \tag{16}$$

where A is an arbitrary constant.

If $y = 0$ when $z = -h$, this becomes

$$e^{\frac{1}{2}\kappa^2 y^2} = \frac{\sinh \kappa z + (\sigma^2/g\kappa) \cosh \kappa z}{-\sinh \kappa h + (\sigma^2/g\kappa) \cosh \kappa h}. \tag{17}$$

Therefore in a canal, whose form is given by the equation

$$e^{\frac{1}{2}\kappa^2 y^2} = \frac{\sinh \kappa z + \lambda \cosh \kappa z}{-\sinh \kappa h + \lambda \cosh \kappa h},$$

we can have waves of length $2\pi/\kappa$ (provided that $\tanh \kappa h$ is less than λ), and the speed is given by the relation

$$\lambda = \frac{\sigma^2}{g\kappa}.$$

5. In particular, if $$\sigma^2 = g\kappa,$$

the equation (15) gives the algebraic form

$$y^2 = \frac{2}{\kappa}(z + h),$$

and $$P = ye^{\kappa z}.$$

The wave-form obtained from the real part of ϕ is given by

$$\zeta = -\frac{\sigma}{g} y \cos \kappa x \sin \sigma t. \tag{18}$$

The section of the wave by a plane perpendicular to the generator is, at any instant, a straight line.

We have, therefore, a train of simple harmonic waves of length $2\pi/\kappa$ moving along a parabolic cylindrical canal of latus rectum $2/\kappa$. The speed is given by

$$\sigma^2 = g\kappa.$$

It may be noted that the train of waves is independent of the depth of the canal, and that, unlike the waves in canals with plane walls, waves of only one length are possible in a canal of given parabolic section. Since

the depth h does not appear in equation (18) the boundary may be composed of arcs of two different parabolas with different depths of the vertices.

6. Putting $$f(y) = y^2$$

in equation (11), we easily deduce that

$$P = y^2 \cosh \kappa z - \frac{z}{\kappa} \sinh \kappa z + \frac{\sigma^2}{g\kappa} \left\{ y^2 \sinh \kappa z - \frac{z}{\kappa} \cosh \kappa z + \frac{1}{\kappa^2} \sinh \kappa z \right\}. \quad (19)$$

We have to determine the boundary appropriate to this value of P, and the only type of boundary which is of hydrodynamical interest is one which is everywhere of finite depth.

The equation (6) may be written, in this instance, in the form

$$\frac{dy}{dz} = \frac{2y\{\cosh \kappa z + (\sigma^2/g\kappa) \sinh \kappa z\}}{\kappa y^2 \{\sinh \kappa z + (\sigma^2/g\kappa) \cosh \kappa z\} - z\{\cosh \kappa z + (\sigma^2/g\kappa) \sinh \kappa z\} - (1/\kappa) \sinh \kappa z}. \quad (20)$$

Now Cauchy's theorem states that in the case of the equation

$$\frac{dy}{dx} = f(x, y),$$

if $f(x, y)$ is holomorphic in the domain $|x - x_0| \leqslant r$, $|y - y_0| \leqslant \rho$, a solution of the equation exists, which reduces to y, when x becomes x_0, and is holomorphic about x_0 within a circle of radius at least equal to $r(1 - e^{-(\rho/2Mr)})$, where M is the maximum value of $|f(x, y)|$ within the region.

Taking now the denominator on the right-hand side of equation (20), we see that the expression is positive for all values of y and all negative values of z, the only values of the variables with which we are concerned in the hydrodynamical problem before us. We can, therefore, find y as a holomorphic function of z such that $y = 0$, when $z = -h$.

The method, therefore, gives a unique boundary*, in which waves of length $2\pi/\kappa$ are propagated unchanged in form with speed σ/κ. It has not, however, been found possible to express the solution in finite terms.

* It may be pointed out that the singular point of the integral given by simultaneous values $y = 0$ when $z = 0$ is excluded from consideration, being irrelevant to the special problem. It gives a canal of zero depth.

7. If we put

$$f(y) = y^3,$$

we get

$$P = y^3 \cosh \kappa z - \frac{3yz}{\kappa} \sinh \kappa z + \frac{\sigma^2}{g\kappa} \left\{ y^3 \sinh \kappa z - \frac{3yz}{\kappa} \cosh \kappa z + \frac{3y}{\kappa^2} \sinh \kappa z \right\}. \tag{21}$$

The boundary must satisfy the relation

$$y \frac{dy}{dz} = \frac{\begin{array}{r} 3y^2 \cosh \kappa z - (3z/\kappa) \sinh \kappa z \\ + (\sigma^2/g\kappa) \{ 3y^2 \sinh \kappa z - (3z/\kappa) \cosh \kappa z + (3/\kappa^2) \sinh \kappa z \} \end{array}}{\begin{array}{r} \kappa y^2 \sinh \kappa z - (3/\kappa) \{ \sinh \kappa z + \kappa z \cosh \kappa z \} \\ + (\sigma^2/g\kappa) \{ \kappa y^2 \cosh \kappa z - 3z \sinh \kappa z \} \end{array}}. \tag{22}$$

The denominator of the expression on the right-hand side can be put in the form

$$\kappa y^2 \left(\sinh \kappa z + \frac{\sigma^2}{g\kappa} \cosh \kappa z \right) - 3z \left(\cosh \kappa z + \frac{\sigma^2}{g\kappa} \sinh \kappa z \right) - \frac{3}{\kappa} \sinh \kappa z.$$

If now the expressions

$$\frac{\sigma^2}{g\kappa} \cosh \kappa z + \sinh \kappa z, \quad \cosh \kappa z + \frac{\sigma^2}{g\kappa} \sinh \kappa z$$

be both positive for negative values of z from 0 to $-h$, so that

$$\coth \kappa h > \frac{\sigma^2}{g\kappa} > \tanh \kappa h,$$

the denominator is always positive for negative values of z from $z = 0$ to $z = -h$. Cauchy's theorem, therefore, applies, and the boundary can be expressed in the form

$$y^2 = \chi(z),$$

where $\chi(z)$ is a holomorphic function of z, which vanishes when

$$z = -h.$$

8. We can similarly deduce values of P for a general polynomial value of $f(y)$, but the expression seems to be too complicated to admit of any interpretation of physical interest.

Part II.

Periodic motion in basins.

1. The origin of coordinates being taken at the undisturbed surface and the z-axis vertically upwards, the velocity-potential will satisfy the three equations, (2), (3), (4). A time-factor $\exp i\sigma t$ being understood, equation (4) may be written in the form

$$\sigma^2\phi = g\frac{\partial\phi}{\partial z}. \tag{23}$$

An expression for ϕ can be written in the form

$$\phi = \left(1+\frac{\sigma^2}{g}z\right) f(x, y)+z^2F(x, y, z), \tag{24}$$

and this satisfies the equation (4). Assuming that the function F can be expanded in ascending powers of z, we may write

$$\phi = \left(1+\frac{\sigma^2}{g}z\right) f(x, y)+z^2\{f_0(x, y)+zf_1(x, y)+z^2f_2(x, y)+\ldots\}. \tag{25}$$

Substituting, in (2), we get

$$\left(1+\frac{\sigma^2}{g}z\right)\nabla_1^2 f(x, y)+z^2[\nabla_1^2 f_0+z\nabla_1^2 f_1+z^2\nabla_1^2 f_2+\ldots] \\ +[2.1f_0+3.2zf_1+4.3z^2f_2+\ldots] = 0, \tag{26}$$

where $$\nabla_1^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

When we equate the various powers of z to zero, this gives

$$\left.\begin{aligned} \nabla_1^2 f+2.1f_0 &= 0 \\ \frac{\sigma^2}{g}\nabla_1^2 f+3.2f_1 &= 0 \\ \nabla_1^2 f_0+4.3f_2 &= 0 \\ \nabla_1^2 f_1+5.4f_2 &= 0 \\ \ldots \quad &\ldots \quad \ldots \end{aligned}\right\} \tag{27}$$

We have, therefore,

$$\phi = \left[\cos z\nabla_1 + \frac{\sigma^2}{g\nabla_1}\sin z\nabla_1\right] f(x, y). \tag{28}$$

2. The function $f(x, y)$ is still arbitrary. Taking the simple value

$$f(x, y) = x,$$

we have

$$\phi = \left(1+\frac{\sigma^2}{g}z\right)x. \tag{29}$$

It now remains to determine the boundary of the surface within which this value of ϕ holds. If the boundary be given by

$$z = \chi(x, y), \tag{30}$$

we have the equation

$$\frac{\sigma^2}{g}x = \left(1+\frac{\sigma^2}{g}z\right)\frac{\partial z}{\partial x}, \tag{31}$$

which leads to the solution

$$\left(z+\frac{g}{\sigma^2}\right)^2 = x^2+\psi(y), \tag{32}$$

where ψ is an arbitrary function of y.

This includes, as particular solutions,

(i) a hyperbolic paraboloid,

(ii) a cone of the second degree,

(iii) a cone of revolution having its vertex at the point $(0, 0, -g/\sigma^2)$, its axis vertical and its semi-vertical angle 45°.

The corresponding elevation at any point ζ is given by the equation

$$\zeta = \frac{i\sigma x}{g}e^{i\sigma t}. \tag{33}$$

3. Another value of $f(x, y)$ which leads to a simple expression for ϕ is

$$f(x, y) = \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}, \tag{34}$$

which gives

$$\nabla_1^2 f(x, y) = -\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \pi^2 f(x, y) = -\kappa^2 f(x, y), \tag{35}$$

where $$\kappa^2 = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \pi^2.$$

We have, therefore,

$$\phi = \left(\cosh \kappa z + \frac{\sigma^2}{g\kappa} \sinh \kappa z\right) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}. \tag{36}$$

If the basin has vertical walls,

$$\frac{\partial \phi}{\partial n} = 0$$

when $$x = a \quad \text{or} \quad y = b,$$

provided that m and n are positive integers. We thus get the motion inside a rectangular basin of sides a and b and depth h, the speed being given by the equation

$$\frac{\sigma^2}{g\kappa} = \tanh \kappa h. \tag{37}$$

In the general case, the boundary corresponding to the above value of ϕ is

$$\kappa \left\{\sinh \kappa z + \frac{\sigma^2}{g\kappa} \cosh \kappa z\right\} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$= -\left\{\cosh \kappa z + \frac{\sigma^2}{g\kappa} \sinh \kappa z\right\}$$

$$\times \left\{\frac{m\pi}{a} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \frac{\partial z}{\partial x} + \frac{n\pi}{b} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \frac{\partial z}{\partial y}\right\}. \tag{38}$$

This is a partial differential equation of Lagrange's type. The subsidiary

equations required by his method of solution are

$$\frac{dz}{\kappa\left\{\dfrac{\sinh\kappa z+(\sigma^2/g\kappa)\cosh\kappa z}{\cosh\kappa z+(\sigma^2/g\kappa)\sinh\kappa z}\right\}}=\frac{-dx}{\dfrac{m\pi}{a}\tan\dfrac{m\pi x}{a}}=\frac{-dy}{\dfrac{n\pi}{b}\tan\dfrac{n\pi y}{b}} \tag{39}$$

Two independent integrals are

$$\left.\begin{aligned}\frac{1}{\kappa^2}\log\left(\sinh\kappa z+\frac{\sigma^2}{g\kappa}\cosh\kappa z\right)&=\left(\frac{a}{m\pi}\right)^2\log\operatorname{cosec}\frac{m\pi x}{a}+A,\\ \frac{1}{\kappa^2}\log\left(\sinh\kappa z+\frac{\sigma^2}{g\kappa}\cosh\kappa z\right)&=\left(\frac{b}{n\pi}\right)^2\log\operatorname{cosec}\frac{n\pi y}{b}+B,\end{aligned}\right\} \tag{40}$$

where A and B are constants.

The appropriate boundary is, therefore,

$$\frac{\{\sinh\kappa z+(\sigma^2/g\kappa)\cosh\kappa z\}^{1/\kappa^2}}{\{\operatorname{cosec}(m\pi x/a)\}^{a^2/(m^2\pi^2)}}=\psi\left[\frac{\{\operatorname{cosec}(n\pi y/b)\}^{b^2/(n^2\pi^2)}}{\{\operatorname{cosec}(m\pi x/a)\}^{a^2/(m^2\pi^2)}}\right]. \tag{41}$$

4. If we take $$\frac{m}{a}=\frac{n}{b}=\lambda,$$

it is simpler to write

$$\phi=\left[\cosh\kappa z+\frac{\sigma^2}{g\kappa}\sinh\kappa z\right]\cos\lambda\pi(x+y), \tag{42}$$

where $$\kappa^2=2\lambda^2\pi^2.$$

The boundary is given by the equation

$$\frac{\kappa\{\sinh\kappa z+(\sigma^2/g\kappa)\cosh\kappa z\}}{\cosh\kappa z+(\sigma^2/g\kappa)\sinh\kappa z}+\lambda\pi\tan\lambda\pi(x+y)\left(\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\right)=0. \tag{43}$$

The subsidiary equations are

$$\frac{dz}{\dfrac{\kappa\{\sinh\kappa z+(\sigma^2/g\kappa)\cosh\kappa z\}}{\cosh\kappa z+(\sigma^2/g\kappa)\sinh\kappa z}}=\frac{-dx}{\lambda\pi\tan\lambda\pi(x+y)}=\frac{-dy}{\lambda\pi\tan\lambda\pi(x+y)}. \tag{44}$$

One integral is $$x-y=A,$$

and a second is $$\left(\sinh\kappa z+\frac{\sigma^2}{g\kappa}\cosh\kappa z\right)\sin\lambda\pi(x+y)=B. \tag{45}$$

The boundary is, therefore,

$$\left(\sinh \kappa z + \frac{\sigma^2}{g\kappa} \cosh \kappa z\right) \sin \frac{\kappa}{\sqrt{2}} (x+y) = \psi(x-y), \tag{46}$$

where ψ is an arbitrary function.

5. For motion symmetrical about the z-axis, we have

$$\phi = \left\{\cos z\nabla_1 + \frac{\sigma^2}{g\nabla_1} \sin z\nabla_1\right\} f(\varpi), \tag{47}$$

where

$$\nabla_1^2 = \frac{d^2}{d\varpi^2} + \frac{1}{\varpi} \frac{d}{d\varpi}. \tag{48}$$

If

$$f(\varpi) = \varpi^n, \quad \nabla_1^2 f(\varpi) = n^2 \varpi^{n-2}. \tag{49}$$

If n be an odd integer or a fraction, ϕ involves negative powers of ϖ, and there is, therefore, a line of singularities at the axis. If n be an even integer, ϕ is a polynomial in z and ϖ.

Putting $n = 2$, we have the simplest value of ϕ,

$$\phi = (\varpi^2 - 2z^2) + \frac{\sigma^2}{g} \left(z\varpi^2 - \frac{2z^3}{3}\right). \tag{50}$$

The corresponding boundary is given by the equation

$$\frac{d\varpi}{dz} = \frac{2\varpi\{1+(\sigma^2 z/g)\}}{-4z + \{\sigma^2(\varpi^2 - 2z\varpi)/g\}} \tag{51}$$

The initial condition may be taken to be

$$z = -h \quad \text{when} \quad \varpi = 0.$$

These simultaneous values keep the denominator of the right-hand expression finite and different from zero. Cauchy's solution, therefore, is $\varpi = 0$.

The implication is that there is no regular function z of ϖ which assumes the value $-h$ when z becomes 0. A solution of equation (51) in finite terms has not been found practicable.

6. Another value of $f(\varpi)$ which leads to a solution in finite terms is obtained when

$$\nabla_1^2 f(\varpi) = -\kappa^2 f(\varpi),$$

that is
$$\frac{d^2 f}{d\varpi^2} + \frac{1}{\varpi}\frac{df}{d\varpi} + \kappa^2 f = 0, \tag{52}$$

which gives
$$f(\varpi) = J_0(\kappa\varpi). \tag{53}$$

This leads to
$$\phi = \left[\cosh \kappa z + \frac{\sigma^2}{g\kappa} \sinh \kappa z\right] J_0(\kappa\varpi). \tag{54}$$

The appropriate boundary is given by the equation

$$\frac{d\varpi}{dz} = \frac{\{\cosh \kappa z + (\sigma^2/g\kappa) \sinh \kappa z\} J_0'(\kappa\varpi)}{\{\sinh \kappa z + (\sigma^2/g\kappa) \cosh \kappa z\} J_0(\kappa\varpi)}. \tag{55}$$

A particular solution of this equation is given by a circular cylinder of uniform depth. The radius of the cylinder is given by the equation

$$J_0'(\kappa a) = 0 \tag{56}$$

and the speed by
$$\frac{\sigma^2}{g\kappa} = \tanh \kappa h. \tag{57}$$

Equation (55) gives

$$\frac{\cosh \kappa z + (\sigma^2/g\kappa) \sinh \kappa z}{\sinh \kappa z + (\sigma^2/g\kappa) \cosh \kappa z}\, dz = \frac{J_0(\kappa\varpi)}{J_0'(\kappa\varpi)}\, d\varpi. \tag{58}$$

Now
$$J_0(\kappa\varpi) = J_1'(\kappa\varpi) + \frac{1}{\kappa\varpi} J_1(\kappa\varpi)$$

and
$$J_0'(\kappa\varpi) = -J_1(\kappa\varpi) \tag{59}$$

We have, therefore,

$$\frac{\cosh \kappa z + (\sigma^2/g\kappa) \sinh \kappa z}{\sinh \kappa z + (\sigma^2/g\kappa) \cosh \kappa z}\, dz = -\left\{\frac{1}{\kappa\varpi} + \frac{J_1'(\kappa\varpi)}{J_1(\kappa\varpi)}\right\} d\varpi, \tag{60}$$

which gives
$$\kappa\varpi J_1(\kappa\varpi)\left\{\sinh \kappa z + \frac{\sigma^2}{g\kappa} \cosh \kappa z\right\} = A. \tag{61}$$

This gives the form of the boundary appropriate to the value of ϕ given by the equation (54). It is obvious that z cannot be finite when ϖ is zero. The basin is, therefore, of infinite depth.

7. If we take the equation (52) in the form

$$\frac{d^2f}{d\varpi^2}+\frac{1}{\varpi}\frac{df}{d\varpi}-\kappa^2 f=0, \tag{62}$$

we get Bessel's functions with imaginary argument. We have

$$\phi=\left[\cos\kappa z+\frac{\sigma^2}{g\kappa}\sin\kappa z\right]I_0(\kappa\varpi) \tag{63}$$

and the corresponding boundary is given by

$$\kappa\varpi I_1(\kappa\varpi)\left\{\sin\kappa z-\frac{\sigma^2}{g\kappa}\cos\kappa z\right\}=B. \tag{64}$$

This basin is also not of finite depth.

Considering the complicated nature of the equation to the boundary which arises from even simple expressions for f, it is not surprising that the motion has been known for only two simple cases.

THE LONDON MATHEMATICAL SOCIETY

BALANCE SHEET, 30th October, 1926.

Liabilities	£	s.	d.	£	s.	d.
Lieut.-Col. Campbell's Fund				499	7	5
Sir A. B. Kempe's Fund				200	0	0
Donations				353	7	0
Life Compositions Fund				2995	16	6
Reserve Fund in 1925	460	3	3			
Depreciation of Investments	103	6	1			
Reserve Fund				356	17	2
Sundry Creditors				19	13	0
Revenue Account balance in hand				566	9	1
				£4991	10	2

Assets.	£	s.	d.
London, Midland & Scottish Railway £385 4 per Cent. Guaranteed Stock at 77 ...	296	9	0
India Stock £700. 7*s.* 1*d.* 3 per Cent. at 59½	416	14	3
Funding Loan £3000 4 per Cent. at 84⅝ ...	2538	15	0
Conversion Loan £1700 3½ per Cent. at 74⅝	1268	12	6
The above-mentioned stocks are valued at the present market price.			
Deposit at Bank	400	0	0
Cash at Bank	70	19	5
	£4991	10	2

REVENUE ACCOUNT for the year 1st November, 1925, to 30th October, 1926.

	Year 1924–25.							Year 1924–25.					
	£	s.	d.	£	s.	d.		£	s.	d.	£	s.	d.
Balance from last year	316	17	7	*137*	*0*	*4*	Printing and Circulating *Proceedings*, &c.	1195	12	0	*874*	*8*	*3*
Income tax returned	20	8	8	*23*	*0*	*3*	Purchases and Binding for Library ...	31	5	0	*44*	*19*	*0*
Dividends on—							Rent	30	0	0	*30*	*0*	*0*
Lord Rayleigh's Fund ...	42	9	8	*38*	*18*	*4*	Postages and Sundries	6	3	5	*4*	*18*	*3*
The Society's Investments	189	3	10	*200*	*18*	*5*	Anniversary Celebration...		—		*4*	*18*	*10*
Entrance Fees	39	18	0	*27*	*6*	*0*	Reserve for printing		—		*300*	*0*	*0*
Annual Subscriptions	441	2	5	*429*	*13*	*8*	Balance in hand on 30th October, 1926...	566	9	1	*316*	*17*	*7*
Sale of *Proceedings*, &c.	464	4	4	*391*	*4*	*11*							
Advertisements revenue	15	5	0	*28*	*0*	*0*							
Grant from Royal Society		—		*300*	*0*	*0*							
Reserve for current year's printing ...	300	0	0		—								
	£1829	9	6	*1576*	*1*	*11*		£1829	9	6	*1576*	*1*	*11*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.		£	s.	d.
Balance in hand 31st October, 1925 ...	6	1	10	Cash at Bank 30th October, 1926	9	2	3
Dividends received .. .	2	8	0				
Income tax returned		12	5				
	£9	2	3		£9	2	3

STATEMENT OF TRUST FUNDS, 30th October, 1926.

Lord Rayleigh's Fund consists of the following Investments:—

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.
£229. 13*s.* 4*d.* 4 per Cent. Funding Stock.

This Fund is held in trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 5 per Cent. Preference Stock.

11th November, 1926.

Signed on behalf of the Council:—

A. L. DIXON, *President*

A. E. WESTERN, *Treasurer.*

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society

ALLAN J. C. CUNNINGHAM, Lt.-Col. R.E., *Auditor.*

24th November, 1926.

CONTRAVARIANT ENVELOPES OF THE CUBIC SURFACE

By WILLIAM P. MILNE.

[Received 15 May, 1926.—Read 10 June, 1926.]

1. *Introduction.*

In the first number of the *Journal of the London Mathematical Society*, I published some investigations into the nature and properties of the harmonic and the equianharmonic envelopes of a cubic surface. The chief results obtained were as follows :—

The 24 *tangent planes to a cubic surface at the* 24 *parabolic points on the* 12 *lines of a double-six of the surface touch the same quadric, and there are* 36 *such quadrics corresponding to the* 36 *double-sixes of the surface. These* 36 *quadrics are contact-quadrics of the harmonic envelope, i.e. they each touch the harmonic envelope at every point where they meet it.*

The method of establishing these results consisted of somewhat detailed algebra, which threw little light on the underlying reasons for the various properties which presented themselves. These defects have been removed in the present paper, where the methods of synthetic geometry have been employed and much new matter has been brought to light.

2. *Mapping of the cubic surface on a plane and on a quadric.*

Let us take a given quadric S and choose upon it five fixed points D, D_3, D_4, D_5, D_6. Any quadric S' through these five fixed points will cut S in a quadri-quadric curve, which can be projected from D on to a fixed plane π_0 into a cubic U through the six points A_1, A_2, A_3, A_4, A_5, A_6, where A_1, A_2 are the points in which the generators through D to S cut

π_0, and A_3, A_4, A_5, A_6 are the projections of D_3, D_4, D_5, D_6 respectively on π_0. If we then express the web of cubics U through the six points A_1, A_2, A_3, A_4, A_5, A_6 by means of the coordinates ξ, η, ζ in the plane π_0, and put $x = U_1$, $y = U_2$, $z = U_3$, $t = U_4$, we obtain Clebsch's method of mapping the general cubic surface on the plane π_0. Thus a (1, 1)-correspondence is obtained between the points of the cubic surface and those of the plane π_0, while to every point P on the plane π_0 corresponds the point of S at which it is cut by the line DP.

Let now the plane $lx+my+nz+pt=0$ cut the cubic surface in a curve U'. The corresponding curve in π_0 will be

$$U \equiv lU_1+mU_2+nU_3+pU_4 = 0.$$

Since between the curves U and U' a (1, 1)-correspondence exists, they may be taken to be identical projectively, in the sense that they have the same absolute invariant and hence must be expressible by the same elliptic functions. Corresponding points on U and U' may be regarded as denoted by elliptic parameters whose sum or difference is constant, and hence points on U whose elliptic parameters differ by k will correspond to points on U' whose elliptic parameters also differ by k. The quadri-quadric SS', of which U is the projection, will possess corresponding properties. Taking the condition that $\lambda S+S'=0$ shall be a cone in Salmon's form, we obtain the equation

$$\Delta\lambda^4+\Theta\lambda^3+\Phi\lambda^2+\Theta'\lambda+\Delta' = 0, \tag{1}$$

which we shall find convenient to write

$$a\lambda^4+4b\lambda^3+6c\lambda^2+4d\lambda+e = 0, \tag{2}$$

so as to obtain the algebraic forms and expressions used in Burnside and Panton's *Theory of equations* in connexion with the biquadratic equation. It is possible to transform the equation (1) into the equation (2) immediately by means of the formulae

$$\frac{a}{\Delta} = \frac{4b}{\Theta} = \frac{6c}{\Phi} = \frac{4d}{\Theta'} = \frac{e}{\Delta'}. \tag{3}$$

In particular, the condition that SS' shall be equianharmonic is given by

$$I \equiv ae-4bd+3c^2 = 0, \tag{4}$$

i.e.

$$12\Delta\Delta'-3\Theta\Theta'+\Phi^2 = 0. \tag{5}$$

Similarly, the condition that SS' shall be harmonic is

$$J \equiv ace+2bcd-c^3-ad^2-eb^2 = 0, \tag{6}$$

i.e. $$72\Delta\Delta'\Phi+9\Theta\Theta'\Phi-2\Phi^3-27\Delta\Theta'^2-27\Delta'\Theta^2 = 0. \tag{7}$$

The equations (4) and (6) involve l, m, n, p to degrees 4 and 6 respectively, and hence give the tangential equations to the equianharmonic and the harmonic envelopes, S_4 and T_6, respectively. The tangential equation to the cubic surface is $I^3 = 27J^2$.

It has also to be noted that points of S are projected into points of π_0, D being the vertex of projection. A cubic U is the projection of the quadri-quadric SS', where S is given, but S' can be equally well replaced by $\lambda S+S'$. Thus, since the envelopes of the cubic surface with which we are concerned are represented by invariant relationships between S and S', it is plain that we need consider only those which are unaltered if we replace the coefficients of S' by those of $\lambda S+S'$. Taking, therefore, S and S' in the canonical forms,

$$S \equiv ax^2+\beta y^2+\gamma z^2+\delta t^2 = 0,$$

$$S' \equiv x^2+y^2+z^2+t^2 = 0,$$

we easily find that there are unique invariants of the second and third degrees respectively in the coefficients of (1), which are combinant in character with respect to the coefficients of S', viz. :

$$3\Theta^2-8\Delta\Phi = 0, \tag{8}$$

$$8\Delta^2\Theta'-4\Delta\Phi\Theta+\Theta^3 = 0. \tag{9}$$

Expressing these in the coefficients of (2), we obtain

$$H \equiv ac-b^2 = 0, \tag{10}$$

$$G \equiv a^2d-3abc+2b^3 = 0, \tag{11}$$

which are each well known expressions in the theory of the biquadratic equation*.

The quadrics S and S' will intersect in a quadri-quadric having a cusp if $I = J = 0$, and hence the invariant equations (5) and (7) give the conditions that planes shall touch the cubic surface at points lying

* Cf. H. W. Turnbull, *Proc. Royal Soc. Edinburgh*, 54 (1923–24), 23–50.

on its parabolic curve. Finally, the quadri-quadric SS' will degenerate into a twisted cubic and one of its chords if the equation (2) possesses two distinct pairs of equal roots, *i.e.* if $G=0$ and $a^2I-12H^2=0$. Such a chord may be (i) a generator of S through D; (ii) a generator of S not passing through one of the points D_3, D_4, D_5, D_6; (iii) a generator of S passing through one of the points D_3, D_4, D_5, D_6. In case (i) the quadri-quadric SS' projects on π_0 into a cubic U having a node at either A_1 or A_2 and passing through the other five base-points. In case (ii) U consists of a line through A_1 or A_2 and the conic through the other five base-points; while in case (iii) U consists of a line joining A_1 or A_2 to any of the other four base-points, together with a conic through the remaining four base-points. We are thus led to consider the double-six

$$\begin{array}{cccccc} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{array}$$

on the cubic surface, and the conics in which planes through the above twelve lines intersect the cubic surface. It is easy to show that, if two quadrics, S and S', intersect each other in a common generator and a twisted cubic touching that generator, the invariant relation

$$3\Theta^2-8\Delta\Phi=0,$$

i.e. $H\equiv ac-b^2=0$, is satisfied. The following result is immediately obtained from the above discussion, since $3\Theta^2-8\Delta\Phi$ involves l, m, n, p to the second degree :—

The 24 planes, containing the lines of a double-six and each intersecting the cubic surface in a residual conic touching the corresponding line of the surface, touch a quadric.

We shall develop this further later on.

3. *Properties of the cubic curve.*

The following properties of the cubic curve must now be established, since they will frequently be required.

We define a triad of points ABC as being apolar to a cubic curve U, if the bipolar (or mixed polar) line of any two of the points with respect to U passes through the third point. If ABC be three points on a cubic curve U apolar to the member $\lambda TU+6\mu SH$ of the Hessian

pencil (H denoting the Hessian of U, and S and T the invariants of the fourth and sixth degrees in the coefficients of U respectively, as given in Salmon's *Higher plane curves*), and if ABC be projected through any arbitrary point of the cubic curve on to the cubic curve again, thus obtaining $A'B'C'$, the triad $A'B'C'$ will also be apolar to* $\lambda TU+6\mu SH$. Let AB meet the cubic curve in O, and let OC meet the curve in C'. It follows from the above theorem that, if the bipolar line of B, C with respect to $\lambda TU+6\mu SH$ passes through A, the bipolar line of A, C' will pass through B, since, if ABC be a triad apolar to $\lambda TU+6\mu SH$, BAC' will also be a triad apolar to $\lambda TU+6\mu SH$. Let now C, C' move up to coincidence with B, A respectively. The bipolar lines of A, C' and B, C then become the polar lines of A and B with respect to $\lambda TU+6\mu SH$ respectively. Hence we obtain the following result :—

If A and B be two points on a cubic curve U such that the polar line of A with respect to the curve of the Hessian pencil $\lambda TU+6\mu SH$ passes through B, the polar line of B with respect to $\lambda TU+6\mu SH$ will pass through A.

We shall designate A, B an "apolar pair" of points on U with respect to $\lambda TU+6\mu SH$.

Let us now consider the general relationship of all "apolar pairs" of points on U with respect to $\lambda TU+6\mu SH$. Let A, B and C, D be two such "apolar pairs". Let AC meet the cubic again at T, and let TB meet the curve again at P. Then, by what we have just proved, CP is an "apolar pair" with respect to $\lambda TU+6\mu SH$. The polar line of C with respect to $\lambda TU+6\mu SH$ passes, therefore, through P. But the polar line of C with respect to $\lambda TU+6\mu SH$ cuts the cubic curve at two points other than the tangential point C_0 of C. Hence P coincides with D or with the third point at which C_0D cuts U. If we use the same letters to denote the points of a cubic curve and also their elliptic parameters, we have

$$A+C+T\equiv 0,$$

$$B+P+T\equiv 0,$$

i.e.

$$A-B\equiv P-C.$$

If, therefore, P coincides with D, we have

$$A-B\equiv D-C.$$

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* W. P. Milne and D. G. Taylor, *Proc. London Math. Soc.* (2), 18 (1920), 375–384.

If P does not coincide with D, we have

$$P+D+C_0 \equiv 0,$$

where $$2C+C_0 \equiv 0,$$

i.e. $$P+D \equiv 2C,$$

i.e. $$P-C \equiv C-D.$$

It follows in this case that

$$A-B \equiv C-D.$$

We have, therefore, the following result:

If A, B and C, D denote the elliptic parameters of two "apolar pairs" on the cubic curve U with respect to the same member of the Hessian pencil $\lambda TU+6\mu SH$, $C-D \equiv \pm(A-B)$. Hence any two "apolar pairs" with respect to the same member of the Hessian pencil can be derived from one another by projection through the same point of the cubic curve on to the cubic curve again.

We proceed now to find the number of "apolar pairs", the line joining which passes through a given point L of the cubic curve U. Let P, Q be the elliptic parameters of such an "apolar pair" P, Q, and let $K_{\lambda,\mu}$ denote the difference between P and Q corresponding to the curve $\lambda TU+6\mu SH$. We have

$$P+Q+L \equiv 0,$$

$$P-Q \equiv K_{\lambda,\mu},$$

i.e. $$2P+L \equiv K_{\lambda,\mu},$$

$$2Q+L \equiv -K_{\lambda,\mu},$$

$$2P+2Q+2L \equiv 0.$$

Let P_0, Q_0, L_0 denote the tangential points of P, Q, L respectively. Hence

$$-P_0+L \equiv K_{\lambda,\mu}, \tag{12}$$

$$-Q_0+L \equiv -K_{\lambda,\mu}, \tag{13}$$

$$P_0+Q_0+L_0 \equiv 0. \tag{14}$$

It follows from (12) and (13) that the line P_0Q_0 is the polar line of L with respect to $\lambda TU+6\mu SH$.

Through any given point L of a cubic curve U pass four lines, each of which cuts the cubic curve in an "apolar pair" with respect to $\lambda TU+6\mu SH$, and each of the four lines has the same satellite line, viz., the polar line of L with respect to $\lambda TU+6\mu SH$.

Plainly a (1, 4)-correspondence exists between the lines $P_0Q_0L_0$ and PQL, and hence the pencils of four lines having the lines $P_0Q_0L_0$ as their common satellite line belong to an involution. If we take the three positions for $P_0Q_0L_0$ (other than LL_0), in which P_0, Q_0 coincide, we obtain three line pairs, each of which, taken twice over, is a member of the above involution of four lines. Thus the above involution is of the type $\rho Jf+6\sigma Ih=0$, where f is a binary quartic, h its Hessian, I and J the invariants of degrees two and three respectively in the coefficients of f. The following result is at once evident:

The pencils of four lines through L on the cubic curve U, each of which has the same satellite line through L_0, the tangential point of L, form an involution of the type $\rho Jf+6\sigma Ih=0$, where f is a binary quartic, h its Hessian, I and J the invariants of degrees two and three respectively in the coefficients of f. The Jacobian sextic of this involution consists of the three pairs of lines through L whose satellites touch U. The line-pairs separating harmonically the four lines constituting a tetrad of the involution regarded as consisting of two line-pairs is; therefore, one or other of the three line-pairs composing the Jacobian sextic.

We shall for convenience take the position of $P_0Q_0L_0$, where P_0, Q_0 coincide with L, as the basic position of the satellite line, and the pencil of four tangents from L to U as the basic tetrad f of the involution of binary quartics.

The satellite line corresponding to the pencil of four tangents from L is therefore the polar line of L with respect to U (i.e. the tangent at L to U), and the satellite lines of the three members of the involution that degenerate into line-pairs, taken twice over, are the three lines through L_0 that touch U (other than LL_0), i.e. the polar lines of L with respect to those three members of the Hessian pencil $\lambda TU+6\mu SH$ having U as their Hessian. Hence, a pair of points on U, the tangents

at which meet on U, constitute an "apolar pair" with respect to one or other of the three cubics having U as their Hessian.

Consider now the pencil of cubic curves $\lambda TU+6\mu SH$ and the point L. The polar lines of L with respect to all the members of this pencil pass through L_0, and hence the lines through L_0 may be taken as defining uniquely the parameter of the member of the Hessian pencil of which the line through L_0 is the polar line with respect to L. Consider the four cases when $\lambda TU+6\mu SH$ degenerates into three straight lines. The polar lines of L with respect to the four triangles thus obtained form a pencil of lines through L_0, viz., l_1, l_2, l_3, l_4. The parameters of the four triangular members of the Hessian pencil $\lambda TU+6\mu SH$ are given by the equation

$$T^4\lambda^4+24S^3T^2\lambda^2\mu^2+8T^2S^3\lambda\mu^3-48S^6\mu^4=0, \tag{15}$$

and the three members having U as their Hessian by

$$T^2\lambda^3+12S^3\lambda\mu^2+2S^3\mu^3=0, \tag{16}$$

S and T being the usual invariants of the cubic curve U.

The first polar of L_0L with respect to the pencil $[l_1\,l_2\,l_3\,l_4]$ consists of the polar lines of L with respect to the three members of the pencil $\lambda TU+6\mu SH$, of which U is the Hessian. The second polar of L_0L with respect to $[l_1\,l_2\,l_3\,l_4]$ consists of the two satellite lines through L_0 of the two pencils of four lines having L as vertex, of which the four tangents from L to U are the Hessian. The third polar of L_0L with respect to $[l_1\,l_2\,l_3\,l_4]$ is the polar line of L with respect to the Hessian H.

Also taking the involution of the pencils of lines through L in the form $6\lambda Jf+\mu Ih=0$, where f represents the pencil of tangents from L to U, we find that the three members of $6\lambda Jf+\mu Ih$, consisting of line-pairs each taken twice over, are given by

$$864J^2\lambda^3-6I^3\lambda\mu^2-I^3\mu^3=0, \tag{17}$$

I and J being the usual invariants of f.

The equations (16) and (17) will be identical if

$$\frac{27J^2}{I^3}=-\frac{T^2}{64S^3}. \tag{18}$$

The pencil of lines $6\lambda Jf+\mu Ih$ have as their common satellite line the polar line of L with respect to $\lambda TU+6\mu SH=0$.

4. *Properties of the quartic curve.*

We proceed next to deal with certain properties of the quartic curve which will also be required.

Let the quartic curve be regarded as the locus of points whose polar lines with respect to the cubic curve U_0 touch the conic S_0', or as the envelope of the polar conics with respect to U_0 of points on S_0'. Let S_0 be the polo-conic of a line p, *i.e.* the envelope of the polar lines with respect to U_0 of points on p, or the locus of points whose polar conics with respect to U_0 touch p. Then the points in which p cuts the quartic curve are the points whose polar lines with respect to U_0 touch both S_0 and S_0'. Now the quadritangent conics of the above system that touch p are the polar conics of the four points of intersection of S_0 and S_0', and hence the polar lines with respect to U_0 of these four points of contact of the quadritangent conics with p touch S_0 at the four points of intersection of S_0 and S_0'. It is, moreover, a known property that the four points of intersection of two conics S_0 and S_0' and the points of contact of their four common tangents form on either conic two binary quartics of the type $6\lambda J_0 f_0 + \mu I_0 h_0 = 0$, where h_0 is the Hessian of f_0, their Jacobian sextic being the points of intersection of either conic with the sides of their self-conjugate triangle. Hence we have the following result :

Any straight line is touched by four members of each of the 63 *systems of quadritangent conics of a quartic curve, and each such set of four points of contact is a member of the Hessian pencil of the binary quartic formed by the four points of intersection of the line with the quartic curve.*

The following results are at once evident from the invariant theory of two conics S_0 and S_0'.

If S_0' (locus) be apolar to S_0 (envelope), *i.e.* if $\Theta_0 = 0$, the points of contact with S_0 of the common tangents of S_0 and S_0' form the Hessian of the four points of intersection of S_0 and S_0'. Since the coefficients of S_0 involve those of the line of which it is the polar conic to the second degree, the condition $\Theta_0 = 0$ may be regarded as a quartic envelope.

If a straight line cuts a quartic curve in four points which form the Hessian of the four points in which the line is touched by the four quadritangent conics of the same system, the line touches the quartic envelope $\Theta_0 = 0$.

The following results can also be easily obtained in a similar manner :

If a straight line is touched by four quadritangent conics of the same system in four points which form the Hessian of the four points of intersection of the line with the quartic curve, the envelope of the line is the conic $\Theta_0' = 0$.

If a straight line is touched by four quadritangent conics of the same system in four points which are apolar to the four points of intersection the line with the quartic curve, the envelope of the line is Jolliffe's class sextic $\Theta_0\Theta_0' - 9\Delta_0\Delta_0' = 0$*.

If a straight line is touched by four quadritangent conics of the same system in four points whose cross-ratio is equianharmonic, the envelope of the line is the class octavic $\Theta_0^2 - 3\Delta_0\Theta_0' = 0$.

Another property of the conic Θ_0' can easily be established :

If T_1 *and* T_2 *be two quadritangent conics of the same system and* T_{12} *the conic through their points of contact with the quartic, the four common tangents of* T_{12} *and the* Φ*-conic (harmonic envelope) of* T_1 *and* T_2 *all touch the conic* Θ_0'.

If $\Theta_{0,1}'$ *and* $\Theta_{0,2}'$ *be two conics of the above type defined by two different systems of quadritangent conics, their four common tangents are each touched by quadritangent conics of the two systems in the same four points, namely the Hessian points of their points of intersection with the quartic curve.*

5. *Properties of the cubic surface.*

Any quadric containing a twisted cubic on the cubic surface cuts the cubic surface in a residual twisted cubic intersecting the former curve in five points. Any quadric containing a twisted cubic of the surface therefore touches the surface in five points. If, in particular, the quadric be a cone having its vertex on the twisted cubic, it will cut the cubic surface in a further twisted cubic passing through its vertex and intersecting the original twisted cubic in four other points. The cone will therefore touch the cubic surface at these four points, which will lie on the polar quadric of the vertex of the cone. The cone will also touch the enveloping cone of the fourth degree drawn from its vertex to the cubic

* W. P. Milne, *Proc. London Math. Soc.* (2), 24 (1925), 335–338.

surface along the four rays joining the vertex to the above four points of contact. The twisted cubics of the same system on the cubic surface through a point E' upon it, project upon an arbitrary plane π into the quadritangent conics of the quartic curve in which the enveloping cone from E' to the cubic surface cuts π. The cones with vertices at E' which cut π in the contravariant envelopes of § 4, we shall speak of as the Θ_0-cone, Θ_0'-cone, etc., of the point E'.

Consider, again, the double six of § 2, viz. :

$$\begin{array}{cccccc} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{array}$$

A degenerate twisted cubic of the system defined by this double-six consists of the straight lines a_1, b_1, c_{12}. Let π' be any plane cutting the cubic surface in the cubic U', and let E' be any point on U'. Let a_1, b_1, c_{12} cut U' at the points A_1', B_1', C_{12}' respectively. Then any quadric containing a_1, b_1, c_{12} will cut the cubic surface in a twisted cubic of the system residual to $a_1\, b_1\, c_{12}$. The four rays of the enveloping cone from E' to the cubic surface lying in π' are the four tangent lines from E' to U'.

Consider now a conic through $A_1'B_1'C_{12}'E'$. It will cut U' again at two points X', Y', whose join passes through a fixed point F' on U'. It will be evident, therefore, that those four rays of the enveloping cone from F' to the cubic surface which lie in π' are the four rays joining F' to the points of contact of the four twisted cubics of the above system through E' that touch π'. Denoting the points and their elliptic parameters by the same letters, we have :

$$\begin{aligned} A_1'+B_1'+C_{12}'+E'+X'+Y' &\equiv 0, \\ A_2'+B_1'+C_{12}' &\equiv 0, \\ F'+X'+Y' &\equiv 0, \end{aligned}$$

whence

$$A_1'-A_2' \equiv F'-E'.$$

Thus the difference between the elliptic parameters of the points E' and F' is the same as the difference between the elliptic parameters of the points in which conjugate members of the double-six cut U'.

E' and F' are "apolar points" with respect to the same member of the Hessian pencil of U' as the points in which U' is cut by the conjugate members of the double-six.

F' is usually called the "point opposé" of the pencil of conics through $A_1'B_1'C_{12}'E'$; and it is plain, in consequence, that E' will be the "point opposé" of the pencil of conics through $A_2'B_2'C_{12}'F'$. Also, the points of contact of the tangents from F' to U' project from E' into the points of contact of the four quadritangent conics of the given system that touch the line of intersection of π' with the plane of projection π. It follows from §§ 3, 4 that *if* $f'=0$ *be the four points of intersection of the line* $\pi\pi'$ *with the quartic curve, and* $6\lambda' J' f'+\mu' I' h'$ *be the four points of contact of the quadritangent conics of the same system with the line* $\pi\pi'$, *then* E', F' *must be an "apolar pair" of points with respect to*

$$\lambda' T' U'+6\mu' S' H'=0$$

where

$$\frac{27J'^2}{I'^3}=-\frac{T'^2}{64S'^3}.$$

6. *The cubic envelope* G.

The geometrical interpretation of the vanishing of G, viz.,

$$8\Delta^2\Theta'-4\Delta\Phi\Theta+\Theta^3=0,$$

is given in Chap. IX of Salmon's *Geometry of three dimensions* as being the condition that two quadrics S, S' shall be so related that tetrahedra can be inscribed in S' having two pairs of opposite edges on the surface of S. If DA_1, DA_2 cut S' in P and Q respectively, it follows that the generators of S through P and Q, other than DA_1 and DA_2, project on π_0 into the tangents at A_1 and A_2 to U, and that they intersect on U. Thus the difference between the elliptic parameters of A_1 and A_2 on U is half a period, and hence the same property will hold as between A_1' and A_2' on U', since a $(1,1)$-correspondence exists between U and U', A_1', A_2' being the points where a_1, a_2 meet U'.

The envelope of planes π' *cutting the cubic surface in sections* U', *such that the tangents to* U' *at the pairs of points where the section is met by conjugate pairs of lines of a double-six intersect on* U', *is a cubic envelope* G.

It follows at once that G contains the double-six, since any plane π' through a_1 contains a_1 as part of U', and hence the tangent where a_2 meets the conic residual to a_1 must intersect a_1 in a point of the curve U', of which a_1 is a part.

G is the cubic envelope which contains the double-six.

The characteristic property of G, which Reye* uses as a means of defining it, can now be deduced, namely, "Any plane intersecting two twisted cubics of the same system on a cubic surface in 6 points lying on a conic envelops a surface of the third class containing the double-six, one half of which consists of the 6 common chords of the two given twisted cubics, and the other half of which is the 6 common chords of any two of the system of conjugate twisted cubics". One twisted cubic of the above double-six consists of the straight lines a_1, b_1, c_{12}. Hence two quadrics containing these lines will cut the cubic curve U' in two further triads of points that will lie on a conic if

$$2A_1'+2B_1'+2C_{12}' \equiv 0.$$

Also

$$A_2'+B_1'+C_{12}' \equiv 0,$$

since a_2, b_1, c_{12} are coplanar. Hence $2A_1'-2A_2' \equiv 0$, which is the definition of G found above.

7. *The quadric envelope H.*

We next proceed to find the geometrical interpretation of the vanishing of $H \equiv ac-b^2$, *i.e.* $3\Theta^2-8\Delta\Phi = 0$. The vanishing of $H \equiv ac-b^2$ signifies that $\lambda = \infty$ is a root of the Hessian h of the quartic

$$f \equiv a\lambda^4+4b\lambda^3+6c\lambda^2+4d\lambda+e = 0$$

which gives the four cones of the pencil $\lambda S+S'$; that is, the tangent plane to S at D is a member of the Hessian pencil h of the four planes f each touching the quadri-quadric SS' at D and at one further point. If the tangent line d at D to SS' meets π_0 in L, we see that LA_1A_2 will be a member of the Hessian pencil h of the four tangent lines f drawn from L to U. Hence, by § 3, A_1A_2 must be an "apolar pair" of points with respect to the Hessian of U. It follows that the corresponding pair of points A_1', A_2 on the section U' of the cubic surface corresponding to U must be an "apolar pair" of points with respect to the Hessian of U'. We obtain, therefore, the following result:

The envelope of planes cutting the cubic surface in sections U', such that the pairs of points in which U' is met by the conjugate pairs of lines

* Th. Reye, *Math. Annalen*, 55 (1902), 257–264.

of a double-six are "apolar pairs" of points with respect to the Hessian of U', is a quadric envelope H.

Consider now a section U' consisting of the line a_1 and a conic. The polar line of any point on the conic with respect to this degenerate cubic U' will be the line a_1 if the conic touch a_1.

The 24 planes through the 12 lines of a double-six that cut the cubic surface in conics touching the respective lines of the double-six all touch the same quadric H.

The above theorem may be otherwise stated :

The tangent planes to a cubic surface at the 24 parabolic points of a double-six all touch the same quadric H.

Consider now any point E' on U', as in § 5. We see that E', F' are an "apolar pair" of points with respect to the Hessian of U' in the case of planes π' that touch the quadric H, and hence that the polar line of E' with respect to the Hessian of U' passes through F'. The pencil of lines joining E' to the points of contact of the tangents from F' to U' is, therefore, the Hessian of the four tangents from E' to U'. Hence, projecting from E' on to the plane π, we see that the points of contact of the four quadritangent conics of the above system that touch the line $\pi\pi'$ are the Hessian of the four points of intersection of $\pi\pi'$ with the quartic.

The enveloping cone from E' to the quadric H is a Θ_0'-cone (see § 4) of the enveloping cone from E' to the cubic surface.

8. *The quartic envelope* $K \equiv a^2I - 12H^2$.

The expression $K \equiv a^2I - 12H^2$ derives its interest from the fact that $G = a^2I - 12H^2 = 0$ are the necessary and sufficient conditions that the biquadratic (2) has two pairs of equal roots. The corresponding quartic envelope is given by

$$16\Delta^2(12\Delta\Delta' - 3\Theta\Theta' + \Phi^2) - (8\Delta\Phi - 3\Theta^2)^2 = 0.$$

The vanishing of $a^2I - 12H^2$ can be interpreted as the condition that $\lambda = \infty$ is a root of $If^2 - 12h^2 = 0$, where f and h have the same interpretations as in § 7, *i.e.* that $\lambda = \infty$ is a member of one or other of the two tetrads of which f is the Hessian.

The vanishing of a^2I-12H^2 signifies geometrically that the tangent plane at D to S is a member of one of the two pencils of four planes having the tangent line d at D to SS' as axis and the four tangent planes to SS' from d as Hessian.

Projecting from D on to the plane π we obtain the following:

LA_1A_2 is a member of the two tetrads of lines having as Hessian the four tangents from L to U.

Since $6\lambda Jf+\mu Ih$ is the pencil of four lines from L whose common satellite line is the polar line of L with respect to $\lambda TU+6\mu SH$, we see, by (18), that A_1, A_2 is an "apolar pair" of points with respect to one or other of the cubic curves $SU^2+9SH^2=0$. Also, if U consists of the straight line a and a conic, any point on the conic will have a as its polar line with respect to either of the cubics SU^2+9H^2.

The envelope of planes meeting the cubic surface in a cubic curve U', such that the pairs of points where U' is met by the conjugate pairs of lines of a double-six are "apolar pairs" of points with respect to one or other of the cubic curves $S'U'^2+9H'^2$, is a quartic envelope containing the double-six, and touching the equianharmonic envelope S_4 at all the points where it is touched by the common tangent planes of S_4 and the quadric H.

Also, since E' and F' possess the property that the pencil of lines joining E' to the points of contact of the tangents from F' to U' have the four tangents from E' to U' as their Hessian, it follows that π' touches the Θ_0-cone of E', as defined in § 4.

The envelope of the tangent planes from E' to the quartic envelope K is the Θ_0-cone (see § 4) of the enveloping cone from E' to the cubic surface.

9. *The sextic envelope* Γ_6.

Consider the expression $\Gamma \equiv 3aJ-2HI$. The equation $\Gamma=0$ expresses the condition that $\lambda=\infty$ is a root of the binary quartic $3Jf-2Ih=0$, and hence by reasoning as before, we deduce that A_1', A_2' constitute an "apolar pair" with respect to $T'U'-24S'H'=0$, which is that cubic of the Hessian pencil of U' apolar to the Cayleyan of U'. If

U' degenerates into the line a_1 and a conic, the polar line of any point on the conic with respect to $T'U'-24S'H'$ is the line a_1. We thus obtain the following result :

The envelope of planes meeting the cubic surface in a cubic curve U' such that the pairs of points in which U' is met by the conjugate pairs of lines of a double-six are "apolar pairs" of points with respect to $T'U'-24S'H'$, i.e. that member of the Hessian pencil of U' which is apolar to the Cayleyan of U', is a sextic envelope Γ_6 containing the double-six and inscribed in the parabolic developable of the cubic surface.

The identity $a^3J \equiv a^2HI-G^2-4H^3$ is proved in treatises on the Theory of Equations, giving :

$$a^2(3aJ-2HI) \equiv H(a^2I-12H^2)-3G^2$$

$$\equiv HK-3G^2.$$

Hence $$a^2\Gamma \equiv HK+3G^2,$$

H is a contact-quadric of the sextic envelope Γ_6, and the planes touching Γ_6 at its points of contact with H touch the cubic envelope G.

K is a contact envelope of class four of Γ_6, and the planes touching Γ_6 at its points of contact with K touch the cubic envelope G.

Since $3Jf-2Ih$ is that tetrad of the Hessian pencil of f which is apolar to f, we see that the lines joining E' to the points of contact of the tangents from F' to the section of the cubic surface U' are apolar to the tangents from E' to U'. The following result is then at once evident from § 4.

The enveloping cone from E' to Γ_6 is the $(\Theta_0\Theta_0'-9\Delta_0\Delta_0')$-cone of Jolliffe from E' to the enveloping quartic cone drawn from E' to the cubic surface.

10. *Case of the Quippian.*

In dealing with the invariant theory of the plane cubic, Cayley obtained two fundamental contravariants, which he called the Pippian and the Quippian respectively. The Pippian is now usually called the Cayleyan, and can be defined as the envelope of a line intersecting a cubic curve in three points constituting a collinear triad apolar to the cubic curve itself. The Quippian is the envelope of a line intersecting

a cubic curve in three points constituting a collinear triad apolar to the Hessian of the cubic curve*. For the curve $x^3+y^3+z^3+6kxyz=0$, its equation is given by Salmon in the form

$$(1-10k^3)(l^3+m^3+n^3)-(30k^2+24k^5)\,lmn=0.$$

Consider now the tetrad belonging to the involution $6\lambda Jf+\mu Ih$ which is apolar to h. Its equation is $I^2f-18Jh=0$. If the tangent plane at D to S belong to this tetrad, the two points A_1, A_2 will constitute an "apolar pair" with respect to $8S^2U+3TH$, which is the curve of the Hessian pencil of U apolar to the Quippian.

The envelope of planes meeting the cubic surface in a cubic curve U', such that the pairs of points in which U' is met by the conjugate pairs of lines of a double-six are "apolar pairs" of points with respect to $8S'^2U'+3T'H'=0$, i.e. the member of the Hessian pencil of U' apolar to the Quippian, is an octavic envelope inscribed in the parabolic developable containing the double-six and having H as a contact quadric. The planes touching this octavic envelope at its points of contact with H touch the equianharmonic envelope S_4. The harmonic envelope T_6 is also a contact envelope of the above octavic, the tangent planes at their points of contact also touching S_4.

Since $I^2f-18Jh$ is that tetrad of the Hessian pencil of f which is apolar to h, we see that *the enveloping cone from E' to the above octavic envelope is the envelope of the planes π' through E' cutting the plane π in straight lines $\pi\pi'$ such that the points of contact of the quadritangent conics of the above system with $\pi\pi'$ are apolar to the Hessian of the points of intersection of $\pi\pi'$ with the quartic curve in which π is cut by the enveloping cone from E' to the cubic surface.*

11. *The harmonic envelope T_6.*

The envelope of planes cutting the cubic surface in a harmonic cubic curve is the class sextic T_6. In virtue of the identity

$$a^3J\equiv a^2HI-G^2-4H^3,$$

we obtain the following result :

* W. P. Milne and D. G. Taylor, *Proc. London Math. Soc.* (2), 20 (1922), 101–106.

The quadric H is a contact quadric of the harmonic envelope T_6, and the tangent planes at the points of contact of H and T_6 touch G. Since an H-quadric presents itself in connexion with each of the 36 double sides of the cubic surface, the harmonic envelope has at least one configuration of 36 contact quadrics.

The common tangent planes of H, G, S_4, *i.e.* the 24 tangent planes to the cubic surface at the parabolic points of the double-six, are plainly singular tangent planes to the harmonic envelope T_6, in virtue of the above identity.

The 54 tangent planes to the cubic surface at the 54 parabolic points on the 27 lines of the surface are singular tangent planes to the harmonic envelope T_6.

12. *Simplification by taking S' (locus) apolar to S (envelope).*

Throughout the foregoing discussion we have considered only the quadri-quadrics SS' passing through the five fixed points D, D_3, D_4, D_5, D_6 on S. It is plain, however, that we may, without loss of generality, take S' (regarded as a locus) to be apolar to S (regarded as an envelope). The invariant formulae will thereby be considerably simplified by putting $\Theta = 0$. Thus the quadric H will correspond to $\Phi = 0$ and the cubic envelope G will correspond to $\Theta' = 0$.

A GENERALIZATION OF AN INTEGRAL EQUATION DUE TO BATEMAN

By C. Fox.

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1. *Introduction.*

In a paper entitled "The inversion of a definite integral"* H. Bateman has proved the formula

$$\int_{-\infty}^{\infty} dx \int_{s}^{\infty} J_0(x-s)\,\frac{J_1(x-t)}{x-t}\,f(t)\,dt = 2f(s), \tag{1.1}$$

which he has stated is equivalent to the pair of formulae

$$\left.\begin{aligned} \phi(x) &= \int_{-\infty}^{\infty} \frac{J_1(x-t)}{x-t}\,f(t)\,dt \\ f(s) &= \tfrac{1}{2}\int_{s}^{\infty} J_0(x-s)\,\phi(x)\,dx \end{aligned}\right\}. \tag{1.2}$$

A sufficient condition for the permissibility of the change of order of integration involved in passing from (1.1) to (1.2) is that $\phi(x)$ should be expressible in the form

$$\phi(x) = \int_{-1}^{+1} e^{ix\mu}\psi(\mu)\,d\mu,$$

where $\psi(\mu)$ is finite and continuous. Bateman's formula has been generalized by Hardy†, who has obtained a formula containing Bessel functions of order ν and $1-\nu$.

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* *Proc. London Math. Soc.* (2), 4 (1906), 484.

† *Ibid.* (2), 7 (1909), 445–472.

The object of this paper is to generalize (1.1) along somewhat different lines. The formulae of which I now give proofs are as follows:

$$\int_s^\infty dx \int_{-\infty}^x \left\{(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t} J_\nu(x-s)+\nu \frac{J_\nu(x-t)}{x-t} J_{\nu-1}(x-s)\right\} f(t)\,dt = 2f(s) \tag{1.3}$$

whenever $\nu > 1$, and

$$\int_s^\infty dx \int_{-\infty}^\infty \left\{(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t} J_\nu(x-s)+\nu \frac{J_\nu(x-t)}{x-t} J_{\nu-1}(x-s)\right\} f(t)\,dt = 2f(s) \tag{1.4}$$

whenever ν is a positive or negative integer or zero.

Hence by subtraction we also get

$$\int_s^\infty dx \int_x^\infty \left\{(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t} J_\nu(x-s)+\nu \frac{J_\nu(x-t)}{x-t} J_{\nu-1}(x-s)\right\} f(t)\,dt = 0 \tag{1.5}$$

whenever ν is a positive or negative integer, the values 0 and 1 being excluded.

When $\nu \to 1$, (1.3) reduces to

$$\lim_{\nu \to 1+0} \int_{-\infty}^x (\nu-1)\frac{J_{\nu-1}(x-t)}{x-t} f(t)\,dt = f(x). \tag{1.6}$$

Equation (1.3) is the direct generalization of (1.1) and can be proved by an extension of Bateman's methods for proving (1.1). It can also be proved by the methods used by Hardy in his paper cited above. Hardy's methods have the twofold advantage of proving (1.3), (1.4), (1.5) and (1.6) all in the same way, and also of giving a set of sufficient conditions for $f(t)$ in a simpler form.

In § 3, I shall give the outline of a proof of (1.3) by Bateman's methods after having used Hardy's methods in §2 to prove the following theorem:

If $f(s)$ is defined by either of the formulae

$$f(s) = \int_a^b \phi(\mu) \begin{matrix}\cos \\ \sin\end{matrix} \mu s\, d\mu,$$

where

$$\int_a^b |\phi(\mu)|\, d\mu$$

exists and $-1 < a < b < 1$, *then* (1.3), (1.4), (1.5), *and* (1.6) *are all valid subject to the restrictions on* ν *following each equation.*

If, in addition, the function $\phi(\mu)$ *is such that* $(1-\mu)^{\alpha}\phi(\mu)$ *is monotonic near* $\mu = 1$ (*where* $0 \leqslant a < \frac{1}{2}$), *then these equations are valid for* $-1 < a < b \leqslant 1$. *A similar additional condition is adequate for the validity of the equations when* $-1 \leqslant a < b < 1$.

Before going on to the proof, it is worth noting that (1.3), and (1.4) can be written so as to give the solutions of two integral equations. Thus (1.3) can be written

$$\left.\begin{aligned} F_\nu(x) &= \nu\int_{-\infty}^{x} \frac{J_\nu(x-t)}{x-t} f(t)\,dt \\ f(s) &= \tfrac{1}{2}\int_{s}^{\infty} \{F_{\nu-1}(x)J_\nu(x-s) + F_\nu(x)J_{\nu-1}(x-s)\}\,dx \end{aligned}\right\} \qquad (1.7)$$

provided that ν is greater than unity, and (1.4) can be written

$$\left.\begin{aligned} F_\nu(x) &= \nu\int_{-\infty}^{\infty} \frac{J_\nu(x-t)}{x-t} f(t)\,dt \\ f(s) &= \tfrac{1}{2}\int_{s}^{\infty} \{F_{\nu-1}(x)J_\nu(x-s) + F_\nu(x)J_{\nu-1}(x-s)\}\,dx \end{aligned}\right\} \qquad (1.8)$$

provided that ν is a positive or negative integer or zero.

2. *Proof of* (1.3) *by Hardy's method.*

Assume first that

$$f(s) = \int_a^b \phi(\mu) \sin \mu s\, d\mu\,;$$

then the absolute convergence of

$$\int_0^\infty \frac{J_{\nu-1}(u)}{u}\,du,$$

combined with the restrictions which have been imposed on $\phi(\mu)$, enables us to change the order of the integrations in the integral

$$\int_a^b \phi(\mu)\,d\mu \int_0^\infty \frac{J_{\nu-1}(u)}{u} \sin \mu(x-u)\,du.$$

Hence

$$\int_{-\infty}^{x} (\nu-1)\,\frac{J_{\nu-1}(x-t)}{x-t}\,dt \int_a^b \phi(\mu)\sin \mu t\, d\mu$$

$$= (\nu-1)\int_a^b \phi(\mu)\,d\mu \int_0^\infty \frac{J_{\nu-1}(u)}{u}\sin \mu(x-u)\,du, \quad (2.1)$$

where, after changing the order of the integrations, t is replaced by $x-u$.

But* $$\lambda\int_0^\infty \frac{J_\lambda(\alpha t)}{t}\begin{matrix}\sin\\ \cos\end{matrix}\beta t\,dt = \begin{matrix}\sin\\ \cos\end{matrix}\{\lambda \sin^{-1}\beta/\alpha\} \quad (\beta<\alpha),$$

and, combining this with (2.1), we get

$$\int_{-\infty}^{x} (\nu-1)\,\frac{J_{\nu-1}(x-t)}{x-t}\,f(t)\,dt = \int_a^b \phi(\mu)\sin\{\mu x-(\nu-1)\sin^{-1}\mu\}\,d\mu. \quad (2.2)$$

From the asymptotic expansion of the Bessel function, it is easily seen that the integral

$$\int_s^\infty J_\nu(x-s)\sin\{\mu x-(\nu-1)\sin^{-1}\mu\}\,dx$$

is uniformly convergent with respect to μ if $\nu>1$ and μ is such that $-1<a\leqslant\mu\leqslant b<1$. Hence, given ϵ, we can find λ_0 so that

$$\left|\int_a^b \phi(\mu)\,d\mu\int_\lambda^\infty J_\nu(x-s)\sin\{\mu x-(\nu-1)\sin^{-1}\mu\}\,dx\right|$$

$$\leqslant \int_a^b |\phi(\mu)|\,d\mu\left|\int_\lambda^\infty J_\nu(x-s)\sin\{\mu x-(\nu-1)\sin^{-1}\mu\}\,dx\right|$$

$$<\epsilon$$

whenever $\lambda>\lambda_0$. It follows that, if λ is sufficiently large, then

$$\left|\int_s^\lambda J_\nu(x-s)\,dx\int_a^b \phi(\mu)\sin\{\mu x-(\nu-1)\sin^{-1}\mu\}\,d\mu\right.$$

$$\left.-\int_a^b \phi(\mu)\,d\mu\int_s^\infty J_\nu(x-s)\sin\{\mu x-(\nu-1)\sin^{-1}\mu\}\,d\mu\right|<\epsilon.$$

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* Watson, *Theory of Bessel functions*, 405 (2) and (3).

Now use (2.2), replace $x-s$ in the second integral inside the modulus sign by v and make $\lambda \to \infty$. The result is

$$\int_s^\infty J_\nu(x-s)\,dx \int_{-\infty}^x (\nu-1)\,\frac{J_{\nu-1}(x-t)}{x-t} f(t)\,dt$$

$$= \int_a^b \phi(\mu)\,d\mu \int_0^\infty J_\nu(v) \sin\{\mu v+\mu s-(\nu-1)\sin^{-1}\mu\}\,dv. \quad (2.3)$$

But it is known that*

$$\int_0^\infty J_\lambda(\alpha t) \begin{matrix}\sin\\ \cos\end{matrix} \beta t\,dt = (\alpha^2-\beta^2)^{-\frac{1}{2}} \begin{matrix}\sin\\ \cos\end{matrix} \{\lambda \sin^{-1}\beta/\alpha\}. \quad (\beta < \alpha) \quad (2.4)$$

Hence

$$\int_s^\infty dx \int_{-\infty}^x (\nu-1)\,\frac{J_{\nu-1}(x-t)}{x-t} J_\nu(x-s) f(t)\,dt$$

$$= \int_a^b \phi(\mu).(1-\mu^2)^{-\frac{1}{2}} \sin\{\mu s+\sin^{-1}\mu\}\,d\mu.$$

In the same way

$$\int_s^\infty dx \int_{-\infty}^x \nu\,\frac{J_\nu(x-t)}{x-t} J_{\nu-1}(x-s) f(t)\,dt = \int_a^b \phi(\mu)(1-\mu^2)^{-\frac{1}{2}} \sin\{\mu s-\sin^{-1}\mu\}\,d\mu,$$

and so, by addition,

$$\int_s^\infty dx \int_{-\infty}^x \left\{(\nu-1)\,\frac{J_{\nu-1}(x-t)}{x-t} J_\nu(x-s)+\nu\,\frac{J_\nu(x-t)}{x-t} J_{\nu-1}(x-s)\right\} f(t)\,dt$$

$$= 2\int_a^b \phi(\mu)(1-\mu^2)^{-\frac{1}{2}} \sin\mu s \cos(\sin^{-1}\mu)\,d\mu$$

$$= 2\int_a^b \phi(\mu) \sin\mu s\,d\mu$$

$$= 2f(s). \quad (2.5)$$

To complete the proof we must show that, if

$$\phi(\mu) = (1-\mu)^{-\alpha}\psi(\mu),$$

where $\psi(\mu)$ is monotonic near $\mu = 1$ and $0 \leqslant \alpha < \frac{1}{2}$, then (2.3) is true for $-1 < a < b \leqslant 1$.

* Watson, *loc. cit.*, 405 (4) and (5).

Choose δ so that $\psi(\mu)$ is monotonic in $(1-\delta, 1)$. Then

$$\int_{1-\delta}^{1} \frac{\psi(\mu)}{(1-\mu)^{\alpha}} \sin\{\mu x-(x-1)\sin^{-1}\mu\}\, d\mu$$

$$= \int_{0}^{\delta} \frac{\psi(1-\lambda)}{\lambda^{\alpha}} \sin\{x(1-\lambda)-(x-1)\sin^{-1}(1-\lambda)\}\, d\lambda.$$

From the second mean value theorem we have

$$\int_{0}^{\delta} \frac{\psi(1-\lambda)}{\lambda^{\alpha}} \sin\{x(1-\lambda)-(x-1)\sin^{-1}(1-\lambda)\}\, d\lambda$$

$$= \psi(1-\delta)\int_{\delta'}^{\delta} \frac{1}{\lambda^{\alpha}} \sin\{x(1-\lambda)-(x-1)\sin^{-1}(1-\lambda)\}\, d\lambda \quad (0 \leqslant \delta' \leqslant \delta)$$

$$= \frac{1}{x^{1-\alpha}}\psi(1-\delta)\int_{\delta' v}^{\delta v} \frac{1}{\mu^{\alpha}} \sin\left\{x-\mu-(x-1)\sin^{-1}\left(1-\frac{\mu}{x}\right)\right\} d\mu.$$

Hence $$\left|\int_{1-\delta}^{1} \phi(\mu)\sin\{\mu x-(x-1)\sin^{-1}\mu\}\, d\mu\right| < \frac{K}{x^{1-\alpha}},$$

where K is a constant independent of x.

But $$J_{\nu}(x-s) = O(x^{-\frac{1}{2}}),$$

where x is large, and so, provided that $-1 < a < b \leqslant 1$ and $\alpha < \frac{1}{2}$, it follows that

$$\int_{s}^{\infty} J_{\nu}(x-s)\, dx \int_{a}^{b} \phi(\mu)\sin\{\mu x-(\nu-1)\sin^{-1}\mu\}\, d\mu$$

is absolutely and uniformly convergent in an interval including $b = 1$.

Hence the order of integration can be changed and we see that (2.3) is still true under the new set of conditions. The rest of the proof follows as before.

The theorem can be proved in a similar manner if $f(s)$ is defined by the equation

$$f(s) = \int_{a}^{b} \phi(\mu)\cos \mu s\, d\mu.$$

(1.4) can be established in exactly the same way for all integral values of ν greater than unity. The proof has to be divided into the two cases ν odd and ν even. The truth of (1.4) for negative integral values of ν other than zero follows from the usual definition

$$J_{-n}(x) = (-1)^{n} J_{n}(x) \qquad (2.6)$$

when n is a positive integer. The cases $\nu = 0$ and 1 in (1.4) are somewhat difficult, since one of the terms on the left becomes the product of zero by a divergent integral. The other term, however, is the same as the integral on the left of Bateman's theorem (1.1), and so we may say, in a sense, that (1.4) is a generalization of (1.1) for values of ν other than 0 or 1.

(1.5) can be proved in the same way as (1.3), or can be obtained by subtraction of (1.3) from (1.4) when ν is an integer greater than unity. When ν is an integer less than zero, it can be deduced from the case $\nu > 1$ with the help of (2.6).

(1.6) follows from (2.2) on making $\nu \to 1$.

Neither (1.4) nor (1.5) can be extended to other than integral values of ν since the Bessel functions in the integrands cease to be one-valued functions.

3. *Proof of* (1.3) *by Bateman's method.*

Bateman proved (1.1) by showing that

$$\int_s^\infty J_0(x-s)\,\frac{J_1(x-t)}{x-t}\,dx = \frac{2}{\pi}\,\frac{\sin(s-t)}{s-t}, \tag{3.1}$$

and then using the equation*

$$f(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(s-t)}{s-t} f(t)\,dt, \tag{3.2}$$

which is true if $f(t)$ satisfies certain conditions.

The generalization from (3.1) to (1.3) consists in showing that for all $\nu > 1$

$$\int_s^\infty \left\{(\nu-1)\,\frac{J_{\nu-1}(x-t)}{x-t}\,J_\nu(x-s) + \nu\,\frac{J_\nu(x-t)}{x-t}\,J_{\nu-1}(x-s)\right\} dx = \frac{2}{\pi}\,\frac{\sin(s-t)}{s-t}, \tag{3.3}$$

using (3.2) and then changing the order of integration, with the help of suitable restrictions on $f(t)$.

To prove (3.3) we observe that the left-hand side is equal to

$$\tfrac{1}{2}\int_0^\infty J_\nu(z)\left\{J_{\nu-1}(z+a)+J_\nu(z+a)\right\} dz + \tfrac{1}{2}\int_0^\infty J_{\nu-1}(z)\left\{J_{\nu+1}(z+a)+J_{\nu+1}(z+a)\right\} dz$$

on writing $x-s=z$, $s-t=a$ and using the recurrence formula for the

* G. H. Hardy, *Proc. London Math. Soc.* (2), 7 (1909), 445.

Bessel functions*. Now use the formula†

$$J_\lambda(z+a) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos(\tfrac{1}{2}\lambda\pi - \nu\theta)\cos\{\tfrac{1}{2}\lambda\pi - (z+a)\sin\theta\}\,d\theta - \frac{\sin\lambda\pi}{\pi}\int_0^\infty e^{-\lambda t-(z+a)\sinh t}\,dt$$

for all the Bessel functions which have $z+a$ as their argument. A change of order of integration can then be justified without much difficulty and an application of (2.4) combined with the result that‡

$$\int_0^\infty e^{-z\sinh t} J_\nu(z)\,dz = e^{-\nu t}\operatorname{sech} t$$

leads to (3.3) after some tedious algebra.

* Watson, *loc. cit.*, 45 (1).
† *Ibid.*, 176 (4).
‡ *Ibid.*, § 13.2, 386.

PERIODIC SIMPLE CONTINUED FRACTIONS

By T. VIJAYARAGHAVAN.

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1. *Introduction.*

Lagrange proved that every quadratic surd corresponds to a periodic simple continued fraction*. In proving this result it is shown that, if we represent any quadratic surd, as indeed we may, in the form $(P+\sqrt{R})/Q$, where $|P|$, $|Q|$, R are positive integers for which it is true that $Q \mid R-P^2$)†, then the number of elements in the periodic part of the corresponding simple continued fraction is less than $2R$. It may be mentioned once for all that in what follows $|P|$, $|Q|$, and R are supposed to be the *least* set of positive integers which satisfy the condition $Q \mid (R-P^2)$. In this paper it is proved by quite an elementary method that the $2R$ of Lagrange may be replaced by $O(R^{\frac{1}{2}+\delta})$, where δ is any positive number.

It is possible to go further, but this requires the use of certain results concerning the minimum values of x and y for which

$$Rx^2 = y^2-4.$$

These results are due to I. Schur‡, and are proved by making use of deep results in the theory of numbers. Schur's results are quoted in § 5. I also found it necessary in § 4 to prove a lemma which seems to have some independent interest. The lemma is that

If x and y exist such that

$$Rx^2 = y^2+1$$

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* O. Perron, *Die Lehre von den Kettenbrüchen*, Berlin (1913), § 20.

† $a \mid b$ means that a is a divisor of b.

‡ I. Schur, *Göttinger Nachrichten*, Heft 1 (1918), 30–36.

and if $(P+\sqrt{R})/Q$ *corresponds to a purely periodic continued fraction, then the continued fraction with an odd number of elements* which corresponds to* $(Px+y)/Qx$ *gives a period of the continued fraction for* $(P+\sqrt{R})/Q$.

A similar result is true regarding periods with even number of elements when $Rx^2 = y^2-1$, and it is implicitly contained in known results concerning automorphs†.

The above lemma and certain simple results lead to a result similar to the well known theorem which states that the "Pellian" equation‡

$$Rx^2 = y^2+1$$

is, or is not, solvable in integers according as the continued fraction for $\sqrt{R}$ has, or has not, a period with an odd number of elements. It is proved in this paper that the continued fraction which corresponds to $(P+\sqrt{R})/Q$ has, or has not, a period with an odd number of elements according as the Pellian equation is, or is not, solvable for R. This is a generalization of previous results, since it asserts for the class of quadratic surds $(P+\sqrt{R})/Q$, what has been asserted for $\sqrt{R}$ only. This is by the way.

The last section of this paper consists of "order" results. Suppose that

$$N(R) = N(P, Q, R)$$

denotes the number of elements in the smallest period of the continued fraction which corresponds to $(P+\sqrt{R})/Q$. Using the lemma mentioned previously, together with Schur's results, I prove, in addition to

$$N(R) = O(R^{\frac{1}{2}+\delta})$$

mentioned above, that $N(R) = O(R^{\frac{1}{2}}\log R)$,

and that, for an infinity of values of R,

$$N(R) > R^{\frac{1}{2}-\delta},$$

where δ is any positive number. It is also proved in an elementary manner that, for an infinity of values of R,

$$N(R) = O(\log R).$$

It may be stated here that the "order" results are uniform with respect to P and Q.

* It is pointed out in § 2 that any rational number can be represented as a simple continued fraction with an odd number of elements.

† G. B. Mathews, *Theory of numbers*, 89–99.

‡ O. Perron, *loc. cit.*, § 26.

2. In this section I collect a few well known results which I shall require later.

Let $[a_1, a_2, \ldots, a_n]$ denote the continued fraction

$$a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n} = \frac{p_n}{q_n}.$$

The following fundamental formulae are well known :

$$p_n = a_n p_{n-1} + p_{n-2}, \tag{1}$$

$$q_n = a_n q_{n-1} + q_{n-2}. \tag{2}$$

When a_1 is an integer, positive, negative, or zero, and $a_2, a_2, \ldots, a_n$ are all positive integers, we have a simple continued fraction, and then

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n. \tag{3}$$

Every number can be represented as a simple continued fraction. When the number is irrational the expression is unending, and the representation is unique. When the number is rational, and not equal to zero, there are just two representations, one containing an odd number of partial quotients, and the other an even number. Thus

$$\tfrac{5}{4} = [1, 4] = [1, 3, 1],$$

and in general

$$a/b = [a_1, a_2, \ldots, a_n] = [a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1], \ldots \quad (a_n \geqslant 2).$$

If a and b be positive and mutually prime, and if c_1/d_1 be the penultimate convergent of the simple continued fraction which corresponds to a/b and contains an odd number of partial quotients, then, from (3), it follows that

$$ad_1 - bc_1 = -1. \tag{4}$$

Similarly, if c_2/d_2 be the penultimate convergent of the corresponding simple continued fraction with an even number of partial quotients, then

$$ad_2 - bc_2 = 1. \tag{5}$$

The pair of values c_1, d_1, such that $0 \leqslant c_1 \leqslant a$, $0 \leqslant d_1 \leqslant b$, and for which the equation (4) holds, is evidently unique. (6)

Similarly c_2 and d_2 are also unique. (7)

Now, coming to quadratic surds and periodic simple continued fractions we have the following theorems. Every quadratic surd can be *uniquely* represented in the form $(P+\sqrt{R})/Q$. Numbers of the type $(P+\sqrt{R})/Q$ can have different periods in their continued fraction (for different values of P and Q). Thus

$$20+\sqrt{(401)} = [\overset{\times}{40}],^*$$

$$\{19+\sqrt{(401)}\}/8 = [\overset{\times}{4}, 1, \overset{\times}{7}],$$

$$\{17+\sqrt{(401)}\}/8 = [\overset{\times}{4}, 1, 1, 1, \overset{\times}{2}],$$

etc. I shall conclude this section by adding an evaluation of the quadratic surd in terms of the elements in the period of the corresponding continued fraction.

If $[a_1, a_2, \ldots a_n] = p_n/q_n, \quad [a_1, a_2, \ldots, a_{n-1}] = p_{n-1}/q_{n-1},$

then
$$\theta = [\overset{\times}{a_1}, a_2, \ldots, \overset{\times}{a_n}] = [a_1, a_2, \ldots, a_n, \theta]$$
$$= (\theta p_n+p_{n-1})/(\theta q_n+q_{n-1}),$$

by (1) and (2). From this we get a quadratic equation for θ, and, on solving it, we have

$$\theta = \frac{p_n-q_{n-1}+\sqrt{\{(p_n-q_{n-1})^2+4p_{n-1}q_n\}}}{2q_n}$$

$$= \frac{p_n-q_{n-1}+\sqrt{\{(p_n+q_{n-1})^2-4(-1)^n\}}}{2q_n} \qquad (8)$$

by (3).

3. THEOREM 1. $N(R) = O(R^{\frac{1}{2}+\delta})$.

Let $(P+\sqrt{R})/Q = \theta = [a_1, a_2, \ldots, a_n, \ldots],$

and let θ_n be defined by the equation

$$\theta = [a_1, a_2, \ldots, a_n, \theta_n].$$

* $[\overset{\times}{a_1}, a_2, \ldots, \overset{\times}{a_n}]$ denotes the number which corresponds to the periodic simple continued fraction the partial quotients in whose period are given in order by $a_1, a_2, \ldots, a_n$.

It is known* that, for all large n,

$$\theta_n = \frac{P_n+\sqrt{R}}{Q_n},$$

where

$$0 < P_n < \sqrt{R}, \quad 0 < Q_n < P_n+\sqrt{R}, \quad Q_n \mid (R-P_n^2), \quad \text{and} \quad \sqrt{R} < P_n+Q_n.$$

Making use of the conditions that $0 < P_n < \sqrt{R}$, and that Q_n is positive and a divisor of $R-P_n^2$, it can be deduced at once that the number of elements in the periodic part cannot be greater than

$$\sum_{x=1}^{\sqrt{R}} d(R-x^2),$$

where $d(n)$ denotes the number divisors of n. Since $d(n) = O(n^\delta)$, where δ is any positive number, it follows that the above expression is of order $R^{\frac{1}{2}+\delta}$ and the theorem is an immediate corollary.

The above result can be slightly sharpened by using more precise results which are known about the order of $d(n)$. In this paper it is proved later that the number of elements in the periodic part is of order $R^{\frac{1}{2}} \log R$, a result which is deeper than any which we can get by using results about the order of $d(n)$.

4. LEMMA 1. *If*

$$\frac{P+\sqrt{R}}{Q} = [\overset{\times}{a_1}, a_2, \ldots, \overset{\times}{a_{2n+1}}],$$

then integers x and y exist such that

$$Rx^2 = y^2+1.$$

Lemma 1 is implicitly contained in certain known results†. From (8) we have

$$\frac{P+\sqrt{R}}{Q} = \frac{A+\sqrt{(B^2+4)}}{C}, \tag{9}$$

where

$$A = p_{2n+1}-q_{2n}, \quad B = p_{2n+1}+q_{2n}, \quad C = 2q_{2n+1}.$$

* O. Perron, *loc. cit.*, § 20.

† G. B. Mathews, *loc. cit.*, 90, 96.

It is evident from (3) that $C \mid (B^2+4-A^2)$, and hence it follows from (9) that

$$\frac{B^2+4}{R} = S^2, \tag{10}$$

where S is a positive integer. Now suppose that B is even. Since $B-A = 2q_{2n}$ it is clear that A is even, and C, being equal to $2q_{2n+1}$, is even. Since P, Q, and R are the *least* numbers for which it is true that $Q \mid (R-P^2)$, and since $2C$ (not merely C) is a divisor of B^2+4-A^2, it follows that S is even. Hence, when B is even, $\frac{1}{2}B$ and $\frac{1}{2}S$ are integers, and in virtue of (10) we shall have

$$R(\tfrac{1}{2}S)^2 = (\tfrac{1}{2}B)^2+1,$$

which proves the lemma when B is even. When B is odd we can find integers x_1 and y_1 such that

$$(B^2+4)\,x_1^2 = y_1^2+1.$$

In fact, if $B = 2m+1$, then

$$(B^2+4)(2m^2+2m+1)^2 = \{2(2m+1)(m^2+m+1)\}^2+1.$$

Therefore

$$R(Sx_1)^2 = y_1^2+1,$$

and this proves the lemma when B is odd, and the lemma is proved.

LEMMA 2. *If $(P+\sqrt{R})/Q$ corresponds to a purely periodic continued fraction, and if integers x and y exist such that*

$$Rx^2 = y^2+1,$$

then the continued fraction which corresponds to $(Px+y)/Qx$, and which has an odd number of elements, gives a period of $(P+\sqrt{R})/Q$.

Since $(P+\sqrt{R})/Q$ corresponds to a purely periodic continued fraction, it follows from (8) that

$$0 < P < \sqrt{R}, \quad 0 < Q < P+\sqrt{R}, \quad 0 < \sqrt{R} < P+Q.$$

Now $$y-Px = \frac{y^2-P^2x^2}{y+Px} = \frac{Rx^2-1-P^2x^2}{y+Px} = \frac{x^2(R-P^2)-1}{y+Px} \geqslant 0,$$

since $\sqrt{R} > P > 0$*. If $y-Px = 0$, then it follows at once that

$$x = 1, \quad R = y^2+1, \quad P = y, \quad Q = 1.$$

* I have naturally supposed that x and y are positive. Similar remarks apply elsewhere too.

In this case $$\frac{P+\sqrt{R}}{Q} = y+\sqrt{(y^2+1)} = [\overset{\times}{2y}],$$

and the theorem is true. The only other case is $y-Px>0$. Let Q' be the integer defined by the equation

$$Q' = \frac{R-P^2}{Q}.$$

Since $$y < \sqrt{R}x < (P+Q)x,$$

it is clear that

$$0 < y-Px < Qx. \tag{11}$$

Further, since $P+Q>\sqrt{R}$, it follows that $\sqrt{R}-P<Q$. Hence

$$Q' = \frac{R-P^2}{Q} \leqslant \left[\frac{R-P^2}{\sqrt{R}-P}\right]^* = [\sqrt{R}+P] = \left[\frac{y}{x}+P\right].$$

Therefore

$$0 < Q'x \leqslant Px+y. \tag{12}$$

Also

$$(Px+y)(y-Px)-Qx\,.\,Q'x = y^2-P^2x^2-QQ'x^2$$

$$= y^2-x^2\{P^2+(R-P^2)\} = y^2-Rx^2 = -1. \tag{13}$$

From (11), (12), (13), (6), and (4) it follows that $Q'x/(y-Px)$ is the penultimate convergent of the odd continued fraction for $(Px+y)/Qx$. Now the surd which has the corresponding period is, as in (8),

$$\frac{2Px+\sqrt{(4y^2+4)}}{2Qx} = \frac{P+\sqrt{R}}{Q}.$$

This proves the lemma.

LEMMA 3. *If $(P+\sqrt{R})/Q$ correspond to a purely periodic continued fraction, and if x and y be integers for which*

$$Rx^2 = y^2-1,$$

then the continued fraction which corresponds to $(Px+y)/Qx$ and which has an even number of elements gives a period of $(P+\sqrt{R})/Q$.

The proof is precisely similar to that of Lemma 2.

* $[x]$ denotes the largest integer which does not exceed x.

LEMMA 4. *If* $\theta = [a_1, a_2, \ldots] = \dfrac{P+\sqrt{R}}{Q}$,

then to every n *there correspond integers* P_n *and* Q_n *such that*

$$Q_n|(R-P_n^2), \quad \theta_n = [a_{n+1}, a_{n+2}, \ldots] = \frac{P_n+\sqrt{R}}{Q_n}.$$

This is quite a well known result, and it has been quoted in the proof of Theorem 1 in § 3.

LEMMA 5. *Conversely, if* a_1 *is an integer,* $a_2, a_3, \ldots, a_n$ *are positive integers,*

$$C_n|(B-A_n^2), \quad \theta_n = \frac{A_n+\sqrt{B}}{C_n},$$

then there are integers A *and* C *such that*

$$C|(B-A^2), \quad \theta = [a_1, a_2, \ldots, a_n, \theta_n] = \frac{A+\sqrt{B}}{C}.$$

To prove this lemma it is sufficient to show that integers A', C' exist such that

$$C'|(B-A'^2), \quad \theta' = [a_n, \theta_n] = \frac{A'+\sqrt{B}}{C'}.$$

Now if $B_n-A_n^2 = C_n d$, then

$$\frac{1}{\theta_n} = \frac{-A_n+\sqrt{B_n}}{d}, \quad [a_n, \theta_n] = \frac{a_n d-A_n+\sqrt{B_n}}{d}.$$

Since d is a divisor of $B_n-A_n^2$, it follows that it is also a divisor of

$$B_n-(a_n d-A)^2,$$

and the required result is evident.

From Lemmas 4 and 5 it is seen that, if $|A|$, B, $|C|$, $|A'|$, B', $|C'|$ are the least integers for which it is true that

$$C|(B-A^2), \quad C'|(B'-A'^2), \quad \frac{A+\sqrt{B}}{C} = [a_1, a_2, \ldots],$$

$$\frac{A'+\sqrt{B'}}{C'} = [a_n, a_{n+1}, \ldots],$$

then $B = B'$. Hence it is clear that, in discussing the relation between the integer R and the periods of continued fractions correspond-

ing to $(P+\sqrt{R})/Q$, it may be assumed without loss of generality that $(P+\sqrt{R})/Q$ corresponds to a purely periodic continued fraction. (14)

From (14) and Lemmas 1 and 2 the following is an immediate corollary.

THEOREM 2. *The continued fraction which corresponds to* $(P+\sqrt{R})/Q$ *has, or has not, a period with an odd number of elements according as the Pellian equation*

$$Rx^2 = y^2+1$$

is, or is not, satisfied for some pair of integers x *and* y.

5. THEOREM 3.

$$N(R) = O(R^{\frac{1}{2}} \log R).$$

It has been proved by Schur* that if x_1 and y_1 be the least integers (the existence of such integers is well known) such that

$$4Rx_1^2 = y_1^2-4,$$

then

$$\log x_1 < \log y_1 < KR^{\frac{1}{2}} \log R. \tag{15}$$

Evidently y_1 is even, and if x and y be the integers x_1 and $\frac{1}{2}y_1$, then

$$Rx^2 = y^2-1.$$

(16) Hence, by Lemma 3, if $(P+\sqrt{R})/Q$ corresponds to a purely periodic continued fraction, and if

$$\frac{Px+y}{Qx} = [a_1, a_2, \ldots, a_{2n}],$$

then

$$\frac{P+\sqrt{R}}{Q} = [\overset{\times}{a}_1, a_2, \ldots, \overset{\times}{a}_{2n}].$$

It is easy to verify that, if

$$[b_1, b_2, \ldots, b_n] = \frac{p_n}{q_n}, \quad [1, 1, \ldots, 1] = \frac{c_n}{d_n},$$

then

$$q_n \geqslant d_n \geqslant \{(5+1)^{n-2}+(-1)^n(\sqrt{5}-1)^{n-2}\} \frac{1}{2^{n-2}} \quad (n \geqslant 4).$$

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* I. Schur, *loc. cit.* In fact, Schur proves that

$$\overline{\lim}\, (\log y_1)/\sqrt{R} \,.\, \log R \leqslant 1.$$

And hence $$n = O(\log q_n).$$

The theorem now follows from (15) and (16).

THEOREM 4. *For an infinity of values of* R,

$$N(R) > R^{\frac{1}{2}-\delta},$$

where δ *is any positive number.*

Let $(a_1, a_2, \ldots, a_{2n})$ correspond to the smallest even period of the continued fraction for $(P+\sqrt{R})/Q$, and let

$$[a_1, a_2, \ldots, a_{2n}] = \frac{p_{2n}}{q_{2n}}.$$

It has been already shown (viz. in the proof of Lemma 1) that

$$(p_{2n}+q_{2n-1})^2-4 = Rs^2,$$

and it was also pointed out that if $p_{2n}+q_{2n-1}$ is even, then s also is even. Hence we see that, when $p_{2n}+q_{2n-1}$ is even,

$$\{\tfrac{1}{2}(p_{2n}+q_{2n-1})\}^2-1 = R(\tfrac{1}{2}s)^2.$$

And since we are considering the smallest even period, it follows from Lemma 3 that $\frac{1}{2}(p_{2n}+q_{2n-1})$ and $\frac{1}{2}s$ are the smallest integers for which it is true that

$$y^2-1 = Rx^2.$$

If $p_{2n}+q_{2n-1}$ be odd, so that $p_{2n}+q_{2n-1} = 2m+1$, then we can infer from Lemma 3 and from the identity

$$\{(2m+1)^2-4\}\{2m(m+1)\}^2 = (4m^3+6m^2-1)^2-1,$$

that the *least* integer y, to which there corresponds an x such that

$$Rx^2 = y^2-1,$$

satisfies the inequality

$$y^2 < (p_{2n}+q_{2n-1})^6 < (2p_{2n})^6. \tag{17}$$

Hence, in any case,

$$\log p_{2n} > (\tfrac{1}{3}\log y - \log 8).$$

Now it has been proved by Schur that, for an infinity of values of R,

$$\log y > R^C,$$

where C is any constant less than $\frac{1}{2}$. Hence there is an infinity of values of R such that

$$\log y > R^{\frac{1}{2}(1-\delta)}. \tag{18}$$

It is also well known that none of the partial quotients $a_1, a_2, \ldots, a_{2n}$ can exceed $2\sqrt{R}$. Therefore, by making use of the recurrence formula (1), we have

$$p_{2n} < (2\sqrt{R}+1)^{2n}. \tag{19}$$

From (17), (18), and (19) it follows that

$$\log 2^6(2\sqrt{R}+1)^{12n} > \log y^2 > R^{\frac{1}{2}(1-\delta)},$$

and it is easy to deduce that, when R is large,

$$n > R^{\frac{1}{2}-\delta}.$$

Since the *minimum even* period consists of $2n$ elements, the smallest period consists of at least n elements, and, since n satisfies the above inequality, the theorem is proved.

THEOREM 5. *For an infinity of values of R,*

$$N(R) = O(\log R).$$

From the proof of Theorem 3 it will be seen that the present theorem will be true if, for an infinity of values of R, integers x and y correspond such that

$$Rx^2 = y^2 \pm 1, \quad \log y = O(\log R).$$

Now if $R = m^2+1$, m^2-1, m^2+4, m^2-4, etc., such is the case, and the theorem is therefore proved.

THEOREM 6. *If $x_1, x_2, \ldots, x_n, \ldots$ are all the numbers in order of magnitude such that*

$$Rx_r^2 = y_r^2 - 1,$$

then the period of $(P+\sqrt{R})/Q$, as given by the even continued fraction of $(Px_r+y_r)/Qx_r$ contains r times, or $3r$ times, the number of elements in the smallest even period of $(P+\sqrt{R})/Q$.

The main results which are necessary to establish the theorem occur in the proof of Theorem 4, and since there is no intrinsic difficulty in the proof I omit it.

A similar theorem can be enunciated when

$$Rx_r^2 = y_r^2+1.$$

Illustration.

$$7+\sqrt{(61)} = [\overset{\times}{14},\ 1,\ 4,\ 3,\ 1,\ 2,\ 2,\ 1,\ 3,\ 4,\ \overset{\times}{1}],$$

and

$$\frac{7+\sqrt{(61)}}{2} = [\overset{\times}{7},\ 2,\ \overset{\times}{2}].$$

If

$$61x_r^2 = y_r^2+1,$$

then the smallest odd period of $7+\sqrt{(61)}$, repeated $2r-1$ times, gives the continued fraction for $(7x_r+y_r)/x_r$. The smallest period of $\{7+\sqrt{(61)}\}/2$, repeated $6r-3$ times, gives the continued fraction for $(7x_r+y_r)/2x_r$.

THE CONVERSES OF THE CAUCHY-HÖLDER INEQUALITY AND THE SOLUTIONS OF THE INEQUALITY $g(x+y) \leqslant g(x)+g(y)$

By R. COOPER.

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1. Except where the contrary is definitely stated, all numbers, variables, and functions used in this paper are real and non-negative.

Converses of the Cauchy-Hölder Inequality.

2. Theorems which can be regarded as converses of the Cauchy-Hölder inequality have been given by Landau and by Riesz*. In considering extensions of these results we use the following as a permanent notation: $\phi(x)$, $\psi(x)$ are continuous, inverse, increasing functions defined in $x \geqslant 0$. We have therefore also $\phi(0) = \psi(0) = 0$, and the functions are strictly increasing and unbounded.

We find it essential to have some condition to replace the identity $(xy)^\alpha = x^\alpha y^\alpha$, and it seems most convenient and natural to assume $\psi(xy) \leqslant \psi(x)\,\psi(y)$. Sometimes we have to assume this for all values of x, y and sometimes only for restricted ranges of x, y.

The condition $\psi(xy) \leqslant \psi(x)\,\psi(y)$ is equivalent to $\phi(tw) \geqslant \phi(t)\,\phi(w)$ for appropriate ranges of t, w.

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* E. Landau, "Über einen Konvergenzsatz", *Göttingen Nachrichten* (1907), 25–27. F. Riesz, "Untersuchungen über Systeme integrierbarer Funktionen", *Math. Annalen*, 69 (1910), 449–497 (457).

THEOREM I. *Let* $\psi(xy) \leqslant \psi(x)\psi(y)$ *for all values of* x, y. *If, for every set of numbers* $b_1, b_2, \ldots$ *which satisfy*

$$\Sigma b_\nu \psi(b_\nu) = V\psi(V) \neq 0,$$

we have

$$\Sigma a_\nu b_\nu \leqslant UV,$$

then

$$\Sigma a_\nu \phi(a_\nu) \leqslant \frac{U}{\phi(1/U)}.$$

If only a finite number of the a_ν's are not zero, let n be the greatest suffix for which $a_\nu \neq 0$. If there is an infinity of non-zero a_ν's, let n be any integer. Consider the set of b's defined by

$$b_\nu = t\phi(a_\nu) \quad (\nu \leqslant n), \quad b_\nu = 0 \quad (\nu > n),$$

where t is a constant so chosen that

$$\Sigma b_\nu \psi(b_\nu) = V\psi(V). \tag{2.11}$$

With this choice of t,

$$t \sum_1^n a_\nu \phi(a_\nu) = \Sigma a_\nu b_\nu \leqslant UV. \tag{2.12}$$

Now

$$\psi(b_\nu) = \psi\{t\phi(a_\nu)\} \leqslant \psi(t)\, a_\nu,$$

and therefore from (2.11)

$$t\psi(t) \sum_1^n a_\nu \phi(a_\nu) \geqslant V\psi(V). \tag{2.13}$$

From (2.12) and (2.13) we find

$$\psi(t) \geqslant \frac{\psi(V)}{U} = \psi(V)\,\psi\left\{\phi\left(\frac{1}{U}\right)\right\} \geqslant \psi\left\{V\phi\left(\frac{1}{U}\right)\right\}$$

and so, since $\psi(x)$ is an increasing function, $t \geqslant V\phi(1/U)$.

Substituting this in (2.12) we find

$$\sum_1^n a_\nu \phi(a_\nu) \leqslant \frac{UV}{t} \leqslant \frac{U}{\phi(1/U)}.$$

Since the right-hand side is independent of n, we must have

$$\Sigma a_\nu \phi(a_\nu) \leqslant \frac{U}{\phi(1/U)}.$$

This result had already been given in the case in which $\phi(x)$, $\psi(x)$ are powers of x, for then $\phi(x) = 1/\phi(x^{-1})$.

We next consider a corresponding result for convergence of series of which the case of powers was given by Landau in the paper referred to above.

THEOREM II. *Let* $\psi(xy) \leqslant \psi(x)\psi(y)$ *for* $x, y \leqslant \delta \neq 0$. *If* $\sum_1^\infty a_\nu b_\nu$ *converges for every sequence* $\{b_\nu\}$ *for which* $\sum_1^\infty b_\nu \psi(b_\nu)$ *is convergent, then* $\sum_1^\infty a_\nu \phi(a_\nu)$ *must also converge.*

We must first show that $a_\nu \to 0$ as $\nu \to \infty$ *. Suppose, if possible, that $a_\nu \not\to 0$, so that we can find an infinite sequence n_i and a number η, such that $a_{n_i} \geqslant \eta > 0$. Then it follows from the second part of the hypothesis that Σb_{n_i} converges whenever $\Sigma b_{n_i}\psi(b_{n_i})$ does. Therefore writing $b_{n_i} = c_i$ we have to show that we can so choose the numbers c_i that Σc_i diverges while $\Sigma c_i \psi(c_i)$ converges.

Take any number $p_1 > 1/\psi(1)$ and choose p_2 so that

$$\psi\left(1 - \frac{p_1}{p_2}\right) = \frac{1}{p_1}.$$

This choice is possible and unique. Next choose p_3 so that

$$\psi\left(1 - \frac{p_2}{p_3}\right) = \frac{1}{p_2},$$

and so on. The sequence p_n is increasing and therefore tends to infinity or to a finite number k. The latter case gives $\psi(0) = 1/k$, which is false; so $p_n \to \infty$.

Now if $c_n = 1 - p_n/p_{n+1}$, the series Σc_n is divergent, while the series

$$\Sigma c_n \psi(c_n) = \Sigma\left(\frac{1}{p_n} - \frac{1}{p_{n+1}}\right)$$

is convergent. We therefore have $a_\nu \to 0$.

* In the original form of the theorem this was part of the hypothesis. I am indebted to the referee for pointing out the following argument which shows that it follows from another part of the hypothesis.

Write $b_\nu = t_\nu \phi(a_\nu)$ and $a_\nu = a_\nu \phi(a_\nu)$. Then it is enough to show that, if Σa_ν is divergent, we can find a sequence $\{t_\nu\}$, such that (i) $\Sigma a_\nu b_\nu = \Sigma t_\nu a_\nu$ diverges while (ii) $\Sigma b_\nu \psi(b_\nu)$ is convergent.

Write $s_n = \sum_1^n a_\nu$. Then it is sufficient to take $t_\nu = 1/s_\nu$.

First $\Sigma t_\nu a_\nu$ diverges because, as $n \to \infty$,

$$\prod_2^n (1 - t_\nu a_\nu) = \prod_2^n \left(\frac{s_{\nu-1}}{s_\nu}\right) = \frac{s_1}{s_n} \to 0.$$

Secondly, if ν is sufficiently large,

$$b_\nu \psi(b_\nu) \leqslant t_\nu \phi(a_\nu) \cdot \psi(t_\nu) a_\nu = \frac{1}{s_\nu} \psi\left(\frac{1}{s_\nu}\right) a_\nu = \frac{1}{s_\nu} \psi\left(\frac{1}{s_\nu}\right)(s_\nu - s_{\nu-1})$$

$$\leqslant \int_{s_{\nu-1}}^{s_\nu} \frac{1}{x} \psi\left(\frac{1}{x}\right) dx.$$

It follows that $\Sigma b_\nu \psi(b_\nu)$ converges if $\int^\infty \frac{1}{x} \psi\left(\frac{1}{x}\right) dx$ does so. Putting $x = 1/y$, we see that we have to prove that $\int_0 \frac{\psi(y)}{y} dy$ is convergent at 0.

Now, by a later theorem*, there exist non-zero numbers A, α such that

$$\psi(y) \leqslant A y^\alpha \quad \text{for} \quad y \leqslant \delta.$$

It follows that, if $0 < y_1 < y_2 \leqslant \delta$,

$$\int_{y_1}^{y_2} \frac{\psi(y)}{y} dy \leqslant \int_{y_1}^{y_2} A y^{\alpha-1} dy \to \frac{A}{\alpha} y_2^\alpha \quad \text{as} \quad y_1 \to 0.$$

We next proceed to the construction of an example which shows that the condition imposed on $\psi(x)$ is "almost necessary" for the result of Theorem II. For this we require a theorem concerning series of positive terms which is of independent interest.

THEOREM III. *If* $S_1 = \Sigma \frac{1}{c_n (\log c_n)^\alpha}$ *is convergent, then for each* $\beta \geqslant \alpha$, *the series* $S_2 = \Sigma \frac{1}{c_n (\log n)^\beta}$ *is also convergent.*

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* Theorem VI.

We may obviously assume $\beta = \alpha \neq 0$. Suppose, if possible, that S_2 is divergent. Then

$$\overline{\lim_{n\to\infty}} \left(\frac{\log c_n}{\log n}\right)^\alpha = +\infty.$$

Take any number $\kappa > 1$, and let $n_1 \geqslant 3$, n_2, n_3, ..., n_i, ... be *the* (maximum) sequence for which

$$\frac{\log c_{n_i}}{\log n_i} > \kappa. \tag{2.2}$$

Then $c_{n_i} \geqslant n_i^\kappa$, so that the series $\sum_{i=1}^{\infty} \frac{1}{c_{n_i}(\log n_i)^\alpha}$ must converge and so the sub-series $\sum_{n\neq n_i} \frac{1}{c_n(\log n)^\alpha}$ diverges while $\sum_{n\neq n_i} \frac{1}{c_n(\log c_n)^\alpha}$ converges.

We must have $$\overline{\lim_{\substack{n\to\infty \\ n\neq n_i}}} \left(\frac{\log c_n}{\log n}\right)^\alpha = +\infty,$$

which contradicts the assumption that the sequence $\{n_i\}$ contains all the integers ($\geqslant 3$) which satisfy (2.2). S_2 must therefore converge.

The result of the theorem obviously fails if $\beta < \alpha$. For S_2 diverges if $c_n = n(\log n)^{1-\beta}$. From this it follows that, if ϵ is given and $0 < \epsilon < 1$, we have

$$\log c_n = \log n + (1-\beta)\log\log n > (1-\epsilon)\log n,$$

for $n > n_0(\epsilon)$.

Therefore $$\frac{1}{c_n(\log c_n)^\alpha} < \frac{1}{(1-\epsilon)^\alpha n(\log n)^{1-\beta+\alpha}}$$

and so, in this case, S_1 is convergent if $\beta < \alpha$.

It should also be observed that even when $\alpha = \beta$ it is *not* true that S_1 and S_2 converge and diverge together. For we can have $c_{n_i} = e$ (say) where $\{n_i\}$ is a sequence for which $\sum_{i=1}^{\infty} \frac{1}{(\log n_i)^\alpha}$ is convergent, and so S_2 can converge when S_1 diverges.

For the example which we wish to construct, define

$$\psi(x) = \frac{1}{\log x^{-1}} \quad \text{for} \quad x \leqslant \frac{1}{e},$$

and define $\psi(x)$ in any manner for $x > e^{-1}$, provided that it is continuous and strictly increasing. Then, however small we take δ, $\psi(x)$ does *not*

satisfy $\psi(xy) \leqslant \psi(x)\psi(y)$ for $x, y \leqslant \delta$. We have, for $x \leqslant 1$,

$$\phi(x) = \exp\left(-\frac{1}{x}\right).$$

Now choose $a_n = 1/\log n$. Then $\Sigma a_n \phi(a_n) = \Sigma(n \log n)^{-1}$ which is divergent. We proceed to show that, nevertheless, $\Sigma a_n b_n$ converges whenever $\Sigma b_n \psi(b_n)$ converges.

Write $b_n = 1/c_n$. Then

$$\Sigma a_n b_n = \Sigma \frac{1}{c_n \log n}$$

and

$$\Sigma b_n \psi(b_n) = \Sigma \frac{1}{c_n \log c_n}.$$

Theorem III with $\alpha = \beta = 1$ gives the required result at once.

It may be observed that Theorem III is easily extended to series of the form

$$\Sigma \frac{1}{c_n l_1(n)\, l_2(n) \ldots (l_k n)^{\alpha}},$$

where $l_1(n) = \log n$ and $l_{k+1}(n) = \log l_k(n)$.

Extensions to Integrals.

3. LEMMA I. *Let $\psi(xy) \leqslant \psi(x)\psi(y)$ for all values of x, y. If, for every set of numbers $b_1, b_2, \ldots$, which satisfy*

$$\Sigma \lambda_\nu b_\nu \psi(b_\nu) = V\psi(V) \neq 0,$$

we have

$$\Sigma \lambda_\nu a_\nu b_\nu \leqslant UV,$$

then

$$\Sigma \lambda_\nu a_\nu \phi(a_\nu) \leqslant \frac{U}{\phi(1/U)}.$$

The proof is practically the same as that of Theorem I.

THEOREM IV. *Let $\psi(xy) \leqslant \psi(x)\psi(y)$ for all values of x, y. Let $f(\theta)$, $g(\theta)$ be functions of a real (not necessarily positive) variable θ. Let E be a bounded, measurable set of values of θ. If the multiplicative axiom*

is true, and if, for all functions $g(\theta)$ *which satisfy*

$$\int_E g(\theta)\,\psi\{g(\theta)\}\,d\theta \leqslant V\psi(V) \neq 0,$$

we have

$$\int_E f(\theta)\,g(\theta)\,d\theta \leqslant UV,$$

then

$$\int_E f(\theta)\,\phi\{f(\theta)\}\,d\theta$$

exists and does not exceed $U/\phi(U^{-1})$.

We are, of course, supposed to be given the existence of the integrals which occur in the hypothesis.

First take $g(\theta)$ equal to a suitable constant. Then we see that $\int_E f(\theta)\,d\theta$ exists, and so, by the multiplicative axiom, $f(\theta)$ is measurable. Since $\phi(x)$ is continuous, $f(\theta)\,\phi\{f(\theta)\}$ is also measurable.

Divide E into n measurable sub-sets $E_1, E_2, \ldots, E_n$, having measures $\lambda_1, \ldots, \lambda_n$. Let a_ν, b_ν be the respective lower bounds of $f(\theta)$, $g(\theta)$ in E_ν. Then the lower bound of $f(\theta)\,g(\theta)$ is not less than $a_\nu b_\nu$. It follows that

$$\int_E f(\theta)\,g(\theta)\,d\theta \geqslant \sum_1^n \lambda_\nu a_\nu b_\nu.$$

Also, if we take any number ϵ with $0 < \epsilon < V\psi(V)$ and take n sufficiently large, we can find a subdivision $(E_1, \ldots, E_n)$ of E, such that for it and for any finer subdivision

$$\int_E g(\theta)\,\psi\{g(\theta)\}\,d\theta \leqslant \epsilon + \sum_1^n \lambda_\nu b_\nu \psi(b_\nu).$$

It follows that, for every set $(b_1, b_2, \ldots, b_n)$, for which

$$\sum_1^n \lambda_\nu b_\nu \psi(b_\nu) \leqslant V\psi(V) - \epsilon = W\psi(W) \quad \text{(say)},$$

we have

$$\sum_1^n \lambda_\nu a_\nu b_\nu \leqslant UV.$$

From Lemma I we deduce that

$$\sum_1^n \lambda_\nu a_\nu \phi(a_\nu) \leqslant \frac{T}{\phi(1/T)},$$

where

$$T = \frac{UV}{W}.$$

Now $\sum_1^n \lambda_\nu a_\nu \phi(a_\nu)$ is a lower approximative sum for $\int_E f(\theta)\,\phi\{f(\theta)\}\,d\theta$. The result follows at once when we observe that $T \to U$ as $\epsilon \to 0$.

LEMMA II. *Let $\psi(xy) \leqslant \psi(x)\,\psi(y)$ for all values of x and for $y \leqslant \delta \neq 0$. Then if $\Sigma\lambda_\nu a_\nu b_\nu$ converges whenever $\Sigma\lambda_\nu b_\nu \psi(b_\nu)$ converges, $\Sigma\lambda_\nu a_\nu \phi(a_\nu)$ will also converge.*

Write $b_\nu = t_\nu \phi(a_\nu)$ and $a_\nu = a_\nu \phi(a_\nu)$. It is enough to prove that, if $\Sigma\lambda_\nu a_\nu$ is divergent, we can choose t_ν so that

(i) $$\Sigma\lambda_\nu a_\nu b_\nu = \Sigma\lambda_\nu t_\nu a_\nu$$

is divergent, while

(ii) $$\Sigma\lambda_\nu b_\nu \psi(b_\nu)$$

converges.

Write $s_n = \sum_1^n \lambda_\nu a_\nu$ and $t_n = 1/s_n$. Then, as in the proof of Theorem II, $\Sigma\lambda_\nu t_\nu a_\nu$ will diverge.

Also, if ν is so large that $t_\nu \leqslant \delta$,

$$\lambda_\nu b_\nu \psi(b_\nu) \leqslant \lambda_\nu t_\nu \phi(a_\nu) \cdot \psi(t_\nu)\, a_\nu = \frac{1}{s_\nu}\,\psi\left(\frac{1}{s_\nu}\right)(s_\nu - s_{\nu-1}).$$

The convergence of $\Sigma\lambda_\nu b_\nu \psi(b_\nu)$ follows as in the proof of Theorem II.

THEOREM V*. *Let $\psi(xy) \leqslant \psi(x)\,\psi(y)$ for all x and for $y \leqslant \delta \neq 0$. Let E be any bounded measurable set of values of a real (not necessarily positive) variable θ. Then, if $\int_E f(\theta)\,g(\theta)\,d\theta$ exists whenever*

$$\int_E g(\theta)\,\psi\{g(\theta)\}\,d\theta \leqslant 1,$$

and if the multiplicative axiom holds, $\int_E f(\theta)\,\phi\{f(\theta)\}\,d\theta$ exists.

---

* Here existence of an integral is taken to include finiteness. For the particular case $\psi(x) = x^\alpha$ of this theorem and of Lemma II, see F. Riesz, *loc. cit.*

First, taking $g(\theta)$ equal to a suitable constant, we see that $f(\theta)$ is integrable in E, and so, by the multiplicative axiom, $f(\theta)$, and therefore also $f(\theta)\,\phi\{f(\theta)\}$, is measurable.

Take any unbounded sequence $a_0 = 0 < a_1 < a_2 < a_3 < \ldots$ which is such that the differences $a_n - a_{n-1}$ are bounded. Let E_n be the sub-set of E in which $a_{n-1} < f(\theta) \leqslant a_n$. E_n is measurable. Let its measure be λ_n and let b_n be the lower bound of $g(\theta)$ in E_n. Then

$$\int_E f(\theta)\,g(\theta)\,d\theta \geqslant \sum_1^\infty \lambda_n a_{n-1} b_n,$$

and, given any positive number ϵ,

$$\int_E g(\theta)\,\psi\{g(\theta)\}\,d\theta \leqslant \epsilon + \sum_1^\infty \lambda_n b_n \psi(b_n),$$

provided that the subdivision $(E_1, E_2, \ldots, E_n, \ldots)$ is sufficiently fine, *i.e.* provided that $a_n - a_{n-1}$ is sufficiently small*.

It follows that $\Sigma\lambda_n a_{n-1} b_n$ converges whenever $\Sigma\lambda_n b_n \psi(b_n)$ does.

Since $\int_E g(\theta)\,\psi\{g(\theta)\}\,d\theta$ exists, $\int_E g(\theta)\,d\theta$ is finite, and so $\Sigma\lambda_n b_n$ is convergent. Since $a_n - a_{n-1}$ is bounded, we conclude that $\Sigma\lambda_n a_n b_n$ converges whenever $\Sigma\lambda_n b_n \psi(b_n)$ does.

From Lemma II we conclude that $\Sigma\lambda_n a_n \phi(a_n)$ converges. Since this is an upper approximative sum for the integral of the measurable function $f(\theta)\,\phi\{f(\theta)\}$, the theorem is proved.

The functional relation $f(xy) \leqslant f(x)f(y)$.

4. We first investigate properties of continuous increasing functions which satisfy $f(xy) \leqslant f(x)f(y)$. We assume generally that the relation holds for all values of the variables†, and point out which results still hold if we assume it only for restricted ranges. We assume also that there are values of x for which $f(x) < 1$.

Taking $y = 1$, we observe that $f(1) \geqslant 1$.

---

* We can omit the case in which $f(\theta)$ is constant in some sub-sets of E of non-zero measure.

† Real and non-negative.

An upper bounding function for small values of the variable. Take any number a such that $f(a) < 1$. Let $x \leqslant 1$ and choose an integer n so that $a^{n+1} < x \leqslant a^n$. Then, since $\log a < 0$,

$$n+1 > \frac{\log x}{\log a} \geqslant n,$$

so that $$f(x) = f(xa^{-n} \cdot a^n) \leqslant f(xa^{-n}) f(a^n) \leqslant f(1)\{f(a)\}^n$$

$$\leqslant \frac{f(1)}{f(a)} \{f(a)\}^{\log x/\log a} = \frac{f(1)}{f(a)} x^\alpha$$

where $$\alpha = \frac{\log f(a)}{\log a}.$$

Incidentally we have proved that $f(0) = 0$.

A slight modification of the argument would give the result used in Theorem II in the case in which $f(xy) \leqslant f(x)f(y)$ for $x, y \leqslant \delta$.

A similar argument shows that we can find $B > 0$, $\beta > 0$, such that

$$f(x) \leqslant Bx^\beta \text{ for } x \geqslant 1.$$

Lower bounding functions. Let $x \geqslant 1$. Then $f(x)f(x^{-1}) \geqslant f(1)$, so that, since $x^{-1} \leqslant 1$,

$$f(x) \geqslant \frac{f(1)}{f(x^{-1})} \geqslant \frac{f(a)}{x^{-\alpha}} = f(a)\, x^\alpha.$$

Similarly $$f(x) \geqslant f(1)\, x^\beta / B \text{ for } x \leqslant 1.$$

Summing up we have the theorem:

THEOREM VI. *If $f(x)$ is continuous and increasing in $x \geqslant 0$, is somewhere less than unity, and satisfies $f(xy) \leqslant f(x)f(y)$, then either*

(i) *for $x \leqslant \delta$, $y \leqslant \delta$ where $\delta \neq 0$*

or (ii) *for all values of x, y,*

then we have respectively

(i) $f(x) \leqslant Ax^\alpha$ *for* $x \leqslant 1$

or (ii) $Ax^\alpha \geqslant f(x) \geqslant f(1)\, x^\beta / B$ *for* $x \leqslant 1$

and $$Bx^\beta \geqslant f(x) \geqslant f(1)x^\alpha/A \quad \textit{for} \quad x \geqslant 1,$$

where $$A > 0, \quad B > 0, \quad 0 < \alpha \leqslant \beta.$$

There are obviously corresponding results for functions which satisfy the opposite inequality $f(xy) \geqslant f(x)f(y)$.

THEOREM VII. *If $f(x)$ is a continuous, strictly increasing function defined in $0 \leqslant x \leqslant 1+\delta$ where $0 < \delta < 1$, which takes values less than unity somewhere, and satisfies $f(xy) \leqslant f(x)f(y)$ for $0 \leqslant x \leqslant 1+\delta$, $0 \leqslant y \leqslant \delta$, then there exists a constant C such that, in any given interval $0 < \delta_1 \leqslant x \leqslant \delta$, $f(x)$ increases more slowly than Cx^α. C is independent of δ_1 and α is as in Theorem VI.*

By saying that $f(x)$ increases more slowly than $g(x)$ in $a \leqslant x \leqslant b$, we mean that we can find $\epsilon > 0$ such that, if $a \leqslant x_1 < x_2 \leqslant b$ and $x_2 - x_1 \leqslant \epsilon$, then

$$0 \leqslant f(x_2) - f(x_1) \leqslant g(x_2) - g(x_1).$$

Write $1+\delta = \lambda$ and take any x_1 with $\delta_1 \leqslant x_1 \leqslant \delta$. Let x_2 be any number satisfying $x_1 < x_2 \leqslant \lambda x_1$.

Then $$f(x_2) - f(x_1) \leqslant f(x_1)\left\{f\left(\frac{x_2}{x_1}\right) - 1\right\} \leqslant Ax_1^\alpha\{f(\lambda) - 1\}.$$

Also $$x_2^\alpha - x_1^\alpha = x_1^\alpha\{(x_2/x_1)^\alpha - 1\}.$$

Write $\xi = x_2/x_1$ so that $1 < \xi \leqslant \lambda$. By the mean value theorem, for some θ in (0, 1),

$$\xi^\alpha - 1 = \xi\alpha\{1+\theta(\xi-1)\}^{\alpha-1} \geqslant \alpha\xi \operatorname{Min}(1, \xi^{\alpha-1}).$$

If $\alpha \geqslant 1$, $\xi^\alpha - 1 \geqslant \alpha\xi > \alpha$. If $\alpha \leqslant 1$, $\xi^\alpha - 1 \geqslant \alpha\xi\lambda^{\alpha-1} \geqslant \alpha\lambda^{\alpha-1}$.

Write k equal to α or $\alpha\lambda^{\alpha-1}$ according as $\alpha \geqslant 1$ or $\alpha < 1$. Then

$$x_2^\alpha - x_1^\alpha \geqslant kx_1^\alpha,$$

and so

$$f(x_2) - f(x_1) \leqslant \frac{A}{k}\{f(\lambda) - 1\}(x_2^\alpha - x_1^\alpha).$$

Taking $C = A\{f(\lambda) - 1\}/k$ we have the result required.

It should be observed that, with our definition of "increasing more slowly than", we cannot take $\delta_1 = 0$.

COROLLARY. *If $\theta(x)$ is any bounded function defined in $a \leqslant x \leqslant b$, where $0 \leqslant a \leqslant b \leqslant \delta$,*

$$\int_a^b \theta(x)\,df(x) \leqslant C \int_a^b \theta(x)\,dx^\alpha.$$

There is also a corresponding result for lower integrals, and for integrals if these exist.

Consider first the case $a > 0$. Divide the range of integration into n parts by dividing-points $x_0 = a < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$. Let M_i be the upper bound of $\theta(x)$ in $x_{i-1} \leqslant x \leqslant x_i$. Then, provided that $x_i - x_{i-1} \leqslant \delta x_{i-1}$,

$$(4.1) \qquad \sum_{i=1}^{n} M_i\{f(x_i) - f(x_{i-1})\} \leqslant \sum_{i=1}^{n} M_i\{Cx_i^\alpha - Cx_{i-1}^\alpha\}.$$

It follows that, if $a > 0$, (4.1) holds for every approximative sum for which $x_i - x_{i-1} \leqslant \delta a$. We must therefore have the result for upper integrals when $a > 0$.

Since C is independent of a, we can let $a \to 0$ in this last result to get the case in which $a = 0$.

The proof for lower integrals differs only by using lower bounds in place of the upper bounds M_i.

Examples of continuous increasing solutions of $f(xy) \leqslant f(x)f(y)$.

5.1. If $\alpha > 0$, $\beta > 0$, then $x^\alpha + x^\beta$ is obviously such a solution. It also vanishes at the origin, is strictly increasing, and tends to infinity with x.

5.2. Consider the function

$$f(x) = ax^\alpha + bx^\beta \left| \sin\left(x + \frac{1}{x}\right) \right| + cx^\gamma$$

$$(\alpha \geqslant \beta \geqslant \gamma > 0, \quad a \geqslant 1, \quad c \geqslant 1).$$

Obviously $f(0) = 0$, $f(x) \to \infty$ with x and $f(x)$ is continuous everywhere.

When $(x+x^{-1})$ is not a multiple of π, we have

$$f'(x) = \alpha a x^{\alpha-1} + \beta b x^{\beta-1} |\sin(x+x^{-1})| \pm b x^{\beta}(1-x^{-2})\cos(x+x^{-1}) + \gamma c x^{\gamma-1}.$$

If we make this positive, $f(x)$ will be an increasing function. This will certainly be the case if

$$\alpha a x^{\alpha-1} + c\gamma x^{\gamma-1} \geqslant b(x^{\beta}+x^{\beta-2}).$$

This will certainly be satisfied for $x \leqslant 1$ if $\gamma-1 \leqslant \beta-2$ and $c\gamma \geqslant 2b$; and for $x \geqslant 1$ if $\alpha-1 \geqslant \beta$ and $a\alpha \geqslant 2b$.

Next

$$f(x)f(y)-f(xy) \geqslant ac(x^{\alpha}y^{\gamma}+x^{\gamma}y^{\alpha}) - bx^{\beta}y^{\beta}\left|\sin\left(xy+\frac{1}{xy}\right)\right|.$$

This is positive or zero if

$$bx^{\beta}y^{\beta} \leqslant ac(x^{\alpha}y^{\gamma}+x^{\gamma}y^{\alpha}),$$

i.e. if

$$b \leqslant ac(x^{\alpha-\beta}y^{\gamma-\beta}+y^{\alpha-\beta}x^{\gamma-\beta}).$$

Take $\alpha-\beta = \beta-\gamma$; then the expression in brackets is a function of x/y which is never less than 2.

It follows that $f(x)$ is a continuous solution which increases from zero to infinity with x if

$$\alpha-1 \geqslant \beta \geqslant \gamma+1 > 1 \quad \{\beta = \tfrac{1}{2}(\alpha+\gamma)\},$$

$$b \leqslant \operatorname{Min}(\tfrac{1}{2}a\alpha,\ \tfrac{1}{2}c\gamma,\ 2ac) \quad (a \geqslant 1,\ c \geqslant 1).$$

5.3. Consider the function

$$f(x) = x^{\alpha}(a+|\sin\log x|) \quad \text{where} \quad a > 0, \quad \alpha > 0.$$

A similar argument shows that this is a continuous solution which increases from zero to infinity with x if

$$\alpha \geqslant 1, \quad 2a \geqslant 1+\sqrt{5}.$$

5.4. Similarly, the function defined by

$$f(x) = ax^{\alpha}+bx^{\beta}|\sin\log x| \quad (x \geqslant 1)$$

$$= ax^{\gamma}+bx^{\beta}|\sin\log x| \quad (x \leqslant 1),$$

is a continuous solution which increases from zero to infinity with x if

$$a \geqslant \beta \geqslant \gamma > 0, \quad a\alpha > b, \quad a\gamma > b, \quad b > 0, \quad 2a \geqslant 1+\sqrt{(4b+1)}.$$

6. It will be observed that as x tends to zero (or to infinity) in each of the above examples $f(x)$ lies between two multiples of a power of x. The following theorem shows that little more than this can occur. In it we consider the inequality $g(x+y) \leqslant g(x)+g(y)$ which is obtained from $f(xy) \leqslant f(x)f(y)$ by taking logarithms and changing the variables.

THEOREM VIII*. *If $g(x)$ satisfies $g(x+y) \leqslant g(x)+g(y)$ and is monotonic for sufficiently large positive values of x, then $g(x)/x$ tends to a finite limit as $x \to +\infty$.*

If $g(x)$ is monotonic for sufficiently large negative values of x, $g(x)/x$ tends to a finite limit as $x \to -\infty$.

Let $g(x)$ be monotonic for $x > x_0 > 0$. Let m, x be any positive integers. Then

$$g(mx+n) \leqslant mg(x)+ng(1).$$

We can find a closed interval I of length m^{-1} in which $g(x)$ is bounded. Let $n \to \infty$ while m is fixed and x varies in I in any way. Then

$$\xi = mx+n \to +\infty$$

arbitrarily, and so

$$(6.1) \qquad a = \overline{\lim_{\xi \to +\infty}} \left\{\frac{g(\xi)}{\xi}\right\} \leqslant \lim_{n \to \infty} \left\{\frac{mg(x)+ng(1)}{mx+n}\right\} = g(1).$$

Again, if x is positive and m is a positive integer,

$$\frac{g(mx)}{mx} \leqslant \frac{mg(x)}{mx},$$

and so

$$(6.2) \qquad \frac{g(x)}{x} \geqslant \overline{\lim_{m \to \infty}} \frac{g(mx)}{mx}.$$

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* Whenever we consider $g(x+y) \leqslant g(x)+g(y)$ the function g and its argument may take any real values (positive or negative).

We prove that the upper limit in (6.2) is a. Let $\xi \to +\infty$ in any way and define a positive integer μ by

$$\mu - 1 < \xi/x \leqslant \mu.$$

If $g(x)$ increases for $x > x_0$, we have

$$\frac{g(\xi)}{\xi} \leqslant \frac{g(\mu x)}{\mu x} \Big/ \frac{\mu - 1}{\mu},$$

and, if $g(x)$ decreases,

$$\frac{g(\xi)}{\xi} \leqslant \frac{g\{(\mu-1)x\}}{(\mu-1)x}.$$

In either case it follows that

$$\overline{\lim_{\xi \to +\infty}} \frac{g(\xi)}{\xi} \leqslant \overline{\lim_{m \to +\infty}} \frac{g(mx)}{mx} \quad (x > 0). \tag{6.3}$$

Since the range of values of ξ includes that of mx, we must have equality in (6.3) and, combining this with (6.2) and (6.1), we find that

$$\frac{g(x)}{x} \geqslant a = \overline{\lim_{x \to +\infty}} \frac{g(x)}{x} \quad (x > 0).$$

Hence as $x \to +\infty$, $g(x)/x \to a$. Theorem VI shows that $a \neq -\infty$.

A similar argument applies to the case in which $x \to -\infty$ and $g(x)$ is monotonic for $x < x_1 < 0$.

Discontinuous solutions of $g(x+y) \leqslant g(x)+g(y)$.

7. Solutions which are everywhere discontinuous and whose oscillation at each point is infinite can be constructed from the known continuous solutions on the lines used by Hamel for the equation $f(x+y) = f(x)+f(y)$*. This, however, depends on the multiplicative axiom.

Unlike the equation $f(x+y) = f(x)+f(y)$, our inequality has discontinuous solutions which can be constructed without the aid of the

* G. Hamel, "Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung $f(x+y) = f(x)+f(y)$", *Mat Annalen*, 60 (1905), 459–462.

multiplicative axiom. As an example we can take*

$$g(x) = m+k \quad \text{when} \quad m \leqslant x < m+1,$$

where m is an integer.

If $n \leqslant y < n+1$, we shall have

$$g(x+y) = m+n+k \quad \text{or} \quad m+n+1+k,$$

and so $g(x)$ will satisfy $g(x+y) \leqslant g(x)+g(y)$ if $k \geqslant 1$.

8. THEOREM IX. *Every odd measurable function which satisfies*

$$g(x+y) \leqslant g(x)+g(y) \tag{8.1}$$

is of the form kx, where k is a constant.

Changing the signs of x, y, in (8.1) and combining the result with (8.1) we see that, if $g(x)$ is an odd solution,

$$g(x+y) = g(x)+g(y).$$

The result follows at once from the corresponding result for this equation†.

THEOREM X. *Every even solution of* (8.1) *is nowhere negative.*

Taking $x = 0$ in (8.1), we see that $g(0) \geqslant 0$. Then, if $g(x)$ is even,

$$2g(x) = g(x)+g(-x) \geqslant g(0) \geqslant 0.$$

We may finally observe that, unlike the equation $f(x+y) = f(x)+f(y)$, the inequality (8.1) has discontinuous measurable solutions: the example in § 7 is of this type.

9. *Note on the Cauchy-Hölder inequality, added* 1 *January*, 1927.— We use the notation of § 2. It can be shown that the obvious extensions

* Mr. A. E. Ingham suggested this example.

† W. Sierpinski, "Sur l'équation fonctionnelle $f(x+y) = f(x)+f(y)$", *Fundamenta Math.*, 1 (1920), 116–122.

of this inequality, namely

$$\frac{1}{n}\sum_1^n a_n b_n \leqslant \Phi^{-1}\left\{\frac{1}{n}\sum_1^n \Phi(a_n)\right\}\ \Psi^{-1}\left\{\frac{1}{n}\sum_1^n \Psi(b_n)\right\},$$

where either $\Phi(x) = \int_0^x \phi(x)\,dx, \quad \Psi(x) = \int_0^x \psi(x)\,dx,$

or $\Phi(x) = x\phi(x), \quad \Psi(x) = x\psi(x),$

and $\phi(x)$, $\psi(x)$ are differentiable everywhere, are only true in general when $\phi(x)$, $\psi(x)$ are powers of x*. We can, however, obtain an extension of the inequality under the same conditions that we used in Theorem I to extend its converse.

Let $\phi(x)$ be convex in $x \geqslant 0$ in addition to the conditions laid down in § 2. Let $x = \theta(y)$ be the function inverse to $y = x/\psi(x)$. We have

$$\phi\left(\frac{\Sigma a_\nu x_\nu}{\Sigma a_\nu}\right) \leqslant \frac{\Sigma a_\nu \phi(x_\nu)}{\Sigma a_\nu},$$

or, writing $b_\nu = a_\nu \phi(x_\nu)$ and substituting for x_ν,

$$\Sigma a_\nu \psi\left(\frac{b_\nu}{a_\nu}\right) \leqslant \Sigma a_\nu \,.\, \psi\left(\frac{\Sigma b_\nu}{\Sigma a_\nu}\right).$$

If $p(x)$, $q(x)$ are measurable functions defined in a measurable set E, we have

$$\int_E p(x)\,\psi\left\{\frac{q(x)}{p(x)}\right\} dx \leqslant \int_E p(x)\,dx \,.\, \psi\left\{\int_E q(x)\,dx \Big/ \int_E p(x)\,dx\right\}.$$

Now write $g(x) = \psi\{q(x)\}$, and assume the condition

(9 . 1) $$\psi(xy) \leqslant \psi(x)\psi(y).$$

The last inequality then becomes

$$\int \frac{p(x)}{\psi\{p(x)\}}\, g(x)\,dx \leqslant \int p(x)\,dx \,.\, \psi\left[\int \phi\{g(x)\}\,dx \Big/ \int p(x)\,dx\right].$$

* Mr. J. E. Littlewood showed me a proof for the case $\Phi(x) = x\phi(x)$, $\Psi(x) = x\psi(x)$, and I constructed a simple one for the other case.

Writing $p(x)/\psi\{p(x)\} = f(x)$, we find

$$\int f(x)\,g(x)\,dx \leqslant \int \theta\{f(x)\}\,dx \,.\, \psi\Big[\int \phi\{g(x)\}\,dx \Big/ \int \theta\{f(x)\}\,dx\Big]$$

$$\leqslant \int \theta\{f(x)\}\,dx \,.\, \psi\Big[1\Big/\int \theta\{f(x)\}\,dx\Big]\,\psi\Big[\int \phi\{g(x)\}\,dx\Big].$$

Now if $z = y\psi\left(\frac{1}{y}\right)$, then $\frac{1}{z} = \frac{1}{y}\Big/\psi\left(\frac{1}{y}\right)$,

and so $$\frac{1}{y} = \theta\left(\frac{1}{z}\right).$$

Hence $$\int f(x)\,g(x)\,dx \leqslant \frac{\psi\Big[\int \phi\{g(x)\}\,dx\Big]}{\theta^{-1}\Big[1\Big/\int \theta\{f(x)\}\,dx\Big]},$$

which may be regarded as a generalization of

$$\int f(x)\,g(x)\,dx \leqslant \Big[\int \{f(x)\}^p\,dx\Big]^{1/p}\Big[\int \{g(x)\}^q\,dx\Big]^{1/q} \quad (p^{-1}+q^{-1}=1),$$

to which it reduces when $\phi(x) = x^q$.

PROPERTIES OF FOURIER SERIES OF ALMOST PERIODIC FUNCTIONS

By S. BOCHNER.

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Part I. Introduction.
,, II. Tests of convergence and summability.
,, III. Operations with Fourier series.
,, IV. Approximation by trigonometric polynomials

PART I.

Introduction.

1. The object of the present paper* is a discussion of Fourier series of almost periodic† functions‡. We are particularly concerned with the question how far the known properties of Fourier series of p.p. functions, which, of course, are special cases of a.p. functions, involve the decisive hypothesis of pure periodicity.

We shall see that, apart from "existence" theorems§, the standard properties of Fourier series hold for general a.p. functions under suitable conditions. In most cases these conditions are seriously restrictive,

* An abstract appeared in *Journal London Math. Soc.*, 1 (1926), 131–133.

† We shall use the abbreviations: a.p. = almost periodic, p.p. = pure periodic.

‡ H. Bohr, *Acta Math.*, 45 (1924), 29–127; 46 (1925), 101–214.

§ These are theorems which establish the existence of an a.p. function whose Fourier series is a given trigonometric series. The latter series is more or less arbitrary, as in the theorems of Riesz-Fischer and its generalization by Hausdorff, or is related to another Fourier series. For references compare E. Hilb and M. Riesz, *Encyk. der Math. Wiss.* II C 10, 1211–13 and 1215–17. The theorems of Part III are also existence theorems, but there the given trigonometric series are very closely connected with Fourier series. We also note that an extension of the Riesz-Fischer theorem was given by N. Wiener, *Math. Zeitschrift*, 24 (1926), 575-616, Theorem XXII.

because they bear on the distribution of the Fourier exponents, but, though the conditions are always satisfied by p.p. functions, they do not demand pure periodicity, or other restrictions of the character of almost periodicity.

2. We write the Fourier series of (any) a.p. function $f(x)$ in the form

$$(1) \qquad f(x) \sim a_0 + \sum_{n=1}^{\infty} a_{i\lambda_n} e^{i\lambda_n x} + a_{-\lambda_n} e^{-i\lambda_n x},$$

where the exponents λ_n are positive. We say that $f(x)$ has an *interval* (μ, ν), if none of the λ_n lie in the interval $(0 \leqslant) \mu < \lambda \leqslant \nu$, and we call $l = \nu - \mu$ the length of the interval. Now, our general hypothesis will be that $f(x)$ has an infinite number of intervals, the length of each being not less than unity*, in which case we call it a *regular* function. For some special cases only we shall require that $f(x)$ be *strictly regular*; by this we shall mean that any two exponents have a distance not less than unity. We note that the conclusions of Part IV do not require any "general" hypothesis, and that the conclusions of Part III are based on assumptions which are less restrictive than regularity. These assumptions affect rather the nature of the function itself than the distribution of its Fourier exponents.

3. If in (1) the exponents are bounded,

$$(2) \qquad 0 < \lambda_n \leqslant K,$$

we call $f(x)$ a *quasi-polynomial*. Quasi-polynomials behave in different respects like polynomials. They are *analytic* a.p. functions†, and have, *a fortiori*, (uniformly continuous) a.p. derivatives of any finite order.

If $f(x)$ has an interval (μ, ν), say of length greater than or equal to 1, then there exists‡ a quasi-polynomial with the Fourier series

$$a_0 + \sum_{\lambda_n \leqslant \mu} a_{\lambda_n} e^{i\lambda_n x} + a_{-\lambda_n} e^{-i\lambda_n x}.$$

* Of course, we could replace the number 1 by any number greater than 0. We note that we do not need any assumption about the density of the exponents outside the intervals.

† H. Bohr, *Comptes rendus*, 181 (1925), 766. In this note only uniformly continuous (compare § 4) a.p. functions are considered; but we shall easily deduce in Part II, § 3 that any quasi-polynomial of the class L (compare § 4) is uniformly continuous.

‡ Compare Part II, § 3.

We call it a *partial sum* of $f(x)$ and denote it by

$$S_\mu(x) = S_\mu(f;\ x);$$

and, in accordance with § 2, we shall speak of a regular or strictly regular sequence

$$S_{\mu_\nu} \quad (\mu_1 < \mu_2 < \mu_3 \ldots \to \infty), \tag{3}$$

of partial sums of $f(x)$.

4. We shall call an a.p. function as defined by H. Bohr a *continuous** a.p. function, as distinguished from (more general) a.p. functions of any class L^σ, $\sigma \geqslant 1$, in accordance with the generalization of W. Stepanoff†. We shall here give a definition of these functions, and for other particulars we refer to the memoir of Stepanoff.

A sequence of integrable functions

$$f_1(x), \quad f_2(x), \quad f_3(x), \quad \ldots \tag{4}$$

is *quasi-convergent* [or quasi-convergent to $f(x)$] of the class L^σ, $\sigma \geqslant 1$, if either

$$\int_\xi^{\xi+1} |f_m(x) - f_n(x)|^\sigma dx \leqslant \epsilon \tag{5}$$

$$[m,\ n \geqslant N(\epsilon);\ -\infty < \xi < +\infty]$$

or

$$\int_\xi^{\xi+1} |f(x) - f_n(x)|^\sigma dx \leqslant \epsilon \tag{6}$$

$$[n \geqslant N(\epsilon);\ -\infty < \xi < +\infty].$$

From general principles any relation (5) involves the existence of a function $f(x)$ satisfying (6). We call $f(x)$ an a.p. function of the class L^σ, if there exists a sequence of *continuous* a.p. functions $f_\nu(x)$ satisfying (6)‡. We add only that every a.p. function of a class L^σ is *quasi-bounded* and

* Strictly speaking, these functions are *uniformly* continuous.

† W. Stepanoff, *Math. Annalen*, 95 (1926), 473–498. Stepanoff's classes (II) and (III) correspond, in our notation, to L^1 and L^2.

‡ For p.p. functions this definition is well known. For references, compare E. Hilb and M. Riesz in *Encyk. der Math. Wiss.* II C 10, § 10, 1213. We shall, further on, quote this paper as "HR", and we refer to it for terminology and theorems about p.p. functions, with which we shall have to deal.

quasi-continuous of this class. The first means that

$$\lim_{\delta \to 0} \operatorname*{Max}_{-\infty < \xi < +\infty} \int_{\xi}^{\xi+\delta} |f(x)|^{\sigma} dx = 0 ; \tag{7}$$

the latter that

$$\lim_{\delta \to 0} \operatorname*{Max}_{-\infty < \xi < +\infty} \int_{\xi}^{\xi+1} |f(x)-f(x+\delta)|^{\sigma} dx = 0. \tag{8}$$

We shall write $f(x)$ for an a.p. function of the class L, with the series

$$a_0 + \sum_{\nu=1}^{\infty} I_{\lambda_\nu}, \tag{9}$$

where

$$I_\lambda = a_\lambda e^{i\lambda x} + a_{-\lambda} e^{-i\lambda x},$$

and

$$\phi(t) = \phi_f(x, t) = f(x+2t)+f(x-2t)-2S \tag{10}$$

for a function of t, in which the value of the constant S is evident from the context and which is, in particular, equal to

$$\lim_{h \to 0} \frac{1}{2h} \int_{-h}^{h} f(x+2t)\, dt,$$

whenever this limit exists.

Furthermore, we shall write

$$\omega_0(f) = \operatorname*{Max}_{-\infty < \xi < +\infty} |f(\xi)|,$$

$$\Omega_0^{\sigma}(f) = \operatorname*{Max}_{-\infty < \xi < +\infty} \left\{ \int_{\xi}^{\xi+1} |f(x)|^{\sigma} dx \right\}^{1/\sigma},$$

$$\Delta_1\{f(x);\ x;\ \delta_1\} = f(x+\delta_1)-f(x),$$

$$\Delta_\nu(f;\ x;\ \delta_1, \delta_2, \ldots, \delta_\nu) = \Delta_{\nu-1}\{f(x+\delta_\nu)-f(x);\ x;\ \delta_1, \delta_2, \ldots, \delta_{\nu-1})\},$$

$$\omega_\nu(f;\ \delta) = \operatorname*{Max}_{-\infty < \xi < +\infty;\ |\delta_\rho| \leq \delta} |\Delta_\nu(f;\ \xi;\ \delta_1, \delta_2, \ldots, \delta_\nu)|,$$

$$\Omega_\nu^{\sigma}(f;\ \delta) = \operatorname*{Max}_{-\infty < \xi < +\infty;\ |\delta_\rho| \leq \delta} \left\{ \int_{\xi}^{\xi+1} |\Delta_\nu(f;\ x;\ \delta_1, \delta_2, \ldots, \delta_\nu)|^{\sigma} dx \right\}^{1/\sigma};$$

and where the index σ is omitted or the class not mentioned, it is tacitly supposed that $\sigma = 1$.

PART II.

Tests of convergence and summability.

1. We start from the fact that, for any $\lambda > 0$, the expression

$$S_\lambda^{(2)}(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x+2t)\,\frac{\sin^2\lambda t}{\lambda t^2}\,dt \tag{12}$$

is a continuous a.p. function, with the expansion

$$S_\lambda^{(2)}(x) \sim a_0 + \sum_{\lambda_\nu \leqslant \lambda}\left(1-\frac{\lambda_\nu}{\lambda}\right) I_{\lambda_\nu}. \tag{12_1}$$

If $f(x)$ is a polynomial this follows immediately from the formulae*

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{i2\mu t}\,\frac{\sin^2\lambda t}{\lambda t^2}\,dt = \begin{cases} 1-(\mu/\lambda) & \text{for} \quad 0<\mu\leqslant\lambda \\ 0 & \text{,,} \quad 0<\lambda\leqslant\mu \end{cases}$$

If $f(x)$ is not a polynomial we quasi-approximate to it by a sequence of polynomials†

$$f^{(\kappa)}(x) \sim a_0^{(\kappa)} + \Sigma I_{\lambda_\mu}^{(\kappa)} \quad (I_\lambda^{(\kappa)} = a_\lambda^{(\kappa)} e^{i\lambda x} + a_{-\lambda}^{(\kappa)} e^{-i\lambda x}).$$

From the quasi-convergence follows

$$\lim_{\kappa\to\infty} a_{\pm\lambda}^{(\kappa)} = a_{\pm\lambda},$$

i.e.

$$\lim_{\kappa\to\infty} I_\lambda^{(\kappa)} = I_\lambda.$$

For any $\lambda > 0$, in the sequence

$$S_\lambda^{(2)}(f_\kappa;\, x) \sim a_0^{(\kappa)} + \sum_{\lambda_\nu\leqslant\lambda}\left(1-\frac{\lambda_\nu}{\lambda}\right) I_{\lambda_\nu}^{(\kappa)} \quad (\kappa = 1, 2, 3, \ldots),$$

the functions on the left converge uniformly to $S_\lambda^{(2)}(f;\, x)$, since $\sin^2\lambda t/\lambda t^2 = O(1/t^2)$, whereas the series on the right converge term by term to the series in (12_1); thus the relation (12_1) is established for a general $f(x)$. Every $S_\lambda^{(2)}(x)$ is therefore analytic (Part I, § 3).

Applying two theorems of Bohr‡ for continuous a.p. functions to the functions $S_\lambda^{(2)}$, we obtain the following extension of these theorems to

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* de la Vallée Poussin, *Leçons sur l'approximation* (1919), 31; W. H. Young, *Quart. J. of Math.*, 48 (1911), 161–177.

† We denote by λ_ν ($\nu = 1, 2, \ldots$), a sequence of the exponents of all functions

$$f(x),\ f^{(1)}(x),\ f^{(2)}(x),\ f^{(3)}(x),\ \ldots.$$

‡ H. Bohr, *Math. Zeitschrift*, 23 (1925), 38–44; also S. Bochner, *Math. Annalen*, 96 (1926), 119–147.

functions of the class L. If, for a given $\Lambda > 0$, either all exponents $\lambda_n \leqslant \Lambda$ are linearly independent or the corresponding a_{λ_n} and $a_{-\lambda_n}$ are positive, then, for any $\epsilon > 0$, the sum $\sum_{\lambda_n \leqslant \Lambda - \epsilon} |a_{\lambda_\nu}| + |a_{-\lambda_\nu}|$ is convergent.

2. We call the Fourier series of $f(x)$ summable $(R\,;\,1)$—the letter "R" indicating that our quasi-polynomials $S_\lambda^{(2)}(x)$ are "means" in the sense of M. Riesz—at x to the sum S, if, for any $\epsilon > 0$,

$$|S_\lambda^{(2)}(x) - S| \leqslant \epsilon \quad [\lambda \geqslant \Lambda(\epsilon)],$$

where $\Lambda(\epsilon)$ is a finite index depending on ϵ. All known convergence tests for Fejér means in the case of a p.p. function $f(x)$ can be derived from the expression (12), without use of their pure periodicity, and thus hold in our general case. *E.g.*, if $f(x)$ is continuous, its Fourier series is uniformly summable $(R\,;\,1)$; and for any $f(x)$ it is summable at x to the sum S if

$$\lim_{h \to 0} \frac{1}{h} \int_0^h |\phi_f(x, t)|\, dt = 0.$$

More generally, for any $\delta > 0$, there exist the means $(R\,;\,\delta)$

$$S_\lambda^{(1+\delta)} \sim a_0 + \sum_{\lambda_n \leqslant \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^\delta I_{\lambda_n},$$

connected with $f(x)$ by *

$$S_\lambda^{(1+\delta)}(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f\left(x + \frac{t}{\lambda}\right) P_{1+\delta}(t)\, dt,$$

where $P_{1+\delta}(t) = P_{1+\delta}(-t)$,

$$P_{1+\delta}(t) = \begin{cases} O(1/t^{1+\delta}) & \text{for } 0 < \delta \leqslant 1 \\ O(1/t^2) & \text{,, } \delta \geqslant 1 \end{cases}$$

and $$\frac{d}{dt}\{t^{2+\delta} P_{2+\delta}(t)\} = t^{1+\delta} \cdot P_{1+\delta}(t).$$

It is still true† that the Fourier series of $f(x)$ is summable $(R,\ \delta)$ at x if

$$\lim_{h \to 0} \frac{1}{h} \int_0^h |\phi(x, t)|\, dt = 0,$$

* W. H. Young, *loc. cit.*

† HR., 1206–7.

and summable $(R\,;\,1+\delta)$ if

$$\lim_{h\to 0}\frac{1}{h}\int_0^h \phi(x,\,t)\,dt = 0.$$

Moreover, we again have the following theorems of Hardy and Littlewood*. The necessary and sufficient condition that the Fourier series of $f(x)$ should be summable (R) for $t = x$ is that there should be a number $k \geqslant 1$ such that

$$\phi_\kappa(t) = \frac{\kappa}{t^\kappa}\int_0^t (t-u)^{\kappa-1}\phi(u)\,du \to 0,$$

when $t \to 0$. If $f(x)$ is bounded in an interval including $t = x$, then its series is either summable by every R-mean or by no R-mean, of positive order. The necessary and sufficient condition that it should be summable is that $\phi_1(t) \to 0$ when $t \to 0$.

3. Let us suppose, now, that $f(x)$ has an interval $(\mu,\ \mu+1)$. The relation

$$(\mu+1)\,S^{(2)}_{\mu+1} - \mu S^{(2)}_{\mu} = \sum_{\lambda_\nu \leqslant \mu}\left\{(\mu+1)\left(1-\frac{\lambda_\nu}{\mu+1}\right)-\mu\left(1-\frac{\lambda_\nu}{\mu}\right)\right\} I_{\lambda_\nu}$$

$$= \sum_{\lambda_\nu \leqslant \mu} I_{\lambda_\nu}$$

asserts the existence and almost periodicity of the function

$$S_\mu(x) \sim a_0 + \sum_{\lambda_\nu \leqslant \mu} I_{\lambda_\nu},$$

which we call a "partial sum" of $f(x)$. Every partial sum, and therefore every quasi-polynomial, since it is its own partial sum, is analytic (Part I, §3). We have, by (12),

$$S_\mu = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x+2t)\,\frac{\sin^2(\mu+1)\,t - \sin^2\mu t}{t^2}\,dt$$

$$= \frac{1}{\pi}\int_0^\infty \{f(x+2t)+f(x-2t)\}\,\frac{\sin(2\mu+1)\,t}{t}\,\frac{\sin t}{t}\,dt,$$

$$\pi(S_\mu - S) = \int_0^\infty \phi(t)\,\frac{\sin t}{t}\,\frac{\sin(2\mu+1)\,t}{t}\,dt.$$

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* G. H. Hardy and J. E. Littlewood, *Math. Zeitschrift*, 19 (1923), 66–96.

Since $f(x)$ is quasi-bounded and quasi-continuous, it is easy to verify that

$$\lim_{\mu\to\infty}\int_{\epsilon}^{\infty}\phi(t)\,\frac{\sin t}{t}\,\frac{\sin(2\mu+1)t}{t}\,dt=0,$$

for every $\epsilon>0$, uniformly in x. Therefore the behaviour of any regular sequence, at x, depends only on the behaviour of $f(x)$ in the neighbourhood of x (Theorem of Riemann-Lebesgue).

From the inequality

$$\left|\int_0^{\epsilon}\phi(t)\,\frac{\sin t}{t}\,\frac{\sin(2\mu+1)t}{t}\,dt-\int_0^{\epsilon}\phi(t)\,\frac{\sin(2\mu+1)t}{t}\,dt\right|\leqslant K\epsilon\int_0^{\epsilon}|\phi(t)|\,dt$$

it follows immediately that all convergence tests for p.p. functions which are based on an analysis of

$$\int_0^{\epsilon}\phi(t)\,\frac{\sin\mu t}{t}\,dt$$

remain valid for any regular function $f(x)$. Such, for example, are the convergence tests of Dini, Dirichlet-Jordan, de la Vallée Poussin, and the most general test of Lebesgue, and the test of Dini-Lipschitz for uniform convergence in partial intervals or in the whole.

For a strictly regular function, either if

$$|a_{\lambda_n}|+|a_{-\lambda_n}|=O\left(\frac{1}{\lambda_n}\right),$$

or if $\Sigma\lambda_n(|a_{\lambda_n}|^2+|a_{-\lambda_n}|^2)$ is convergent, a necessary and sufficient condition for convergence is the limit-relation

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\phi(t)\,dt=0;$$

another necessary and sufficient condition is summability (R), and yet another is summability by Poisson's method. This may be explained in the following way. The Poisson integral

$$F(s)=F(\sigma+it)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sigma f(t+\tau)\,d\tau}{\sigma^2+\tau^2}$$

represents for any a.p. $f(x)$, in the half-plane $\Re(s)>0$, a harmonic a.p.

function*, with the expansion

$$F(\sigma+ix) \sim a_0+\Sigma e^{-\sigma\lambda_\nu} I_{\lambda_\nu}.$$

We call $f(x)$ summable Poisson for $x = x_0$ if

$$\lim_{\sigma \to +0} F(\sigma+ix_0)$$

exists and is finite.

4. The theorems about absolute convergence may all be extended to strictly regular functions. Thus, if

$$\omega_1(f;\ \delta) = O(\delta^\alpha)$$

or if, more generally, for a suitable k, with $f(x)$ belonging to L^2,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\ \Delta_\kappa(f;\ t;\ \delta, \delta, \ldots, \delta)\ |^2 = O(\delta^{2\alpha}), \tag{13}$$

then the series $\qquad \Sigma\, |\ c_{\lambda_n}\ |^\beta \quad (|\ c_\lambda\ |^2 = |\ a_\lambda\ |^2 + |\ a_{-\lambda}\ |^2)$

is convergent for $\beta > 2/(2\alpha+1)$. The proof of Szász† can be adapted to our general case. Formally, the theorem can be given an enunciation which does not require any regularity of $f(x)$ at all. If we introduce the "average coefficient"

$$C_n^2 = \sum_{n \leqslant \lambda_\nu < n+1} |\ a_{\lambda_\nu}\ |^2 + |\ a_{-\lambda_\nu}\ |^2,$$

the series $\qquad \Sigma\, C_n^\beta$

is convergent for any $f(x)$ satisfying (13).

Finally, we remark that for a regular sequence we also have the theorem of Hardy-Littlewood (and generalizations) about "strong" summability, viz.

$$\lim_{n \to \infty} \frac{|\ s_{\lambda_1} - f\ |^2 + |\ s_{\lambda_2} - f\ |^2 + \ldots |\ s_{\lambda_n} - f\ |^2}{\lambda_n} = 0,$$

when $\qquad \lim_{h \to 0} \{f(x+h)+f(x-h)-2f(x)\} = 0.$

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* I. Favard, *Comptes rendus*, 182 (1926), 757.

† O. Szász, *Münchener Sitzungsberichte* (1922), 135–154.

PART III.

Operations with Fourier series.

1. As regards differentiation, it follows immediately from the mean-value theorem

$$\frac{f(x+h)-f(x)}{h} = f'(x+\theta h) \quad (|\theta| \leqslant 1),$$

that the derivative of an (absolutely) continuous a.p. function $f(x)$ is certainly almost periodic if it is uniformly continuous. More generally, if $f'(x)$ is quasi-continuous of the class σ, it is almost periodic of this class. The proof is as follows. We approximate to $f(x)$ by a sequence $S^{(z)}_{2\lambda}$,

$$\pi S^{(2)}_{2\lambda}(x) = \int_{-\infty}^{+\infty} f\left(x+\frac{t}{\lambda}\right)\frac{\sin^2 t}{t^2}\,dt$$

$$= \int_{-G}^{+G} + \int_{G}^{+\infty} + \int_{-\infty}^{G}$$

$$= M_\lambda + N_\lambda + L_\lambda.$$

We have

$$N'_\lambda(x) = \frac{d}{dx}\int_{G+x}^{+\infty} f(t)\frac{\sin^2\lambda(t-x)}{\lambda(t-x)^2}\,dt$$

$$= -f(G+x)\frac{\sin^2\lambda G}{\lambda G^2} + \int_{G+x}^{\infty} f(t)\,\frac{\sin 2\lambda(t-x)}{(t-x)^2}\,dt$$

$$-2\int_{G+x}^{\infty} f(t)\,\frac{\sin^2\lambda(t-x)}{\lambda(t-x)^3}\,dt,$$

and therefore $N'_\lambda(x)$, and similarly $L'_\lambda(x)$, tends to 0, as $G\to\infty$, uniformly in x and λ ($\geqslant 1$). But, for a fixed G, any sequence

$$M'_{\lambda_\nu}(x) \quad (\lambda_\nu \to \infty)$$

is quasi-convergent of the class σ, since

$$\pi|M'_\lambda - M'_\mu| \leqslant \int_{-G}^{G}\left|f'\left(x+\frac{t}{\lambda}\right)-f'\left(x+\frac{t}{\mu}\right)\right|\frac{\sin^2 t}{t^2}\,dt,$$

$$\Omega^\sigma_0(|M'_\lambda - M'_\mu|) \leqslant K\,\Omega^\sigma_1\left(f';\ \frac{G}{\lambda}+\frac{G}{\mu}\right);$$

and this proves the proposition. We add the remark that in the case of linearly independent exponents or positive coefficients, the almost periodicity (an d uniform continuity) of $f(x)$ follows from its boundedness.

2. As regards integration, we shall prove that $F(x) = \int_0^x f(t)\,dt$ is almost periodic if $a_0 = 0$ and the exponents λ_n have not the point 0 as a limit point, so that

$$\lambda_n \geqslant g > 0, \tag{14}$$

where g is independent of n*. We make the unessential assumption that $g \geqslant 4$. It is known† that the numbers p_ρ ($\rho \geqslant 0$) defined by

$$p_{\lambda/\mu} = \int_{-\infty}^{+\infty} \cos \lambda t \, \frac{(\sin \mu t)^4}{\mu^3 t^4}\, dt,$$

$$p_0 = \int_{-\infty}^{+\infty} \frac{(\sin \mu t)^4}{\mu^3 t^4}\, dt = \int_{-\infty}^{+\infty} \left(\frac{\sin t}{t}\right)^4 dt$$

vanish for $\rho \geqslant 4$ (and that $|p_\rho/p_0| \leqslant 1$). The functions

$$f_\lambda(x) = \frac{1}{p_0} \int_{-\infty}^{+\infty} f(x+t)\, \frac{(\sin \lambda t)^4}{\lambda^3 t^4}\, dt$$

$$\sim \frac{1}{p_0} \sum_{\lambda_n \leqslant 4\lambda} p_{\lambda_n} I_{\lambda_n}$$

are easily seen to be quasi-convergent to $f(x)$ when $\lambda \to \infty$. If we write

$$K_\lambda(\xi) = \frac{\sin^4 \lambda\xi}{p_0 \lambda^3 \xi^4},$$

$$L_\lambda(\xi) = K_\lambda(\xi) - K_1(\xi),$$

$$M_\lambda(\xi) = \int_{-\infty}^{\xi} K_\lambda(t)\, dt,$$

$$D_\lambda(\xi) = M_\lambda(\xi) - M_1(\xi),$$

there exists an absolute constant C, such that for $\lambda \geqslant 1$,

$$|D_\lambda(t)| \leqslant C \,.\, \mathrm{Min}\left(1, \frac{1}{|t|^3}\right), \tag{15}$$

From $D_\lambda(0) = 0$ it follows that

$$|D_\lambda(t)| \leqslant 2$$

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* This theorem was proved by H. Bohr, *Acta Math.*, 47 (1926), 237–281, for analytic a.p. functions in the interior of their strips of existence.

† de la Vallée Poussin, *loc. cit.*, 44.

for $|t| \leqslant 1$; for $t \leqslant -1$ we have

$$|D_\lambda(t)| \leqslant 2M_1(|t|) \leqslant \frac{1}{p_0|t|^3}$$

since

$$|M_\lambda(t)| \leqslant \int_{|t|}^{\infty} \frac{(\sin \lambda\xi)^4}{p_0 \lambda^3 \xi^4}\, d\xi \leqslant M_1(t);$$

and in the case of $t > 1$ we have only to observe* that

$$-D_\lambda(t) = \int_t^\infty K_\lambda(\xi)\, d\xi - \int_t^\infty K_1(\xi)\, d\xi,$$

and, therefore,

$$|D_\lambda(t)| \leqslant \frac{1}{p_0 t^3},$$

whereby (15) is proved.

From our hypothesis (14) it follows that

$$f_1(x) \equiv 0,$$

and therefore

$$f_\lambda(x) = f_\lambda(x) - f_1(x) = \int_{-\infty}^{+\infty} f(t)\, L_\lambda(t-x)\, dt,$$

$$\int_0^x f_\lambda(\xi)\, d\xi = \int_{-\infty}^{+\infty} dt\, f(t) \left[\int_0^x L_\lambda(t-\xi)\, d\xi\right]$$

$$= -\int_{-\infty}^{+\infty} dt\, f(t) \left[\int_t^{t-x} L_\lambda(\xi)\, d\xi\right],$$

and in consequence of (15)

$$-\int_0^x f_\lambda(\xi)\, d\xi = \int_{-\infty}^{+\infty} f(t)\, D_\lambda(t-x)\, dt - \int_{-\infty}^{+\infty} f(t)\, D_\lambda(t)\, dt$$

$$= \int_{-\infty}^{+\infty} f(x+t)\, D_\lambda(t)\, dt - \int_{-\infty}^{+\infty} f(t)\, D_\lambda(t)\, dt.$$

The latter formula shows that, for $\lambda \geqslant 1$, the function $\int_0^x f_\lambda(t)\, dt$ is almost periodic.

The almost periodicity of $\int_0^x f(x)\, dx$ itself follows now from a theorem of H. Bohr† applied to

$$\phi_m(x) = \int_0^x f_m(t)\, dt.$$

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* Where we need the assumption (14).

† *Acta Math.*, 46 (1925), 110, "Konvergenzhilfssatz".

Strictly speaking we have proved that every a.p. function becomes almost periodically integrable after the removal of any $f_\lambda(x)$, $\lambda > 0$. Thus, if, for $\Lambda > 0$, all λ_n which do not exceed Λ are linearly independent or if the corresponding coefficients are positive, for the almost periodicity of $F(x)$ it is not only sufficient but also necessary that

$$\sum_{\lambda_\nu < \Lambda'} \frac{|a_{\lambda_\nu}| + |a_{-\lambda_\nu}|}{\lambda_\nu}$$

should be finite for any Λ' which is less than Λ.

3. We turn now to the question of the existence of an a.p. function conjugate to a given a.p. function. If two a.p. functions $f(x)$ and $\overline{f(x)}$ have the expansions

$$f(x) \sim \Sigma(a_{\lambda_n} e^{i\lambda_n x} + a_{-\lambda_n} e^{-i\lambda_n x}),$$

$$\overline{f(x)} \sim i\Sigma(a_{\lambda_n} e^{i\lambda_n x} - a_{-\lambda_n} e^{-i\lambda_n x})^*,$$

then we call $\overline{f(x)}$ the allied function of $f(x)$. Every p.p. function $f(x)$ of a class L^σ $(\sigma > 1)$ has an allied function, belonging to the same class†, and a function of the class L possesses a conjugate under very general conditions.

In the case of an a.p. function $f(x)$ these existence theorems cannot hold for a quite general $f(x)$. *E.g.* the function

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} \phi_n(\sigma_n x),$$

where

$$\phi_\kappa(t) = \frac{\sin t}{1} + \frac{\sin 2t}{2} + \ldots \frac{\sin \kappa t}{\kappa},$$

and the numbers $\sigma_1, \sigma_2, \ldots$ are linearly independent with $\sigma_n = O(1/n)$, has no allied function. The conjugate, if it existed, would have the Fourier series

$$\overline{f(x)} \sim \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} \psi_n(\sigma_n x),$$

where

$$\psi_\kappa(t) = \frac{\cos t}{1} + \frac{\cos 2t}{2} + \ldots + \frac{\cos \kappa t}{\kappa}.$$

* Or, what is the same, if they have the corresponding expansions

$$(A_n \cos \lambda_n x + B_n \sin \lambda_n x\,;\ \ B_n \cos \lambda_n x - A_n \sin \lambda_n x).$$

† M. Riesz, *Comptes rendus*, 178 (1924), 1464. The case $\sigma = 2$ follows directly from the Riesz-Fischer theorem.

Therefore, $\overline{f(x)}$ being a quasi-polynomial (and hence, *a fortiori*, continuous) and the σ_n being linearly independent, the series

$$\Sigma \frac{1}{n(\log n)^2} \operatorname{Max} |\psi_n(\sigma_n t)|$$

would be convergent*; but this is not possible since

$$\operatorname{Max} |\psi_n(\sigma_n t)| \sim \log n.$$

But the same existence theorems do hold if $f(x)$ is almost periodically integrable, which for a p.p. function is automatically satisfied. For a polynomial $p(x)$ the allied function is easily seen to be

$$\overline{p(x)} = \frac{1}{\pi} \int_0^\infty \frac{p(x+t)-p(x-t)}{t}\, dt.$$

Now a function of a class L^σ has an allied function, belonging to the same class, if, for a sequence of quasi-approximating polynomials $p_n(x)$, the functions

$$\overline{p_n}(x) = \frac{1}{\pi} \int_0^\infty \frac{p_n(x+t)-p_n(x-t)}{t}\, dt$$

are quasi-convergent of this class. Introducing an upper boundary H of $\left| \int_0^x f(t)\, dt \right|$, we obtain, by partial integration,

$$\left| \int_G^\infty \frac{f(x+t)-f(x-t)}{t}\, dt \right| \leqslant \frac{4H}{G} + 4H \int_G^\infty \frac{dt}{t^2}.$$

Therefore, for a sequence of Fejér polynomials† $p_n(x)$ of $f(x)$, we have

$$\lim_{G\to\infty} \int_G^\infty \frac{p_n(x+t)-p_n(x-t)}{t}\, dt = 0,$$

uniformly in x and n. But for every fixed G, writing

$$q_n(G;\ x) = \int_0^G \frac{p_n(x+t)-p_n(x-t)}{t}\, dt,$$

we have‡

$$\int_\xi^{\xi+G} |q_m(x)-q_n(x)|^\sigma dx \leqslant K_\sigma(G) \int_{\xi-3G}^{\xi+3G} |p_m(x)-p_n(x)|^\sigma dx,$$

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* S. Bochner, *loc. cit.*, Satz VIII.

† S. Bochner, *loc. cit.*, § 2.

‡ Compare E. C. Titchmarsh, *Math. Zeitschrift*, 25 (1926), 321.

where $\sigma > 1$ denotes the class to which $f(x)$ belongs and $K_\sigma(G)$ is an absolute constant. Therefore, for $\sigma > 1$, $\overline{f(x)}$ exists and is of the class L^σ; and, further, it can be put into the form

$$\overline{f(x)} = \frac{1}{\pi} \int_0^\infty \frac{f(x+t)-f(x-t)}{t}\, dt.$$

A general function of the class L has not necessarily an allied function, even if it is pure periodic; therefore we need some additional assumption about $f(x)$. From

$$\int_\xi^{\xi+1} |q_n(\epsilon, x)|\, dx \leqslant \int_0^\epsilon \frac{\Omega_1(f;\ t)}{t}\, dt$$

it follows that $\overline{f(x)}$ exists if $f(x)$ satisfies a *quasi-Lipschitz condition*

$$\Omega_1(t) = O(t^\alpha) \quad (\alpha > 0),$$

or, *e.g.*, if only $$\Omega_1(t) = O\left(\frac{1}{\log (1/t)^2}\right).$$

The function $\overline{f(x)}$ is continuous if $f(x)$ satisfies a Lipschitz condition, or even if

$$\lim_{t\to 0} \int_0^t \frac{\omega_1(\tau)}{\tau}\, d\tau = 0.$$

In addition, we have the theorem of Fatou-Privaloff* that from

$$\omega_f(\delta) = O(\delta^\alpha)$$

it follows that $$\omega_{\bar{f}}(\delta) = O(\delta^\alpha) \qquad \text{if} \quad \alpha < 1,$$

$$= O\left(\frac{\delta}{\log (1/\delta)}\right) \quad \text{if} \quad \alpha = 1.$$

4. Let us now assume that $f(x)$ has an a.p. integral, and that it has an interval $(\mu, \mu+1)$.

For any ξ, the function

$$f(\xi+x) \cos (\mu+\tfrac{1}{2})\, x,$$

as a function of x, has all its exponents greater than or equal to $\frac{1}{2}$. Therefore, by § 2, its integral

$$\int_0^x f(\xi+t) \cos (\mu+\tfrac{1}{2})\, t\, dt$$

* HR, p. 1213.

is almost periodic and has an upper boundary independent of ξ and μ. From this it follows easily that the partial sum $\bar{s}_\mu(x)$ of the allied $\overline{f(x)}$ can be represented in the form

$$\bar{s}_\mu(x) = \frac{1}{\pi}\int_0^\infty \{f(x+t)-f(x-t)\}\left[\frac{1}{t}-\frac{\cos(\mu+\frac{1}{2})t}{t}\right]dt;$$

moreover, for any regular sequence $\bar{s}_{\mu_\nu}(x)$, the expression

$$\int_G^\infty \{f(x+t)-f(x-t)\}\left\{\frac{1}{t}-\frac{\cos(\mu+\frac{1}{2})t}{t}\right\}dt$$

tends to 0 for $G\to\infty$, uniformly in x and μ. And now it is very easy to deduce, as Lukács* did for p.p. functions, that where the limit

$$D_x = \lim_{h\to\infty}\{f(x+h)-f(x-h)\},$$

or, more generally, the limit

$$\lim_{h\to 0}\frac{1}{h}\int_0^h |f(x+t)-f(x-t)-D_x|\,dt = 0$$

exists, we have

$$\lim_{\nu\to\infty}\frac{\bar{s}_{\mu_\nu}(x)}{\log\mu_\nu} = \frac{D_x}{\pi}$$

PART IV.

Approximation by trigonometric polynomials.

In this last part we propose to point out in what form theorems about "best" approximation to p.p. functions by (trigonometric) polynomials hold in our general case.

1. To begin with, for any quasi-polynomial $p(x)$ of order λ, from the hypothesis that

$$|p(x)|\leqslant 1 \quad (-\infty<x<+\infty)$$

it follows that

$$|p'(x)|\leqslant\lambda \quad (-\infty<x<+\infty).$$

In order to prove this generalization of a well known theorem of S. Bern-

* HR, p. 1199.

stein, we may follow closely the proof of this theorem, as given by F. Riesz*, to which we refer the reader. We can write

$$p'(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} p(x+t)\, g(\lambda t)\, \frac{(\sin \frac{1}{2}\lambda t)^2}{(\sin \frac{1}{2}t)^2}\, dt,$$

where $g(t)$ is any function with period 2π whose expansion begins with

$$\sin t + (a_2 \cos 2t + b_2 \sin 2t) + \ldots,$$

and, as in the paper referred to, the substitution

$$g(u) = \sin u - r^2 \sin 3u + r^4 \sin 5u - r^6 \sin 7u + \ldots$$

gives, in the limit when $r \to 1$,

$$|\, p'(x)\,| \leqslant \lambda.$$

We add that, more generally, for any $\rho \geqslant 1$, from the inequality

$$\int_{\xi}^{\xi+1} |\, p(x)\,|^{\rho}\, dx \leqslant 1 \quad (-\infty < \xi < +\infty),$$

it follows that

$$\int_{\xi}^{\xi+1} |\, p'(x)\,|^{\rho}\, dx \leqslant \lambda.$$

2. We turn now to theorems of Jackson, S. Bernstein, and Lebesgue. The constants A_r, B_r, D_r, and $c^r_{\lambda,\mu}$ which we shall have to introduce are absolute constants depending only on their indices.

Using the quasi-polynomials $f_\lambda(x)$, introduced already in Part 3, §2, it is easily seen† that for any continuous a.p. function $f(x)$ and any λ, there exists a quasi-polynomial of order λ

$$P_{\lambda,1}(f;\ x) \sim a_0 + \sum_{\mu \leqslant \lambda} c^1_{\lambda,\mu} I_\mu \quad (c^1_{\lambda,\mu} = 0 \text{ if } \mu \geqslant \lambda),$$

such that

$$|\, f(x) - P_{\lambda,1}(x)\,| \leqslant A_1 \omega_1\Big(f;\ \frac{1}{\lambda}\Big). \tag{16}$$

In particular, for a function $g(x)$ with exponents greater than or equal to Λ, we have

$$|\, g(x)\,| \leqslant A_1 \omega_1\Big(g;\ \frac{1}{\Lambda}\Big). \tag{17}$$

* F. Riesz, *Comptes rendus*, 158 (1914), 1657.

† Compare the treatment of p.p. functions by de la Vallée Poussin, *loc. cit.*, 43.

From (16) and (17) we deduce by recurrence that, for

$$r = 1,\ 2,\ 3,\ \ldots,$$

there exist quasi-polynomials $P_{\lambda, r}(f;\ x)$ of order λ,

$$P_{\lambda, r} = a_0 + \sum_{\mu \leqq \lambda} c^r_{\lambda, \mu} I_\mu,$$

such that

$$|f(x) - P_{\lambda, r}(x)| \leqq A_r \omega_r\left(f;\ \frac{1}{\lambda}\right), \tag{18}$$

and that

$$|g(x)| \leqq A_r \omega_r\left(g;\ \frac{1}{\Lambda}\right). \tag{19}$$

It will be sufficient to deal only with the case $r = 2$. Writing

$$f(x) - P_{\lambda, 1}(f;\ x) = Q_{\lambda, 1}(f;\ x),$$

and applying (16) to the function $f(x) - f(x+\rho)$, where ρ is a constant, we obtain

$$|Q_{\lambda, 1}(f;\ x) - Q_{\lambda, 1}(f;\ x+\rho)| \leqq A_1 \omega_1\left(f(x) - f(x+\rho);\ \frac{1}{\lambda}\right).$$

Thus

$$\omega_1\left(Q_{\lambda, 1};\ \frac{1}{\lambda}\right) \leqq A_1 \omega_2\left(f;\ \frac{1}{\lambda}\right).$$

Another reference to (16) gives

$$\begin{aligned} |Q_{\lambda, 1}(f;\ x) - P_{\lambda, 1}(Q_{\lambda, 1};\ x)| &= |f(x) - \{P_{\lambda, 1} + P_{\lambda, 1}(Q_{\lambda, 1})\}| \\ &\leqq A_1 \omega_1\left(Q_{\lambda, 1};\ \frac{1}{\lambda}\right) \leqq A_1^2 \omega_2\left(f;\ \frac{1}{\lambda}\right). \end{aligned}$$

Thus, writing

$$P_{\lambda, 2} = P_{\lambda, 1} + P_{\lambda, 1}(Q_{\lambda, 1})$$

$$A_2 = A_1^2,$$

we obtain expressions $P_{\lambda, 2}$ and A_2, which have the required properties.

Generally, we obtain in this way

$$c^r_{\lambda, \mu} = 1 - (1 - c^1_{\lambda, \mu})^r, \quad A_r = A_1^r.$$

If, in particular, we assume that $f(x)$ has an a.p. k-th derivative, we obtain directly from (18),

$$|f(x) - P_{\lambda, \kappa+1}(f;\ x)| \leqq A_{\kappa+1} D_\kappa \frac{\omega_1(f^{(\kappa)};\ 1/\lambda)}{\lambda^\kappa},$$

and this is the form in which Jackson stated his theorem. But it is possible to state the general theorem (18) without even assuming that the function is continuous, *i.e.* our analysis gives, more exactly,

$$\Omega_1^\sigma(f-P_{\lambda_n}) \leqslant A_\sigma \Omega_1^\sigma\left(f;\ \frac{1}{\lambda}\right),$$

if $\sigma \geqslant 1$ is the class of the function $f(x)$.

3. Conversely*, in consequence of § 1, for any integer $r > 0$, if $f(x)$ can be approximated to by quasi-polynomials $p_{\lambda_1}, p_{\lambda_2}, \ldots$, of the respective orders $\lambda_1, \lambda_2, \ldots$, with $0 < \lambda_{n+1}-\lambda_n < g$, in the form

$$|f(x)-p_{\lambda_n}(x)| \leqslant \frac{\alpha(\lambda_n)}{\lambda_n^r},$$

where $$\frac{\alpha(\lambda)-\alpha(\mu)}{\lambda-\mu} \leqslant 0, \quad \int_0^\infty \frac{\alpha(\lambda)}{\lambda}\, d\lambda \leqslant K,$$

then $f(x)$ has a continuous a.p. r-th derivative.

4. Now let us assume that $f(x)$ has an interval $(\lambda_\alpha, \lambda_\beta)$, $l = \lambda_\beta - \lambda_\alpha$. Then we have

$$S_{\lambda_\alpha} = \frac{\lambda_\beta S_{\lambda_\beta}^{(2)} - \lambda_\alpha S_{\lambda_\alpha}^{(2)}}{\lambda_\beta - \lambda_\alpha} = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x+2t)\, \frac{\sin^2 \lambda_\beta t - \sin^2 \lambda_\alpha t}{lt^2}\, dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x+2t)\, \frac{\sin (2\lambda_\alpha + l)\, t \sin lt}{lt^2}\, dt;$$

and, if $|f(x)| \leqslant G$,

$$|S_{\lambda_\alpha}(x)| \leqslant \frac{G}{\pi}\int_{-\infty}^{+\infty} \left|\frac{\sin \{(2\lambda_\alpha + l)\, t/l\} \sin t}{t^2}\right| dt$$

$$\leqslant GB_0'\left(1+\log \frac{\lambda_\beta}{\lambda_\beta - \lambda_\alpha}\right),$$

$$|R_{\lambda_\alpha}(x)| = |f(x)-S_{\lambda_\alpha}(x)| \leqslant GB_0\left(1+\log \frac{\lambda_\beta}{\lambda_\beta - \lambda_\alpha}\right).$$

* de la Vallée Poussin, *loc. cit.*, 53.

Substituting $\Delta_r(f;\ x;\ \delta_1, \delta_2, \ldots, \delta_r)$, with $|\delta_r| \leqslant 1/\lambda_a$, for $f(x)$ and combining with (19), we obtain

$$(20)\quad |R_{\lambda_a}(f;\ x)| \leqslant A_r \omega_r\left(R_{\lambda_a};\ \frac{1}{\lambda_a}\right) \leqslant B_r\left(1+\log\frac{\lambda_\beta}{\lambda_\beta-\lambda_a}\right)\omega_r\left(f:\ \frac{1}{\lambda_a}\right)$$

$$(B_r = A_r B_0).$$

In particular, if $f(x)$ has a κ-th a.p. derivative, we obtain

$$|R_{\lambda_a}(f;\ x)| \leqslant B_{\kappa+1} D_\kappa \left(1+\log\frac{\lambda_\beta}{\lambda_\beta-\lambda_a}\right)\frac{\omega_1(f^{(\kappa)};\ 1/\lambda_a)}{\lambda_a^\kappa}.$$

For p.p. functions, with $\lambda_\beta-\lambda_a = 1$, this is a known theorem of Lebesgue. From (20) we can get the Lipschitz-Dini test for uniform convergence of a regular sequence. Furthermore, if *e.g.*, $\lambda_\nu = \nu^\kappa$, with $\kappa > 0$,

$$\log\frac{n^\kappa}{n^\kappa-(n-1)^\kappa} \sim \kappa \log n,$$

and thus the Lipschitz-Dini condition still holds, though, for $\kappa < 1$, the function is not "regular". And in Dirichlet's case $\lambda_n = \log n$,

$$\log\frac{\lambda_n}{\lambda_n-\lambda_{n-1}} \sim \log n = \lambda_n,$$

the (uniform) continuity of $f'(x)$ is sufficient to secure the uniform convergence of the series. Finally, for any continuous $f(x)$, any regular sequence $s_{\lambda_{a_\nu}}$, with

$$\frac{\lambda_{\beta_\nu}-\lambda_{a_\nu}}{\lambda_{a_\nu}} \geqslant \sigma\ (>0),$$

is uniformly convergent.

INCIDENCE RELATIONS FOR CREMONA SPACE TRANSFORMATIONS

By HILDA P. HUDSON.

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References.

O. Chisini, *Bologna Mem.* (7), 8 (1921), 3-42.
R. F. Clebsch, *Math. Ann.*, 4 (1871), 490-496.
G. Jung, *Lomb. Ist. Rend.* (2), 19 (1886), 161–166.
B. Levi, *Torino Atti*, 33 (1897), 56–76.
M. Noether, *Ann. di Mat.* (2), 5 (1871), 163–177.
M. Pannelli, *Roma Acc. Linc. Rend.* (5), 19 sem. 1 (1910), 449–455;
(5), 20 sem. 1 (1911), 404–409;
—— *Lomb. Ist. Rend.* (2), 47 (1914), 1041–1052.
C. Segre, *Ann. di Mat.* (2), 25 (1897), 1–54.
A. Tummarello, *Soc. Ital. Prog. Sci.*, 13 (1924), 326–327.

1. *Plane incidence relations.*

A Cremona plane transformation is determined by its fundamental system (F-system), that is, the set of σ fixed points O_α of fixed multiplicities i_α, the base of the first homaloidal net (ϕ_n) of curves in the first plane S that $\sim$ (correspond to) the lines of the second plane S'; their degree n is the same as that of the second homaloidal net (ϕ'), in S', that corresponds to the lines of S.

Each F-point O_α corresponds to a principal curve (P-curve) j'_α of degree i_α; these curves together form the Jacobian $J' \equiv \prod_{\alpha=1}^{\sigma} j'_\alpha$ of (ϕ'). Similarly there is the second F-system with accented symbols, and the first P-system $\sum_{\alpha'} j_{\alpha'}$. Let $i_{\alpha\alpha'}$ be the multiplicity of O_α on $j_{\alpha'}$.

If some of the F-points are adjacent, then part of the definition of (ϕ) may be that O_α is a singularity of any assigned nature. For the purpose of the incidence relations, this can always be regarded as the limit of a set of distinct ordinary multiple and simple F-points, and the

formulae to be stated hold without modification; they must be properly interpreted when a P-curve breaks up.

Clebsch's incidence relations are

$$\sigma = \sigma', \tag{1}$$

$$\sum_\alpha i_\alpha = 3(n-1), \tag{2}$$

$$\sum_\alpha i_\alpha^2 = n^2-1, \tag{3}$$

$$i_{\alpha\alpha'} = i''_{\alpha'\alpha}, \tag{4}$$

$$\sum_\alpha i_{\alpha\alpha'} = 3i''_{\alpha'}-1, \tag{5}$$

$$\sum_\alpha i_\alpha i_{\alpha\alpha'} = ni''_{\alpha'}, \tag{6}$$

$$\sum_\alpha i_{\alpha\alpha'} i_{\alpha\beta'} = i''_{\alpha'} i''_{\beta'} \quad (\alpha' \neq \beta'), \tag{7}$$

$$\sum_\alpha i_{\alpha\alpha'}^2 = i''^2_{\alpha'} + 1, \tag{8}$$

and the similar equations with the planes interchanged.

Of these, (2)–(7) are proved geometrically by considering the determination, genus, and intersections of ϕ and $\Sigma j_{\alpha'}$; then (1) follows from (5), and (8) from (7), by summation. Jung also finds (5), (6) and (8) by applying (2) and (3) to a transformation compounded of two others having one common F-point.

We wish to find analogues of these relations for space transformations. The chief new results of this paper are, first, that (1) holds without change: the total number, properly reckoned, of F-elements of a Cremona space transformation is the same in each space; and, secondly, the proof of the theorem merely stated by Tummarello, that the irrational curves of the two F-systems are associated in a (1, 1) relation in pairs of equal genera.

2. *Reduction of space transformations.*

In space, a Cremona transformation is also determined by the F-system, the base of the homaloidal web (ϕ_n) of surfaces in S that correspond to the planes of S'. The F-elements may be points or curves; contact cannot, in general, be treated as a limit, and the order of contact, as well as the multiplicity, is an essential characteristic of each F-element, as are also the numbers of incidences of the F-curves with each other and with the F-points of higher multiplicities.

Since the multiplicities of a plane F-system are replaced by this host of numbers, any general space relations answering to those of § 1 are

ponderous. Our attempt must be restricted to cases when only the simplest types of F-elements are present.

For these, as for singularities of a surface, it is to some extent arbitrary what we consider as simple. Segre calls a singular point of a surface f extraordinary or ordinary, according as the tangent cone has, or has not, a repeated sheet. Levi has shown that, by a series of auxiliary Cremona transformations of well known types, f can be so reduced as to have ordinary multiple points and curves only.

Let O be an intersection of two curves ω_α, ω_β of f, of multiplicities i_α, i_β, where $1 < i_\alpha \leqslant i_\beta$; then O is of multiplicity at least i_β, and if it is not higher, the tangent cone to f at O consists of i_β planes through t_β, the tangent line to ω_β, of which i_α coincide with the plane $p_{\alpha\beta}$ of t_α, t_β; and O is extraordinary. Hence, when f is reduced, all intersections of multiple curves fall at hypermultiple points.

On these lines, a Cremona transformation would be regarded as simple when the F-system consists of ordinary points and curves without contact, and all intersections of F-curves, whether simple or multiple, fall at F-points of higher multiplicity, where the tangent cone to ϕ has no fixed sheet. For this case a few general formulae have been given by Noether and Pannelli.

In Chisini's more recent work, a surface f is reduced to have ordinary multiple curves only, each curve by itself free from singularities, and f has no singular points at all, except the two kinds necessitated by the curves: (i) elementary pinch-points, where two of the distinct branches of the plane section are replaced by a cusp of first species; (ii) elementary intersections of two different multiple curves, where f satisfies no additional condition; the multiplicity of f at each is the greater of those of the two curves through it; such a point is extraordinary, the tangent cone having a repeated plane.

This point of view is artificial; for example, a quadric cone is regarded as requiring reduction. But it has the great advantage of retaining only one type of singular element, the curve, instead of both curve and point.

An immediate extension of Chisini's method reduces any Cremona transformation, by compounding it with well known auxiliary transformations, to a simple type $T_{n-n'}$ in which there are no isolated F-points or contact conditions. The first F-system consists of σ ordinary curves such as ω_α of multiplicity i_α, degree m_α, genus p_α, with d_α intersections with each of the homaloidal curves $c_{n'}$ that correspond to the lines of S'; also ω_α is free from multiple points, and has D_α simple intersections $O_{\alpha\beta}$ with ω_β, no other F-curve passing through $O_{\alpha\beta}$, and here the homaloidal

surface ϕ_n satisfies no other condition than those imposed by $\omega_\alpha^{i_\alpha}$, $\omega_\beta^{i_\beta}$. This is an F-point of partial or total contact, i_α sheets of (ϕ) touching the fixed plane $p_{\alpha\beta}$ if $i_\alpha \leqslant i_\beta$, but it does not present any additional contact *conditions.*

The pinch-points on ω_α are variable with ϕ, and are not F-elements. None of these special points lie on the general c.

The second F-system is of the same nature. The degrees n, n' of the homaloids in S, S' are different in general.

Such a transformation will be called *ordinary.*

3. *Known properties of the transformation T.*

An F-curve ω_α is of the first or second species according as $d_\alpha > 0$ or $= 0$. If ω_α is of first species, it corresponds to a P-surface j'_α of degree d_α, whose genus, as given by Noether's formula, is $-p_\alpha$; let j'_α have $i''_{\alpha'\alpha}$ sheets through $\omega'_{\alpha'}$. A point Q_α of ω_α corresponds to a P-curve κ'_α of degree i_α, whose locus is j'_α. The Jacobian of (ϕ') is $J' \equiv \overset{\sigma}{\underset{\alpha=1}{\amalg}} j'_\alpha$, and $\omega'_{\alpha'}$ is $(4i''_{\alpha'}-1)$-fold on J'.

If ω_α is of the second species, it is rational, and corresponds to an F-curve $\omega'_{\alpha'}$, also of the second species, and these are of multiplicities $4i_\alpha$, $4i''_{\alpha'}$ on J, J' respectively. A point Q_α of ω_α corresponds to the whole of $\omega'_{\alpha'}$ taken μ_α times, in the sense that any surface through Q_α corresponds to a surface, μ_α of whose sheets through $\omega'_{\alpha'}$ touch a fixed developable associated with Q_α, where

$$\mu_\alpha = \mu_{\alpha'} = i_\alpha / m'_{\alpha'} = i''_{\alpha'} / m_\alpha.$$

Since now there is no P-surface which corresponds to ω_α, the symbols $i''_{\alpha'\alpha}$, $i''_{\beta'\alpha}$ are at our disposal, and we arbitrarily define

$$i''_{\alpha'\alpha} = -1, \quad i''_{\beta'\alpha} = 0 \quad (\beta' \neq \alpha'),$$

when ω_α, $\omega'_{\alpha'}$ are corresponding F-curves of the second species. By this device, it is found that all our formulae hold for F-curves of either species without modification.

The free intersection (not absorbed by F-curves) of two general homaloids is one curve c; that of ϕ, $j_{\alpha'}$ is a variable set of $m'_{\alpha'}$ curves $\kappa_{\alpha'}$; a general $\kappa_{\alpha'}$ has no free intersection with ϕ or with $j_{\beta'}$.

4. *Transformation of an intersection of F-curves.*

The free intersection of j'_α, j'_β corresponds to the set of $D_{\alpha\beta}$ points $O_{\alpha\beta}$ of intersection of ω_α, ω_β. Let $i_\alpha \leqslant i_\beta$, and let Q_α, $Q_\beta \to O$ along ω_α, ω_β respectively. The P-curve κ'_α that corresponds to Q_α coincides in the

limit with the whole or part of κ'_β, lying on both j'_α and j'_β; and if $i_\alpha < i_\beta$, there is another part $\kappa'_{\beta\alpha}$ of κ'_β, lying on j'_β but not on j'_α, and meeting κ'_α in one point X'.

We assume that X' is not an F-point, and in particular that neither κ'_α nor $\kappa'_{\beta\alpha}$ is coincident with an F-curve $\omega'_{\alpha'}$, that is, adjacent to $\omega'_{\alpha'}$, lying on j'_β. If this occurred, we should compound T with another auxiliary transformation, having $\omega'_{\alpha'}$ as F-curve, when the adjacent elements would be scattered and the peculiarity would disappear.

Both of ω_α, ω_β cannot be of the second species if T is ordinary; for if they were, part of $\kappa'_\beta \equiv \mu_\beta \omega'_{\beta'}$ would coincide with $\kappa'_\alpha \equiv \mu_\alpha \omega'_{\alpha'}$, and $\omega'_{\alpha'}, \omega'_{\beta'}$ would coincide in an F-curve of contact.

As long as Q_α is different from O, a plane p through t_α corresponds to a ϕ' having a double point at one point P' of κ'_α, and any plane through P' corresponds to a ϕ having one sheet touching p at Q_α. This sets up a (1, 1) relation between p, P'. As P' describes κ'_α once, p rotates once about t_α; when P' meets $\omega'_{\alpha'}$, then p touches $j_{\alpha'}$, and conversely: the number of incidences of κ'_α, $\omega'_{\alpha'}$ is $i_{\alpha\alpha'}$, being the number of sheets of $j_{\alpha'}$ through ω_α. These incidences fall at points simple on $\omega'_{\alpha'}$, which has no multiple point; they may be simple or multiple on κ'_α. Similarly, all the sheets of J through ω_α correspond to the F-points on κ'_α.

If ω_α is a line, p always contains the whole of it, including O; the argument holds, p being replaced by a surface of higher degree, touching ω_α at Q_α, but not containing the whole of it.

As $Q_\alpha \to O$, all the tangent planes to ϕ through $t_\alpha \to p_{\alpha\beta}$; the (1, 1) relation between p, P' degenerates: all the points of κ'_α, including F-points, correspond to the one plane $p_{\alpha\beta}$, with the exception of one point Y' of κ'_α, corresponding to every other plane through t_α. Every general plane of the pencil corresponds to a ϕ' having Y' as a double point, and $p_{\alpha\beta}$ corresponds to a ϕ' having κ'_α as a double curve; and there is through ω_α either one or no sheet of J which does not touch $p_{\alpha\beta}$, according as Y' is or is not an F-point.

(i) If $i_\alpha < i_\beta$, any plane through O corresponds to a ϕ' containing both κ'_α and $\kappa'_{\beta\alpha}$, and therefore touching j'_β at X'. If the plane contains t_α, it touches ω_α, and ϕ' touches j'_α along κ'_α and therefore at X'. But j'_α, j'_β do not touch, since ω_α, ω_β do not; hence this ϕ' has a double point at X', which coincides with Y'. Now ϕ has $i_\beta - i_\alpha$ variable tangent planes at O through t_β, in a (1, 1) relation to the points of $\kappa'_{\beta\alpha}$, in which $p_{\alpha\beta}$ answers to X'. A plane through X' corresponds to a ϕ having $i_\alpha + 1$ sheets touching $p_{\alpha\beta}$ at O, but not containing any fixed curve, since X' is assumed not to be an F-point; and no sheet of J through ω_α touches any plane but $p_{\alpha\beta}$. But

the behaviour of $j_{\alpha'}$ at O is determined solely by its two multiplicities $i_{\alpha\alpha'}$, $i_{\beta\alpha'}$; hence

$$i_{\alpha\alpha'} \leqslant i_{\beta\alpha'}, \quad \text{if} \quad i_\alpha < i_\beta,$$

and similarly

$$i_{\alpha\alpha'} \geqslant i_{\beta\alpha'}, \quad \text{if} \quad i_\alpha > i_\beta,$$

An analytical proof is given in § 9.

(ii) If $i_\alpha = i_\beta$, the (1, 1) relation degenerates for ω_β also, and there is a point X' corresponding to every plane through t_β. This again $\equiv Y'$, for a general point P near O corresponds to a point near both X' and Y'.

Now O is a point of total contact of (ϕ), the fixed tangent cone being $p_{\alpha\beta}$ taken i_α times. The planes of the star (X') correspond to a net of (ϕ) containing points adjacent to O in general directions; on these therefore O is $(i_\alpha+1)$-fold, the tangent cones having t_α, t_β as i_α-fold edges, and so breaking up into $p_{\alpha\beta}$ taken $i_\alpha-1$ times and a net (q) of quadric cones through t_α, t_β. If ϕ, ψ are two homaloids of this net, their free intersection c touches the two other common generators of q_ϕ, q_ψ, and has an additional double point at O; being already rational, it breaks up, and one branch through O is a fixed P-curve κ, which corresponds to X'. Hence X' is an F-point; it is double on the ϕ' which corresponds to any plane of each of the pencils through t_α, t_β, and therefore double on the net (ϕ') that corresponds to the star (O); hence X' is a point of total contact of the whole web (ϕ'), and is therefore an intersection $O'_{\alpha'\beta'}$ of two F-curves $\omega'_{\alpha'}$, $\omega'_{\beta'}$ of equal multiplicities, and is of the same nature as O; and κ is a common P-curve of $j_{\alpha'}$, $j_{\beta'}$, each of which therefore has one sheet through O not touching $p_{\alpha\beta}$.

Now there is only one such sheet through each of ω_α, ω_β; hence one of $j_{\alpha'}$, $j_{\beta'}$, say $j_{\alpha'}$, has one such sheet through ω_α, and $j_{\beta'}$ has one through ω_β, and no other P-surface has any such sheet:

$$i_\alpha = i_\beta, \quad i_{\alpha\alpha'} = i_{\beta\alpha'}+1, \quad i_{\alpha\beta'} = i_{\beta\beta'}-1, \quad i_{\alpha\gamma'} = i_{\beta\gamma'}.$$

Further, κ is continuous with a series of curves $\kappa_{\alpha'}$ through adjacent points Q_α of ω_α, but not with a series of $\kappa_{\alpha'}$ through Q_β, and with a series of $\kappa_{\beta'}$ through Q_β but not through Q_α. Hence the one sheet of j'_α which does not touch $p'_{\alpha'\beta'}$ passes through $\omega'_{\alpha'}$, and that of j'_β through $\omega'_{\beta'}$; and we have

$$i'_{\alpha'} = i'_{\beta'}, \quad i'_{\alpha'\alpha} = i'_{\beta'\alpha}+1, \quad i'_{\alpha'\beta} = i'_{\beta'\beta}-1, \quad i'_{\alpha'\gamma} = i'_{\beta'\gamma},$$

as a necessary consequence of the other relations.

Hence the pair of intersecting F-curves ω_α, ω_β in S, of equal multiplicities, selects a similar pair $\omega'_{\alpha'}$, $\omega'_{\beta'}$ in S', and of these, ω_α selects $\omega'_{\alpha'}$ and ω_β selects $\omega'_{\beta'}$, and the above relations hold; also the points of intersection of the first pair are in (1, 1) relation to those of the second:

$$D_{\alpha\beta} = D'_{\alpha'\beta'}.$$

If ω_α is the second species, parts of the argument are modified, and the results hold.

None of this applies to F-curves which do not meet; then $D_{\alpha\beta} = 0$ and j'_α, j''_β have no free intersection.

[*Added* 13 *August*, 1926.

5. *Proof of Tummarello's theorem: the genera of the irrational F-curves are the same in each space.*

Instead of an intersection of two distinct F-curves, let O be a d.p. of one F-curve ω_α; then H ceases to be ordinary to this extent. As before, O is an i_α-fold point of total contact of (ϕ), and is an $(i_\alpha+1)$-fold point on a net of (ϕ) which corresponds to the net of planes through a point X', a d.p. of $\Sigma\omega'$; and X' corresponds to a P-curve κ with one branch through O, a double curve on Σj, not touching the plane p of the two tangents to ω_α.

Any $j_{\beta'}$ has $i_{\alpha\beta'}$ sheets through each branch of ω_α at O; if it contains κ, it has an $(i_{\alpha\beta'}+1)$-fold point at O, and its tangent cone consists of p taken $i_{\alpha\beta'}-1$ times and a quadric cone $q_{\beta'}$ through the tangents to ω_α and κ. If κ were an intersection of two different P-surfaces $j_{\beta'}$, $j_{\gamma'}$, it would have two branches through O, touching the common generators of $q_{\beta'}$, $q_{\gamma'}$, other than the tangents to ω_α. Hence κ is not an intersection of $j_{\beta'}$, $j_{\gamma'}$, but a double curve on one P-surface $j_{\beta'}$, and X' is a d.p. on $w'_{\beta'}$; H' ceases to be ordinary to this extent.

Thus if, in an ordinary transformation T, one F-curve ω_α of H acquires a d.p. without breaking up, there is one F-curve $\omega'_{\beta'}$ of H' which also acquires a d.p. without breaking up. Both ω_α and $\omega'_{\beta'}$ are irrational.

Now the equations proved below for an ordinary transformation, and assumed here,

$$\sigma = \sigma', \quad \Sigma p = \Sigma p',$$

hold also for T when modified, as can be proved by reducing it to one of ordinary type. Now σ is unaltered when ω_α acquires a d.p.; hence σ is also unaltered, and no ω' breaks up. Also Σp is lowered by 1, and therefore $\Sigma p'$ also; but p'_β is lowered by 1, hence no other d.p. of $\Sigma\omega'$ can arise.

The two curves ω_α, $\omega'_{\beta'}$ are associated by the fact that if either acquires a d.p., so does the other. This sets up a (1, 1) relation between the sets of irrational curves of H, H'.

Now let ω_α acquire p_α d.p.'s; it does not break up, σ and therefore σ' are unaltered, and ω'_β does not break up when it acquires p_α d.p.'s. Hence $p'_{\beta'} \geqslant p_\alpha$, and similarly $p_\alpha \geqslant p'_{\beta'}$, that is

$$p_\alpha = p'_\beta.$$

In the (1, 1) relation, associated F-curves of H, H' have the same genus, which proves Tummarello's theorem.

Thus the numbers of irrational curves in H, H' are equal, and also the total numbers of curves; it follows that the numbers of rational curves are equal also.

If ω_α is a rational curve of degree greater than 1, it breaks up on acquiring a d.p. This may cause (ϕ) to break up, and lower n; or it may entail an additional F-curve of second species, and lower n'; or it may cause a P-surface to break up, one part j meeting (ϕ) in F-curves only, then j corresponds to an isolated F-point of (ϕ') and T ceases to be ordinary.

If T is still ordinary and of unaltered degrees, σ and therefore σ' are increased by 1, and one ω' breaks up on acquiring a d.p., and is therefore also a rational curve of degree greater than 1. But the particular ω' that breaks up may depend on the particular way in which ω_α breaks up, that is, on the degrees of the two components of ω_α and the distribution between them of the intersections of ω_α with the other F-curves. We do not thus obtain any (1, 1) relation between the rational curves of H, H'.]

6. *Incidence relations for space transformations.*

(1) $\sigma = \sigma'.$

(2) $\sum_\alpha p_\alpha = \sum_{\alpha'} p'_{\alpha'}.$

(3) $\sum_\alpha d_\alpha = 4n'-4.$

(4) $\sum_\alpha i'_{\alpha'\alpha} = 4i'_{\alpha'}-1.$

(5) $\sum_\alpha d_\alpha i_\alpha = nn'-1.$

(6) $\sum_\alpha d_\alpha i_{\alpha\alpha'} = n' d'_{\alpha'}.$

(7) $\sum_\alpha i_\alpha i'_{\alpha'\alpha} = n i'_{\alpha'}.$

(8) $\sum_\alpha i_{\alpha\alpha'} i'_{\beta'\alpha} = d'_{\alpha'} i'_{\beta'} \quad (\alpha' \neq \beta').$

(9) $\sum_\alpha i_{\alpha\alpha'} i'_{\alpha'\alpha} = d'_{\alpha'} i'_{\alpha'}+1.$

(10) $\sum_\alpha i_\alpha m_\alpha - 4n = \sum_{\alpha'} i'_{\alpha'} m'_{\alpha'} - 4n'.$

(11) $\sum_\alpha i_\alpha^2 m_\alpha = n^2-n'.$

(12) $\sum_\alpha i_\alpha i_{\alpha\alpha'} m_\alpha = n d'_{\alpha'} - i'_{\alpha'} m'_{\alpha'}.$

(13) $\sum_\alpha i_{\alpha\alpha'} i_{\alpha\beta'} m_\alpha = d'_{\alpha'} d'_{\beta'} - i'_{\alpha'} D'_{\alpha\beta'} \quad (\alpha' \neq \beta').$

(13)′ $$i_l D_{\alpha\beta} = -\sum_{\alpha'} i'_{\alpha'\alpha} i'_{\alpha'\beta} m'_{\alpha'} + d_\alpha d_\beta \quad (\alpha \neq \beta).$$

(14) $$\sum_\beta i_l D_{\alpha\beta} = \sum_{\alpha'} i'_{\alpha'\alpha}(i'_{\alpha'\alpha}+1)\, m'_{\alpha'} + 4 i_\alpha m_\alpha - d_\alpha(d_\alpha+4).$$

(15) $$\sum_\beta i_\beta D_{\alpha\beta} = \sum_{\alpha'} i'_{\alpha'\alpha} m'_{\alpha'} + n m_\alpha - 2 i_\alpha (p_\alpha - 1) - 4 d_\alpha.$$

(16) $$\sum_\beta i_l i_\beta D_{\alpha\beta} = i_\alpha \sum_{\alpha'} i'^2_{\alpha'\alpha} m'_{\alpha'} + n i_\alpha m_\alpha - i_\alpha d_\alpha^2 - d_\alpha.$$

(17) $$\sum_\beta i_l^2 D_{\alpha\beta} = 2 i_\alpha \sum_{\alpha'} i'^2_{\alpha'\alpha} m'_{\alpha'} + i_\alpha^2 (4 m_\alpha + 2 p_\alpha - 2) - 2 i_\alpha d_\alpha^2 - d_\alpha.$$

(18) $$\sum_\beta i_{\beta\alpha'} D_{\alpha\beta} - d'_{\alpha'} m_\alpha + 2 i_{\alpha\alpha'}(p_\alpha - 1) = \sum_{\beta'} i'_{\beta'\alpha} D'_{\alpha'\beta'} - d_\alpha m'_{\alpha'} + 2 i'_{\alpha'\alpha}(p'_{\alpha'} - 1).$$

(19) $$\sum_\beta i_\beta i_{\beta\alpha'} D_{\alpha\beta} - \sum_\beta (i_\beta - i_\alpha)(i_{\beta\alpha'} - i_{\alpha\alpha'}) D_{\alpha h}$$
$$= (n i_{\alpha\alpha'} + i_\alpha d'_{\alpha'}) m_\alpha - i_\alpha i_{\alpha\alpha'}(4 m_\alpha + 2 p_\alpha - 2) - i'_{\alpha'\alpha} m'_{\alpha'}.$$

(20) $$\sum_\beta i_{\beta\alpha'} i_{\beta\beta'} D_{\alpha\beta} - \sum_\beta (i_{\beta\alpha'} - i_{\alpha\alpha'})(i_{\beta\beta'} - i_{\alpha\beta'}) D_{\alpha h}$$
$$= (d'_{\alpha'} i_{\alpha\beta'} + d'_{\beta'} i_{\alpha\alpha'}) m_\alpha - i_{\alpha\alpha'} i_{\alpha\beta'}(4 m_\alpha + 2 p_\alpha - 2) - i'_{l'\alpha} D'_{\alpha'\beta} \quad (\alpha' \neq \beta').$$

(21) $$\sum_\beta i_{\beta\alpha'}^2 D_{\alpha\beta} - \sum_\beta (i_{\beta\alpha'} - i_{\alpha\alpha'})^2 D_{\alpha h}$$
$$= 2 d'_{\alpha'} i_{\alpha\alpha'} m_\alpha - i_{\alpha\alpha'}^2 (4 m_\alpha + 2 p_\alpha - 2) + d_\alpha m'_{\alpha'} - i'_{\alpha'\alpha}(4 m'_{\alpha'} + 2 p'_{\alpha'} - 2)$$
$$- \sum_{\beta'} (i'_{\beta'\alpha} - i'_{l'\alpha}) D'_{\alpha'\beta'}$$

(22) $$\sum_\alpha \sum_\beta D_{\alpha\beta} - 4 \sum_\alpha m_\alpha = \sum_{\alpha'} \sum_{\beta'} D'_{\alpha'\beta'} - 4 \sum_{\alpha'} m'_{\alpha'}.$$

(23) $$\sum_\alpha \sum_\beta i_l D_{\alpha\beta} = \sum_\alpha i_\alpha (4 m_\alpha - p_\alpha + 1) - 11(n-1).$$

(24) $$\sum_\alpha \sum_\beta i_h D_{\alpha\beta} = \sum_\alpha \{ n m_\alpha - i_\alpha (p_\alpha - 1) \} - \sum_{\alpha'} m'_{\alpha'} - 5(n-1).$$

(25) $$\sum_\alpha \sum_\beta i_l^2 D_{\alpha\beta} = \sum_\alpha \sum_{\alpha'} i_\alpha i'^2_{\alpha'\alpha} m'_{\alpha'} - \sum_\alpha \{ i_\alpha^2 (p_\alpha - 1) + d_\alpha^2 i_\alpha \}$$
$$+ 2n^2 - 4n' + 2.$$

(26) $$\sum_\alpha \sum_\beta i_\alpha i_\beta D_{\alpha\beta} = \sum_\alpha \{ n i_\alpha m_\alpha - i_\alpha^2 (p_\alpha - 1) \} - 2n^2 + 2.$$

(27) $$2 \sum_\alpha \sum_\beta i_l^3 D_{\alpha\beta} = 3 \sum_\alpha \sum_{\alpha'} i_\alpha^2 i'^2_{\alpha'\alpha} m'_{\alpha'} + \sum_\alpha \{ 4 i_\alpha^3 (2 m_\alpha + p_\alpha - 1) - 3 d_\alpha^2 i_\alpha^2 \}$$
$$- n^3 + 1.$$

(28) $$2 \sum_\alpha \sum_\beta i_l^2 i_h D_{\alpha\beta} = \sum_\alpha \sum_{\alpha'} i_\alpha^2 i'^2_{\alpha'\alpha} m'_{\alpha'} - \sum d_\alpha^2 i_\alpha^2 + n^3 - 2nn' + 1.$$

(29) $$2 \sum_\alpha \sum_\beta i_l i_h^2 D_{\alpha\beta} = \sum_\alpha \sum_{\alpha'} i_\alpha^2 i'_{\alpha'\alpha}(2 - i'_{\alpha'\alpha})\, m'_{\alpha'}$$
$$- \sum_\alpha \{ 4 i_\alpha^3 (p_\alpha - 1) - i_\alpha^2 (d_\alpha - 8) \} + n^3 - 1.$$

(30) $$\sum_\alpha \sum_\beta i_{l\alpha'} D_{\alpha\beta} + \sum_{\beta'} D'_{\alpha' h'} = \sum_\alpha i_{\alpha\alpha'}(4 m_\alpha - p_\alpha + 1) + 4 m'_{\alpha'} - p'_{\alpha'} + 1 - 11 d'_{\alpha'}.$$

(31) $$\sum_\alpha \sum_\beta i_{h\alpha'} D_{\alpha\beta} + \sum_{\beta'} D'_{\alpha' l'} = \sum_\alpha \{ d'_{\alpha'} m_\alpha - i_{\alpha\alpha'}(p_\alpha - 1) \} - p'_{\alpha'} + 1 - 5 d'_{\alpha'}.$$

For an ordinary space transformation, the plane characteristics n, i_α of the F-system are replaced by

$$n, \quad n', \quad i_\alpha, \quad m_\alpha, \quad p_\alpha, \quad D_{\alpha\beta}, \quad d_\alpha;$$

and the incidence numbers $i_{\alpha\alpha'}$ of the P-system by $i_{\alpha\alpha'}$, $i_{\acute{\alpha}'\alpha}$. We shall prove the above relations, and the corresponding (1)′ ... (31)′ with the spaces interchanged, where

i_l, i_h stand for the lower and higher of i_α, i_β, or both for their common value if they are equal; and similarly $i_{l\alpha'}$, $i_{h\alpha'}$ for the lower and higher of $i_{\alpha\alpha'}$, $i_{\beta\alpha'}$;

$D_{\alpha h}$ means $D_{\alpha\beta}$ if $i_\alpha \leqslant i_\beta$, and means 0 if $i_\alpha > i_\beta$;

$D_{\alpha l}$ means $D_{\alpha\beta}$ if $i_\alpha > i_\beta$, and means 0 if $i_\alpha \leqslant i_\beta$.

7. *Noether's and Pannelli's relations.*

To prove all these we have, in the first place, the simplified forms of Noether's and Pannelli's relations; the first three express postulation, genus and equivalence for (ϕ):

$$(n+1)(n+2)(n+3)-24 = \sum_\alpha i_\alpha(i_\alpha+1)\{(3n+6)m_\alpha-(2i_\alpha+1)(2m_\alpha+p_\alpha-1)\} - \sum_\alpha\sum_\beta i_l(i_l+1)(3i_h-i_l+1)\,D_{\alpha\beta},$$

$$(n-1)(n-2)(n-3) = \sum_\alpha i_\alpha(i_\alpha-1)\{(3n-6)m_\alpha-(2i_\alpha-1)(2m_\alpha+p_\alpha-1)\} - \sum_\alpha\sum_\beta i_l(i_l-1)(3i_h-i_l-1)\,D_{\alpha\beta},$$

$$n^3-1 = \sum_\alpha i_\alpha^2\{3nm_\alpha-2i_\alpha(2m_\alpha+p_\alpha-1)\}-\sum_\alpha\sum_\beta i_l^2(3i_h-i_l)\,D_{\alpha\beta},$$

$$\sum_\alpha(p_\alpha-1) = \sum_{\alpha'}(p'_{\alpha'}-1),$$

$$\sum_\alpha\sum_\beta D_{\alpha\beta}-4\sum_\alpha m_\alpha = \sum_{\alpha'}\sum_{\beta'}D'_{\alpha'\beta'}-4\sum_{\alpha'}m'_{\alpha'},$$

$$4\sum_\alpha\sum_\beta i_l D_{\alpha\beta}-\sum_\alpha i_\alpha(5m_\alpha-4p_\alpha+4) = 4\sum_{\alpha'}\sum_{\beta'}i''_{l'}D'_{\alpha'\beta'}-\sum_{\alpha'}i''_{\alpha'}(5m'_{\alpha'}-4p'_{\alpha'}+4).$$

Next we express the facts that the degree of c is that of ϕ':

$$n^2-\sum_\alpha i_\alpha^2 m_\alpha = n';$$

and that the genera of plane sections of ϕ, ϕ' are equal:

$$\tfrac{1}{2}(n-1)(n-2)-\sum_\alpha \tfrac{1}{2}i_\alpha(i_\alpha-1)\,m_\alpha = \tfrac{1}{2}(n'-1)(n'-2)-\sum_{\alpha'}\tfrac{1}{2}i''_{\alpha'}(i''_{\alpha'}-1)\,m'_{\alpha'}.$$

By combining these, and the same with the spaces interchanged, we can prove some of our formulae, and combinations of others, namely

(10), (11), (22), (23), (26), (2)–(1), $3\times(28)$–(27).

Others can be proved, for curves of first species only, by simple geometrical considerations :

(3) expresses the degree of J;

(4) ,, ,, multiplicity of $\omega'_{\alpha'}$ on J';

(5) ,, ,, number of intersections of ϕ, c;

(6) ,, ,, ,, ,, ,, ,, $j_{\alpha'}$, c;

(7) ,, ,, ,, ,, ,, ,, ϕ, $\kappa_{\alpha'}$;

(8) ,, ,, ,, ,, ,, ,, $j_{\alpha'}$, $\kappa_{\beta'}$;

(12) ,, ,, degree of the intersection of ϕ, $j_{\alpha'}$;

(13) ,, ,, ,, ,, ,, ,, $j_{\alpha'}$, $j_{\beta'}$.

This also holds for (4) when $\omega'_{\alpha'}$ is of second species, and is $4i'_{\alpha'}$-fold on J'; for then $\sum_{\alpha} i_{\alpha'\alpha}$ contains a term -1, arising from the corresponding F-curve of second species, which does not answer to any sheet of J'.

8. *Transformation of certain sets of points.*

I. Let k'_ξ be a curve of degree ξ' and genus p, having $\eta'_{\alpha'}$ simple intersections with $\omega'_{\alpha'}$; and f'_x a surface of degree x' on which k' is simple and $\omega'_{\alpha'}$ is $y'_{\alpha'}$-fold. Let $f'_{z'}$ be another such surface, of degree z' on which k' is simple and $\omega'_{\alpha'}$ is $v'_{\alpha'}$-fold, and let each of η', y', v' be greater than or equal to 1. If $\omega'_{\alpha'}$ is of the first species, from the homologues of k', $f_{x'}$ there fall away $\eta'_{\alpha'}$ P-curves $\kappa_{\alpha'}$ and $y'_{\alpha'}$ P-surfaces $j_{\alpha'}$ respectively. If $\omega'_{\alpha'}$ is of the second species, the corresponding F-curve ω_α of the second species falls away $\mu'_{\alpha'}\eta'_\alpha$ times from k, reducing its degree by $\mu'_{\alpha'}\eta'_{\alpha'}m_\alpha = \eta'_{\alpha'}i'_{\alpha'}$, and the reduction of η_α is as before. No P-surface is dropped from f_x on account of $\omega'_{\alpha'}$, and the multiplicity of ω_α on f_x is increased by η_α. Now $d'_{\alpha'} = 0$, and, with our conventions, $i_{\alpha\beta'} = i'_{\alpha'\beta} = 0$, $i_{\alpha\alpha'} = i'_{\alpha'\alpha} = -1$; we have for the homologues

$$\xi = \xi' n' - \sum_{\alpha'} \eta'_{\alpha'} i'_{\alpha'}, \qquad \eta_\alpha = \xi' d_\alpha - \sum_{\alpha'} \eta'_{\alpha'} i'_{\alpha'\alpha},$$

$$x = x' n - \sum_{\alpha'} y'_{\alpha'} d'_{\alpha'}, \qquad y_\alpha = x' i_\alpha - \sum_{\alpha'} y'_{\alpha'} i_{\alpha\alpha},$$

$$z = z' n - \sum_{\alpha'} v'_{\alpha'} d'_{\alpha'}, \qquad v_\alpha = z' i_\alpha - \sum_{\alpha'} v'_{\alpha'} i_{\alpha\alpha'},$$

the sums extending to F-curves of both species.

We can take ξ', x', z' to be large compared with η', y', v'; then all of ξ, η, x, y, z, v are greater than 0.

Now k meets its total residual, in the intersection of f_x, f_z, in

$$\xi(x+z-4)-2(p-1)$$

points; since ω_a is of multiplicity greater than or equal to that of the simple curve k on both f_x and f_z, each of its intersections with k absorbs y_a+v_a-1 of these points. The free intersections of k with its residual correspond to the points of the same nature for k', $f'_{x'}$, $f'_{z'}$:

$$\xi(x+z-4)-2(p-1)-\sum_a \eta_a(y_a+v_a-1)$$
$$= \xi'(x'+z'-4)-2(p-1)-\sum_{a'} \eta'_{a'}(y'_{a'}+v'_{a'}-1).$$

Now ξ', η', x', y', z', v' are arbitrary. Substituting for $\xi \ldots v$, and equating coefficients, we have the relations (3)–(9).

Sum (9) with regard to a':

$$\sum_{a'}\sum_{a} i_{aa'}i'_{a'a} = nn'-1+\sigma, \quad \text{by (5)}.$$

The double sum and the product nn' are symmetrical in S, S'. Hence (1), which signifies that the total number of F-curves is the same in the two spaces. Since F-curves of the second species occur in corresponding pairs, and therefore in equal numbers in S, S', the number of F-curves of the first species is also the same in the two spaces.

In the process of reducing an arbitrary transformation V to one T of ordinary type, the F-elements of each auxiliary transformation give rise to equal numbers of F-curves of T in S, S'. Hence the equality of F-elements holds for V also, provided that any isolated F-point or F-curve of contact of V is counted as that number of distinct F-elements to which it gives rise when V is reduced to T.

II. Next, let $f'_{x'}$ be such that $y'_{a'} < y'_{\beta'}$ if $i'_{a'} < i'_{\beta'}$. The number of free intersections of $\omega'_{a'}$ with its residual, in the intersection of $f'_{x'}$, ϕ', is

$$m'_{a'}(x'i'_{a'}+y'_{a'}n')-i'_{a'}y'_{a'}(4m'_{a'}+2p'_{a'}-2)$$
$$-\sum_{\beta'}(y'_{a'}i'_{\beta'}+y'_{\beta'}i'_{a'}-y'_{a'}i'_{a'})D'_{a'h'}-\sum_{\beta'}y'_{\beta'}i'_{\beta'}D'_{a'l'}.$$

If $\omega'_{a'}$ is of the first species, these points correspond to the free intersections of $j_{a'}$ with a general plane section of f_x, in number

$$xd'_{a'}-\sum_a y_a i_{aa'}m_a.$$

If $\omega'_{\alpha'}$ is of the second species, they correspond to points of the plane section of f_x adjacent to the corresponding F-curve ω_α, in number $y_\alpha m_\alpha$, to which the last expression reduces in this case.

The equality holds for $\omega'_{\alpha'}$ of either species; substituting for x, y, and equating coefficients of x', $y'_{\beta'}$, $y'_{\alpha'}$, we have (12), (13), and (15)′–(14)′.

III. Again, let f_x be an arbitrary surface, with x large compared with each y, and y_α greater than each y_β; let f_z be of the same nature. Then

$$y'_{\alpha'} - y'_{\beta'} = x(i''_{\alpha'} - i''_{\beta'}) - \sum_{\gamma=1}^{\sigma} y_\gamma (i''_{\beta'\gamma} - i''_{\alpha'\gamma}).$$

If $i''_{\alpha'} < i''_{\beta'}$, it follows that $y'_{\alpha'} < y'_{\beta'}$.

If $i''_{\alpha'} = i''_{\beta'}$, and ω'_α, $\omega'_{\beta'}$ intersect, so that $D'_{\alpha'\beta'} > 0$, then every term in $\sum_\gamma y_\gamma (i''_{\beta'\gamma} - i''_{\alpha'\gamma})$ vanishes except the two arising from the two F-curves ω_γ, ω_β, of equal multiplicities $i_\gamma = i_\beta$, selected by $\omega'_{\alpha'}$, $\omega'_{\beta'}$ respectively; and

$$y'_{\alpha'} - y'_{\beta'} = -(y_\gamma - y_\beta).$$

In particular, if $\gamma = \alpha$, we have

$$y_\alpha > y_\beta, \quad y'_{\alpha'} < y'_{\beta'}.$$

In every case, $v'_{\alpha'} >$, $=$, or $< v'_{\beta'}$ according as $y'_\alpha >$, $=$, or $< y'_{\beta'}$.

The number of free intersections of ω_α with its residual, in the intersection of f_x, f_z, is

$$m_\alpha(xv_\alpha + zy_\alpha) - y_\alpha v_\alpha(4m_\alpha + 2p_\alpha - 2) - \sum_\beta y_\beta v_\beta D_{\alpha\beta}.$$

These points correspond to some of the free intersections of $j'_\alpha, f'_{x'}, f'_{z'}$; also, corresponding to each point $O_{\alpha\beta}$ for which $i_\alpha < i_\beta$, there lies on $f'_{\alpha'}$ a P-curve $\kappa'_{\beta\alpha}$, of multiplicity $y_\alpha - y_\beta$ for $f'_{x'}$, which lies on j'_β and meets j'_α once, at the point X' of § 4; this is $(v_\alpha - v_\beta)$-fold on $f'_{z'}$, and it accounts for $(y_\alpha - y_\beta)(v_\alpha - v_\beta)$ free intersections of $j'_\alpha, f'_{x'}, f'_{z'}$.

By Noether's general equivalence formula the remaining number is

$$x'z'd_\alpha - \sum_{\alpha'} [(x'v'_{\alpha'} + z'y'_{\alpha'})\, i''_{\alpha'\alpha} m'_{\alpha'} + y'_{\alpha'} v'_{\alpha'} \{ d_\alpha m'_{\alpha'} - i''_{\alpha'\alpha}(4m'_{\alpha'} + 2p'_{\alpha'} - 2) \}]$$

$$+ \sum_{\alpha'} \sum_{\beta'} \begin{cases} \{(y'_{\alpha'} v'_{\beta'} + y'_{\beta'} v'_{\alpha'})\, i''_{\alpha'\alpha} + y'_{\alpha'} v'_{\alpha'} (i''_{\beta'\alpha} - i''_{\alpha'\alpha})\} & \text{if } y'_{\alpha'} \leqslant y'_{\beta'} \\ \text{or } \{(y'_{\alpha'} v'_{\beta'} + y'_{\beta'} v'_{\alpha'})\, i''_{\beta'\alpha} + y'_{\beta'} v'_{\beta'} (i''_{\alpha'\alpha} - i''_{\beta'\alpha})\} & \text{if } y'_{\alpha'} > y'_{\beta'} \end{cases} D_{\alpha'\beta'}$$

$$- \sum_\beta (y_\alpha - y_\beta)(v_\alpha - v_\beta) D_{\alpha\beta} \quad (\text{for } i_\alpha < i_\beta).$$

In the second sum, every pair of curves which has $i''_{\alpha'} < i''_{\beta'}$ has also $y'_{\alpha'} < y'_{\beta'}$, and belongs to the first line; then $i''_{\alpha'\alpha} \leqslant i''_{\beta'\alpha}$, and we can write $i''_{\iota'\alpha}$ for $i''_{\alpha'\alpha}$. Every pair with $i_\alpha > i_\beta$ belongs to the second line, and for it we can write $i''_{\iota'\alpha}$ for $i''_{\beta'\alpha}$. If $i''_{\alpha'} = i''_{\beta'}$ and also $i''_{\alpha'\alpha} = i''_{\beta'\alpha}$, the two lines are identical, and it is indifferent to which line we assign the pair.

For all these pairs the summand can be written in the common form

$$\{(y'_{\alpha'}v'_{\beta'}+y'_{\beta'}v'_{\alpha'})\, i''_{\iota'\alpha}+y'_{\alpha'}v'_{\alpha'}(i''_{\beta'\alpha}-i''_{\iota'\alpha})+y'_{\beta'}v'_{\beta'}(i''_{\alpha'\alpha}-i''_{\iota'\alpha})\}\, D'_{\alpha'\beta'}.$$

There remain the pairs for which $i''_{\alpha'} = i''_{\beta'}$ and $i''_{\alpha'\alpha} \neq i''_{\beta'\alpha}$, that is, those selected by ω_α and the curves ω_β of equal multiplicity which meet it. If such a pair belongs to the first line,

$$i''_{\beta'\alpha} = i''_{\iota'\alpha} = i''_{\alpha'\alpha}-1,$$

$$y'_{\beta'}-y'_{\alpha'} = y_\alpha-y_\beta, \quad v'_{\beta'}-v'_{\alpha'} = v_\alpha-v_\beta, \quad D'_{\alpha'\beta'} = D_{\alpha\beta}.$$

The summand is equal to the common form just given with the correction

$$(y'_{\alpha'}v'_{\beta'}+y'_{\beta'}v'_{\alpha'}-y'_{\alpha'}v'_{\alpha'}-y'_{\beta'}v'_{\beta'})(i''_{\alpha'\alpha}-i''_{\beta'\alpha})\, D'_{\alpha'\beta'} = -(y_\alpha-y_\beta)(v_\alpha-v_\beta)\, D_{\alpha\beta},$$

and similarly for a pair belonging to the second line. These corrections can be included in the third sum if this is extended to curves for which $i_\alpha = i_\beta$ as well as those for which $i_\alpha < i_\beta$; this we denote by writing $D_{\alpha h}$ for $D_{\alpha\beta}$. The second and third sums become

$$\sum_{\alpha'}\sum_{\beta'}\{(y'_{\alpha'}v'_{\beta'}+y'_{\beta'}v'_{\alpha'})\, i''_{\iota'\alpha}+y'_{\alpha'}v'_{\alpha'}(i''_{\beta'\alpha}-i''_{\iota'\alpha})+y'_{\beta'}v'_{\beta'}(i''_{\alpha'\alpha}-i''_{\iota'\alpha})\}\, D'_{\alpha'\beta'}$$
$$-\sum_{\beta}(y_\alpha-y_\beta)(v_\alpha-v_\beta)\, D_{\alpha h}.$$

If ω_α is of the second species, we find, as before, that the formula remains correct. Substituting for x, y, z, v, and equating coefficients, we have (17)$-2\times$(16), 19, 20, 21.

9. *The compound transformation* W.

Let $V_{x-x''}$ be another ordinary transformation, between S and a third space S'', which has in common with T one F-curve in S of the first species, say ω_α of multiplicity y_α, the other F-curves $\Sigma\omega_\delta$ of multiplicities Σy_δ being in as general positions as possible with regard to T, so that ω_δ meets ω_α in $D_{\alpha\delta}$ points, but does not meet any ω_β ($\beta \neq \alpha$). To construct V, we take a monoidal transformation of arbitrary degree, with an arbitrary simple

F-curve ω, reduce this to ordinary type, and identify one of the spaces with S and the homologue of ω with ω_a. We can take x large compared with y_a. Let f_x be the general homaloid of V.

We have thus set up the compound transformation $W \equiv V.T^{-1}$ between S', S'', whose homaloid in S' is the surface $f'_{x'}$, which $\overset{T}{\sim} f_x$ (corresponds to f_x in the transformation T), where

$$x' = xn' - y_a d_a,$$

and the F-system of W consists of four sets of ordinary curves :

(i) An F-curve $\omega'_{a'}$ of T is of degree m'_a, and of multiplicity $xi'_{a'} - y_a i'_{a'a}$ for W.

(ii) The curve ω'_δ which $\overset{T}{\sim} \omega_\delta$ is of degree $nm_\delta - i_a D_{a\delta}$, and of multiplicity y_δ for W; it meets another curve ω'_ϵ of the same set in $D_{\delta\epsilon}$ points, and meets $\omega'_{a'}$ in the $d'_{a'} m_\delta - i_{aa'} D_{a\delta}$ points which $\overset{T}{\sim}$ those intersections of ω_δ, $j_{a'}$ which do not lie on ω_a.

(iii) If $y_a < y_\delta$, each intersection $O_{a\delta}$ of ω_a, ω_δ is a hypermultiple point of f_x on ω_a, and $\overset{T}{\sim}$ a curve κ'_a of degree i_a. Each of the $D_{a\delta}$ such curves is an F-curve of W of second species of multiplicity $y_\delta - y_a$; it meets $\omega'_{a'}$ in $i_{aa'}$ points, and meets once the one ω'_δ with which it is associated.

(iv) If $i_a < i_\beta$, each $O_{a\beta} \overset{T}{\sim}$ a curve κ'_β breaking up into κ'_a, $\kappa'_{\beta a}$; the first component lies on j'_a and is dropped from $f'_{x'}$, but the second remains as an F-curve of W of second species, of degree $i_\beta - i_a$, and of multiplicity y_a. It meets $\omega'_{a'}$ in $i_{\beta a'} - i_{aa'}$ points, and meets no F-curve of W of the other sets.

The F-system of W in S'' is of the same nature, and consists of

$$\Sigma\omega''_{\delta''}, \qquad \Sigma\omega''_\beta, \qquad \Sigma\kappa''_a \ (i_a < i_\beta), \qquad \Sigma\kappa''_{\delta a} \ (y_a < y_\delta),$$

of multiplicities $ny''_{\delta''} - i_a y''_{\delta''a}$, $\quad i_\beta$, $\quad i_\beta - i_a$, $\qquad\qquad$ i_a, respectively.

There are two kinds of P-surfaces of W in S''. We have $\omega'_a \overset{W}{\sim}$ the surface $j''_{a'}$ which $\overset{V}{\sim} j_{a'}$; on this, the multiplicities of

$$\omega''_{\delta''}, \qquad \omega''_\beta, \qquad \kappa''_a, \qquad\qquad \kappa''_{\delta a}$$

are $\qquad d'_{a'} y''_{\delta''} - i_{aa'} y''_{\delta''a}$, $\quad i_{\beta a}$, $\quad i_{\beta a'} - i_{aa'}$ (if positive), $\quad i_{aa'}$ respectively.

Also $\omega'_\delta \overset{W}{\sim} j''_\delta$, on which $\omega''_{\delta''}$ is $y''_{\delta''\delta}$-fold; and, if $y_a < y_\delta$, it contains simply the $D_{a\delta}$ curves $\kappa''_{\delta a}$ associated with ω_δ.

10. *Incidence relations for W.*

Thus W is an ordinary transformation, and we can apply to it the formulae already proved for T. In particular, apply (4) to κ''_α:

$$4(i_\beta - i_\alpha) - 1 = \sum_{\alpha'} (i_{\beta\alpha'} - i_{\alpha\alpha'}) - 1,$$

where on the right the sum extends to those values of α' for which the summand is positive, and the term -1 arises from the corresponding F-curve $\kappa'_{\beta\alpha}$ of the second species.

But, by the same formula for T,

$$4(i_\beta - i_\alpha) = \sum_{\alpha'} (i_{\beta\alpha'} - i_{\alpha\alpha'}),$$

summed for all values of α', whether the summand is positive, zero, or negative. Hence there are no negative terms in the last sum, that is, $i_{\alpha\alpha'} \leqslant i_{\beta\alpha'}$ provided that $i_\alpha < i_\beta$ and ω_α meets ω_β, as was proved geometrically in § 4.

We have here assumed that ω_α is of the first species for T; if it is of the second species, every κ'_α is adjacent to the curve $\omega'_{\alpha'}$ which $\underset{T}{\sim} \omega_\alpha$, and which is an F-curve of contact for W. But though W is no longer ordinary, κ''_α is an ordinary F-curve of W, for which (4) still holds.

We shall next apply to W the formula (23), which can be deduced from those already proved. We need to know which has the greater multiplicity of each pair of intersecting F-curves of W.

The difference of multiplicities of $\omega'_{\beta'}$, $\omega'_{\alpha'}$ is

$$x(i'_{\beta'} - i'_{\alpha'}) - y_\alpha(i'_{\beta'\alpha} - i'_{\alpha'\alpha}).$$

Since x is large compared with y_α, we have $\omega'_{\alpha'}$ of lower multiplicity provided that either

$$i''_{\alpha'} < i''_{\beta'}$$

or else

$$i''_{\alpha'} = i''_{\beta'} \text{ and also } i''_{\alpha'\alpha} > i''_{\beta'\alpha}.$$

The last alternative occurs if, and only if, $\omega'_{\alpha'}$, $\omega'_{\beta'}$ are a pair of F-curves of T selected by ω_α and an ω_β of equal multiplicity.

The multiplicity for W of ω'_δ is less than that of $\omega'_{\alpha'}$; its relation to another curve ω'_ϵ of the same set as itself is the same as that of ω_δ to ω_ϵ for V. The F-curves of W of the second species are all of lower multiplicities than any F-curves which they meet.

Hence the term in (23) arising from the intersection of $\omega'_{\alpha'}$, $\omega'_{\beta'}$ is

$$(xi''_{\lambda'} - y_\alpha i''_{\lambda'\alpha}) D'_{\alpha'\beta'} \text{ if } i''_{\alpha'} \neq i''_{\beta'}, \text{ or } i''_{\alpha'} = i''_{\beta'} \text{ and } i''_{\alpha'\alpha} = i''_{\beta'\alpha};$$

and it is $\{xi''_{\lambda'} - y_\alpha(i''_{\lambda'\alpha} + 1)\} D'_{\alpha'\beta}$ if $i''_{\alpha'} = i''_{\beta'}$ and $i''_{\alpha'\alpha} \neq i''_{\beta'\alpha}$,

$$= (xi''_{\lambda'} - y_\alpha i''_{\lambda'\alpha}) D'_{\alpha'\beta'} - y_\alpha D_{\alpha\beta} \text{ where } i_\alpha = i_\beta.$$

The equation is therefore :

$$\sum_{\alpha'}\sum_{\beta'}(xi''_{l'}-y_\alpha i'_{l'\alpha})\,D'_{\alpha'\beta'}-y_\alpha\sum_{\beta}D_{\alpha\beta}\ (\text{for } i_\alpha=i_\beta)+\sum_{\delta}\sum_{\epsilon}y_l D_{\delta\epsilon}$$

$$+\sum_{\delta}\sum_{\alpha'}y_\delta(d'_{\alpha'}m_\delta-i_{\alpha\alpha'}D_{\alpha\delta})+\sum_{\beta}\sum_{\alpha'}y_\alpha(i_{\beta\alpha'}-i_{\alpha\alpha'})\,D_{\alpha\beta}\ (\text{for } i_\alpha<i_\beta)$$

$$+\sum_{\delta}(y_\delta-y_\alpha)(\sum_{\alpha'}i_{\alpha\alpha'}+1)\,D_{\alpha\delta}\ (\text{for } y_\alpha<y_\delta)$$

$$-\sum_{\alpha'}(xi''_{\alpha'}-y_\alpha i'_{\alpha'\alpha})(4m'_{\alpha'}-p'_{\alpha'}+1)-\sum_{\delta}y_\delta\{4(nm_\delta-i_\alpha D_{\alpha\delta})-p_\delta+1\}$$

$$-\sum_{\beta}\{4(i_\beta-i_\alpha)+1\}\,y_\alpha D_{\alpha\beta}\ (\text{for } i_\alpha<i_\beta)$$

$$-\sum_{\delta}(4i_\alpha+1)(y_\delta-y_\alpha)\,D_{\alpha\beta}\ (\text{for } y_\alpha<y_\delta)$$

$$+11(xn'-y_\alpha d_\alpha-1)=0.$$

By means of equations already proved, applied to T and to V, this reduces to (30)'. If ω_α is of the second species for T, (30) becomes the same as (21), already proved for either species.

The rest of the formulae of § 5 now follow by combination and summation of those already proved. Many others can be obtained by pursuing the methods further. For example, Noether's formula for the genus $-p'_{\alpha'}$ of $j_{\alpha'}$ gives

$$(d'_{\alpha'}-1)(d'_{\alpha'}-2)(d'_{\alpha'}-3)+6p'_{\alpha'}$$

$$=\sum_{\alpha}i_{\alpha\alpha'}(i_{\alpha\alpha'}-1)\{(3d'_{\alpha'}-6)\,m_\alpha-(2i_{\alpha\alpha'}-1)(2m_\alpha+p_\alpha-1)\}$$

$$-\sum_{\alpha}\sum_{\beta}i_{l\alpha'}(i_{l\alpha'}-1)(3i_{h\alpha'}-i_{l\alpha'}-1)\,D_{\alpha\beta}.$$

ÜBER DIE VERALLGEMEINERUNG EINIGER THEOREME VON HARDY IN DER THEORIE DER FOURIER'SCHEN REIHEN

Von M. JACOB.

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1. Die folgenden Untersuchungen knüpfen an eine bekannte Abhandlung von Hardy* an. Hardy hat dortselbst bewiesen :

Ist $f(\xi)$ eine integrierbare Funktion von der Periode 2π und

$$(1) \qquad \frac{a_0}{2}+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu x+b_\nu\sin\nu x)\equiv\mathfrak{A}_0+\sum_{\nu=1}^{\infty}\mathfrak{A}_\nu(x)$$

ihre Fourierreihe, dann gilt in jedem Punkte x, wo die Beziehung :

$$(A.0) \qquad \lim_{h\to 0}\frac{1}{h}\int_0^h|g(t)|\,dt=0$$

$$\{g(t)=f(x+t)+f(x-t)-2f(x)\}$$

erfüllt ist :

(a) $C^{(\delta)}\{s_n(x)\}\to f(x)\ \ (\delta>0), \qquad C^{(\delta)}\{s_n(x)\}=\mathfrak{A}_0+\sum_{\nu=1}^{n-1}\left(1-\frac{\nu}{n}\right)^{\delta}\mathfrak{A}_\nu(x),$

(b) $$s_n(x)=o(\log n),$$

(c) $$\sum_{\nu=2}^{\infty}\frac{\mathfrak{A}_\nu(x)}{\log\nu} \text{ ist konvergent.}$$

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* G. H. Hardy, "On the summability of Fourier's series", *Proc. London Math. Soc.* (2), 12 (1913), 365–372.

Es sei :

$$(2)\qquad \begin{cases} g_1(t) = \int_0^t g_0(u)\,du, \ \ldots, \ g_r(t) = \int_0^t g_{r-1}(u)\,du, \\ G_0(t) = g_0(t) = g(t), \\ G_1(t) = \dfrac{1}{t}\, g_1(t), \quad \ldots, \ G^r(t) = \dfrac{r!}{t^r}\, g_r(t). \end{cases}$$

In Hinsicht auf die obige Bezeichnungsweise wird sich nun unsere Untersuchung in zwei Richtungen bewegen. Einerseits werden wir die Bedingung (A . 0) für ein beliebiges ganzzahliges r verallgemeinern, woraus sich mit Hilfe der Cesàro'schen Summierung die entsprechende Erweiterung der Sätze (a), (b), und (c) ergeben wird; anderseits wird es uns gelingen, den Satz (b) für jede beliebige Ordnung $r' > -1$ der C-Summierung und den Satz (c) auf den Fall $-1 < \delta < 0$ auszudehnen.

Wir behaupten : Ist die nachstehende Bedingung :

$$(\mathrm{A}\,.\,r)\qquad \lim_{h\to 0} \frac{1}{h} \int_0^h | G_r(t) |\, dt = 0 \quad (r \geqslant 0)^*$$

erfüllt, dann gelten die folgenden Sätze :

$$(\mathrm{a}\,.\,1)\qquad C^{(r+\delta)}\{s_n(x)\} \to f(x) \quad (\delta > 0),$$

$$(\mathrm{b}\,.\,1)\qquad C^{(r)}\{s_n(x) = o(\log n),$$

$$(\mathrm{b}\,.\,2)\qquad C^{(r-\delta)}\{s_n(x)\} = o(n^\delta) \quad (0 < \delta < 1),$$

$$(\mathrm{c}\,.\,1)\qquad \sum_{\nu=2}^{\infty} \frac{\mathfrak{A}_\nu(x)}{\log \nu} \text{ ist } (C, r) \text{ summierbar},$$

$$(\mathrm{c}\,.\,2)\qquad \sum_{\nu=1}^{\infty} \frac{\mathfrak{A}_\nu(x)}{\nu^\delta\, \phi_\nu} \text{ ist } (C, -\delta) \text{ summierbar für } r = 0 \text{ und } 0 < \delta < 1,$$

wobei $\{1/\phi_\nu\}$ eine konvexe† Nullfolge bedeutet.

Bemerkung zu (c . 12); Für quadratisch integrierbare Funktionen erhält man den Satz

(c. 21). *1st* $\sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2)\, \nu^{2\delta}$ *konvergent, dann ist die Fourierreihe* (1) *fast überall* $(C, -\delta)$ *summierbar.*

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* (Zusatz während der Korrektur.) Diese Voraussetzung kann für die Giltigkeit des Satzes (a. 1) verallgemeinert werden; die entsprechende Formulierung findet man am Schlusse dieser Arbeit.

† Wir sagen, dass eine Folge $\{\lambda_\nu\}$ konvex (konkav) ist, wenn die zweiten Differenzen $\{\Delta^2 \lambda_\nu\}$ von einem bestimmten Gliede angefangen, beständig positiv (negativ) sind.

Der Satz (c . 21) hängt bekanntlich mit der ganzen Kette* von Untersuchungen zusammen, deren Anfang auf Fatou, Jerosch und Weyl zurückgeht und an welche sich die Arbeiten von W. H. Young, Hardy, Kolmogoroff und Seliverstoff und von Plessner anschliessen. In anderer Hinsicht steht der Satz (c. 121) in engem Zusammenhange mit einem Satze von A. Zygmund† über die $(C, -\gamma)$ Summierbarkeit der Fourierreihe einer stetigen Funktion, welche im ganzen Intervalle $[0, 2\pi]$ einer Lipschitzbedingung von der Ordnung $\delta > \gamma$ genügt.

In Anlehnung an einen Satz von Hardy und Littlewood‡ und von Zygmund§ lassen sich die Analoga der Sätze (b . 2) und (c . 2) für die konjugierten Reihen im Falle $r = 0$ durchführen.

Ersetzt man die Bedingung (A . r) durch die folgende:

$$(\mathrm{B}\,.\,r) \qquad \lim_{h \to 0} \frac{1}{h} \int_0^h G_r(t)\,dt = 0,$$

so gelten die Sätze (a . 1) bis (c . 1), wenn man in denselben r durch $r+1$ ersetzt. Denn, wie man leicht beweisen kann, ist die Behauptung (B . r) aequivalent mit der Voraussetzung:

$$\lim_{t \to 0} G_{r+1}(t) = 0$$

und folglich gilt $(A\,.\,r+1)$||.

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* P. Fatou, "Séries trigonométriques et séries de Taylor", *Acta Math.*, 30 (1906), 335–400. F. Jerosch und H. Weyl, "Über die Konvergenz von Reihen, die nach periodischen Funktionen fortschreiten", *Math. Annalen*, 66 (1908), 67–80. H. Weyl, "Über die Konvergenz von Reihen, die nach Orthogonalfunktionen fortschreiten", *Math. Annalen*, 67 (1909), 225–245. G. H. Hardy, *loc. cit.*, 1. W. H. Young, "Sur les séries de Fourier convergentes presque partout", *Comptes rendus*, 155 (1912), 1480–1482. A. Kolmogoroff und G. Seliverstoff, "Sur la convergence des séries de Fourier", *Comptes rendus*, 178 (1924), 303-306. A. Plessner, Über die Konvergenz von trigonometrischen Reihen", *Journal für Math.*, 155 (1925), 15–25.

† A. Zygmund, "Sur la sommabilité des séries de Fourier des fonctions vérifiant la condition de Lipschitz", *Bull. Intern. de l'Acad. de Cracovie*, Série A : Sc. math. (1925), 1-9.

‡ G. H. Hardy and J. E. Littlewood, "The allied series of a Fourier series", *Proc. London Math. Soc.* (2), 24 (1925), 211–246.

§ A. Zygmund, "Sur la sommation des series trigonométriques conjuguées aux séries de Fourier", *Bull. Intern. de l'Acad. de Cracovie*, Série A : Sc. math., (1924), 250-258.

|| G. H. Hardy and J. E. Littlewood, "Solution of the Cesàro summability problem for power series and Fourier series", *Math. Zeitschrift*, 19 (1924), 67–96; "The equivalence of certain integral means", *Proc. London Math. Soc.* (2), 22 (1923), xl-xliii (*Records* of Proceedings at Meetings, 14 June, 1923).

In anderer Hinsicht kann aus der Voraussetzung (A . r) der folgende Satz hergeleitet werden:

$$(a.2) \qquad R^{(1)}\{C^{(r-1)}\{s_n(x)\}\} \to f(x) \quad (r \geqslant 1),$$

wobei $R^{(1)}$ das erste logarithmische Mittel von M. Riesz* bedeutet. Daher ergibt sich aus (B . r):

$$R^{(1)}[C^{(r)}\{s_n(x)\}] \to f(x) \quad (r \geqslant 0).\dagger$$

Endlich wollen wir in diesem Zusammenhange auf die Lebesgue'schen Konstanten und die Landau'sche‡ obere Grenze für beschränkte Potenzreihen für $(C, -\delta)$, $0 < \delta < 1$, ein wenig eingehen.

Beweis der Sätze (a . 1) *und* (a . 2).

2. Im folgenden sei immer, wenn nicht anderes angegeben wird, die Voraussetzung (A . r) als erfüllt angenommen.

Satz (a . 1): $$C^{(r+\delta)}\{s_n(x)\} \to f(x) \quad (\delta > 0).$$

Den Beweis des obigen Satzes werden wir mit Hilfe der Young'schen§ Funktion:

$$C_p(t) = \frac{t^p}{\Gamma(p+1)}\left\{1 - \frac{t^2}{(p+1)(p+2)} + \frac{t^4}{(p+1)(p+2)(p+2)(p+4)} - \cdots\right\}$$

$$(p \geqslant 0)$$

durchführen, wobei wir uns einen Gedanken von Fejér‖ zu Nutze machen werden. Wir stützen uns also auf die diesbezüglichen Resultate von

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* M. Riesz, "Sur la sommation des séries de Dirichlet", *Comptes rendus*, 149 (1909), 18–21.

† Im Falle $r = 0$ ist dieser Satz von A. Zygmund bewiesen worden. A. Zygmund, "Sur la dérivation des séries de Fourier", *Bull. Intern. de l'Acad. de Cracovie*, Série A: Sc. math. (1924), 243–249; "Sur un théorème de M. Gronwall", *ibid.* (1925), 207–217.

‡ E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie* (1916), Kap. I.

§ W. H. Young, "On infinite integrals involving a generalization of the sine and cosine functions", *Quart. J. of Math.*, 43 (1911), 161–177.

‖ L. Fejér, "Über die arithmetischen Mittel erster Ordnung der Fourierreihe", *Göttinger Nachrichten* (1925), 13–17.

W. H. Young. Nach ihm findet man :

$$C^{(r+\delta)}\{s_n(x)\} - f(x) = \int_0^\infty g(t)\,\phi_{r+1+\delta}(nt)\,dt \tag{3}$$

wobei $\phi_p(nt) = \dfrac{\Gamma(p)}{\pi}\, n\, C_p(nt)(nt)^{-p}$ bedeutet. Aus den Young'schen Abschätzungen ergibt sich :

$$\phi_p^{(0)}(nt) = \phi_p(nt) = \begin{cases} O\left[\dfrac{n}{(1+nt)^p}\right] & \text{für} \quad 0 \leqslant p \leqslant 2, \\ O\left[\dfrac{n}{(1+nt)^2}\right] & \text{für} \quad p > 2, \end{cases} \qquad 0 \leqslant t \leqslant \infty. \tag{4}$$

Bezeichnet man : $\phi_p^{(r)}(nt) = \dfrac{d^r}{dt^r}\,\phi_p(nt)$

so findet man für* $p \geqslant r \geqslant 1$.

$$\phi_p^{(r)}(nt) = \frac{(-1)^r}{t_r} \sum_{\nu=0}^{r} c_\nu^{p,r}\,\phi_{p-\nu}(nt), \tag{5}$$

$$c_\nu^{p,r} = (-1)^\nu \binom{r}{\nu} \frac{\Gamma(p+r-\nu)}{\Gamma(p-\nu)}, \qquad c_r^{r,r} = (-1)^r\,\Gamma(r).$$

In Hinsicht auf (4) und (5) ergibt sich also für $t \neq 0$:

$$\begin{cases} \phi_p^{(r)}(nt) = o(1) \quad \text{sobald} \quad p > r+1 \\ \phi_{r+1}^{(r)}(nt) = O(1), \quad \phi_r^{(r)}(nt) = O(n) \end{cases} \qquad (r \geqslant 0). \tag{6}$$

Nach diesen Bemerkungen wollen wir zeigen, daß das Integral (3) mit wachsendem n verschwindet; wir beschränken uns hierbei auf den Fall $0 < \delta \leqslant 1$.

$$I_n = \int_0^\infty g(t)\,\phi_{r+1+\delta}(nt)\,dt = \int_0^h + \int_h^\infty = I_n^{(1)} + I_n^{(2)} \quad (h > 0).$$

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* $\phi_0(nt) = \dfrac{1}{\pi}\, n \cos nt, \quad \phi_1(nt) = \dfrac{1}{\pi}\,\dfrac{\sin nt}{t}, \quad \phi_2(nt) = \dfrac{1}{2n\pi}\left(\dfrac{\sin \frac{1}{2}nt}{\frac{1}{2}t}\right)^2,$

z.B. für $r \geqslant 3$:

$$\phi_r^{(r-1)}(nt) = \frac{(-1)^{r-1}}{t^{r-1}} \left\{ \sum_{\nu=0}^{r-3} c_\nu^{r,r-1}\,\phi_{r-\nu}(nt) + c_{r-2}^{r,r-1}\,\frac{1}{2n\pi}\left(\frac{\sin \frac{1}{2}nt}{\frac{1}{2}t}\right)^2 + c_{r-1}^{r,r-1}\,\frac{1}{\pi}\,\frac{\sin nt}{t} \right\},$$

$$\phi_r^{(r)}(nt) = \frac{(-1)^r}{t^r} \left\{ \sum_{\nu=1}^{r-3} c_\nu^{r,r}\,\phi_{r-\nu}(nt) + c_{r-2}^{r,r}\,\frac{1}{2n\pi}\left(\frac{\sin \frac{1}{2}nt}{\frac{1}{2}t}\right)^2 + c_{r-1}^{r,r}\,\frac{1}{\pi}\,\frac{\sin nt}{t} + c_r^{r,r}\,\frac{1}{\pi}\, n \cos nt \right\}.$$

Mit Rücksicht auf (4) erhalten wir für $I_n^{(2)}$:

$$(7)\quad I_n^{(2)} = \int_0^\infty g(t)\,\phi_{r+1+\delta}(nt)\,dt = \begin{cases} O\left[n\int_h^\infty \frac{|g(t)|}{(1+nt)^{1+\delta}}\,dt\right] & \text{für } r=0, \\ O\left[n\int_h^\infty \frac{|g(t)|}{(1+nt)^{2}}\,dt\right] & \text{für } r\geqslant 1. \end{cases}$$

Nun ist:

$$n\int_h^\infty \frac{|g(t)|}{(1+nt)^{1+\delta}}\,dt = n\int_h^\pi \frac{|g(t)|}{(1+nt)^{1+\delta}}\,dt + \sum_{\nu=1}^{\infty} n\int_{\pi\nu}^{\pi(\nu+1)} \frac{|g(t)|}{(1+nt)^{1+\delta}}\,dt = O\left(\frac{1}{n^\delta}\right)$$

und somit:
$$I_n^{(2)} = o(1). \qquad \text{(I)}$$

Für $I_n^{(1)}$ bekommt man durch partielle Integration:

$$(8)\quad I_n^{(1)} = \int_0^h g(t)\,\phi_{r+1+\delta}(nt)\,dt = \sum_{\nu=1}^{r} (-1)^{\nu-1}\left[g_\nu(t)\,\phi_{r+1+\delta}^{(\nu-1)}(nt)\right]_0^h$$
$$+(-1)^r \int_0^h g_r(t)\,\phi_{r+1+\delta}^{(r)}(nt)\,dt = I_n^{(3)} + I_n^{(4)*}.$$

Zufolge (6) ist

$$\phi_{r+1+\delta}^{(\nu-1)}(nh) = o(1) \quad \text{für} \quad \nu = 1, 2, \ldots, r,$$

und daher:
$$I_n^{(3)} = o(1). \qquad \text{(II)}$$

Schliesslich erhält man für $I_n^{(4)}$ mit Berücksichtigung von (2), (4), (5) und (6):

$$(9)\quad I_n^{(4)} = (-1)^r \int_0^h g_r(t)\,\phi_{r+1+\delta}^{(r)}(nt)\,dt$$
$$= \frac{1}{r!}\int_0^h G_r(t)\left(\sum_{\nu=0}^{r} c_\nu^{r+1+\delta,\,r}\,\phi_{r+1+\delta-\nu}(nt)\right)dt$$
$$= O\left\{\sum_{\nu=0}^{r-1}\int_0^h |G_r(t)|\,\phi_{r+1+\delta-\nu}(nt)\,dt + \int_0^h |G_r(t)|\,|\phi_{1+\delta}(nt)|\,dt\right\}$$
$$= O\left\{\int_0^h |G_r(t)|\,\frac{n}{(1+nt)^{1+\delta}}\,dt\right\}\dagger.$$

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* Für $r=0$ fehlt das Glied $I_n^{(3)}$.

† Durch Einführung dieses Majorantenkernes wird bei allen folgenden Beweisen die übliche Einteilung des Integrales $\int_0^h$ in $\int_0^{1/n} + \int_{1/n}^h$ vermieden. Diese Idee verdankt man Herrn Fejér, *loc. cit.*

Unter der Voraussetzung (A . r), wonach

$$G_r^*(h) = \int_0^h |\, G_r(t)\,|\, dt = o(h)$$

liefert das letzte Integral in (9) :

$$(10)\quad \int_0^h |\, G_r(t)\,|\, \frac{n}{(1+nt)^{1+\delta}}\, dt = \left[G_r^*(t)\, \frac{n}{(1+nt)^{1+\delta}}\right]_0^h + \int_0^h G_r^*(t)\, \frac{(1+\delta)\, n^2}{(1+nt)^{2+\delta}}\, dt$$

$$= O\left(\frac{1}{n^\delta}\right) + (1+\delta) \int_0^h \frac{G_r^*(t)}{t}\, \frac{nt}{(1+nt)}\, \frac{n}{(1+nt)^{1+\delta}}\, dt$$

$$= o(1) + o(1)\left(\int_0^h \frac{n}{(1+nt)^{1+\delta}}\, dt\right) = o(1).$$

Hiermit ist : $$I_n^{(4)} = o(1). \qquad \text{(III)}$$

w.z.b.w.

3. *Satz* (a . 2) $\qquad R^{(1)}\left[C^{(r-1)}\{s_n(x)\}\right] \to f(x) \quad (r \geqslant 1).$

Für $r = 1$ ist bekanntlich :

$$u_n^{(0)} = s_n(x) - f(x) = \frac{1}{\pi} \int_0^h g(t)\, \frac{\sin nt}{t}\, dt + o(1).$$

Für $r \geqslant 2$ ist nach (3) :

$$u_n^{(r-1)} = C^{(r-1)}\{s_n(x)\} - f(x) = \int_0^h g(t)\, \phi_r(nt)\, dt + o(1).$$

Es ist also zu beweisen, daβ

$$(11)\qquad R^{(1)}(u_n^{(r-1)}) = \frac{1}{\log n}\left[\frac{u_1^{(r-1)}}{1} + \frac{u_2^{(r-1)}}{2} + \ldots + \frac{u_n^{(r-1)}}{n}\right] = o(1).$$

Durch r-malige partielle Integration erhält man für $u_n^{(r-1)}$:

$$(12)\quad u_n^{(r-1)} = \left[g_1(t)\, \phi_r(nt)\right]_0^h + \sum_{\nu=2}^{r-1} (-1)^{\nu-1} \left[g_\nu(t)\, \phi_r^{(\nu-1)}(nt)\right]_0^h$$

$$+ (-1)^{r-1} \left[g_r(t)\, \phi_r^{(r-1)}(nt)\right]_0^h + (-1)^r \int_0^h g_r(t)\, \phi_r^{(r)}(nt)\, dt$$

$$= v_{n,1}^{(r-1)} + v_{n,2}^{(r-1)} + v_{n,3}^{(r-1)} + v_{n,}^{(r-1)*}.$$

---

* Im Falle $r = 1$ kommen in (12) die Glieder $v_{n,1}^{(0)}$, $v_{n,2}^{(0)}$, $v_{n\,3}^{(0)}$ nicht vor, für $r = 2$ fehlen $v_{n,1}^{(1)}$ und $v_{n,2}^{(1)}$, für $r = 3$ fehlt noch $v_{n,2}^{(2)}$.

Zufolge (6) verschwinden alle Glieder in $v_{n,1}^{(r-1)}$ und $v_{n,2}^{(r-1)}$ mit wachsendem n und daher ist:

$$R^{(1)}(v_{n,1}^{(r-1)}) = o(1), \tag{I}$$

$$R^{(1)}(v_{n,2}^{(r-1)}) = o(1). \tag{II}$$

$$(13) \qquad v_{n,3}^{(r-1)} = (-1)^{r-1} g_r(h)\, \phi_r^{(r-1)}(nh)$$

$$= \frac{g_r(h)}{h^{r-1}} \sum_{\nu=0}^{r-1} c_\nu^{r,r-1}\, \phi_{r-\nu}(nh)$$

$$= G_r(h) \left\{ \frac{h}{r!} \sum_{\nu=0}^{r-2} c_\nu^{r,r-1}\, \phi_{r-\nu}(nh) + c_{r-1}^{r,r-1} \frac{1}{\pi} \frac{\sin nh}{h} \right\}.$$

In (13) verschwinden zufolge (6) alle Summenausdrücke mit wachsendem n bis auf das Glied:

$$\frac{1}{r!} \frac{1}{\pi} G_r(h) \sin nh.$$

Nun ist aber:

$$R^{(1)}(\sin nh) = \frac{1}{\log n} \left[\sum_{\nu=1}^{n} \frac{\sin \nu h}{\nu} \right] = O\left(\frac{1}{\log n}\right) = o(1)$$

und somit

$$R^{(1)}(v_{n,3}^{(r-1)}) = o(1). \tag{III}$$

$$(14) \qquad v_{n,4}^{(r-1)} = (-1)^r \int_0^h g_r(t)\, \phi_r^{(r)}(nt)\, dt = \frac{1}{r!} \int_0^h G_r(t) \left(\sum_{\nu=0}^{r} c_\nu^{r,r}\, \phi_{r-\nu}(nt) \right) dt$$

$$= \sum_{\nu=0}^{r-2} \frac{c_\nu^{r,r}}{r!} \int_0^h G_r(t)\, \phi_{r-\nu}(nt)\, dt + \frac{c_{r-1}^{r,r}}{r!\,\pi} \int_0^h G_r(t) \frac{\sin nt}{t}\, dt$$

$$+ \frac{c_r^{r,r}}{r!\,\pi} \int_0^h G_r(t)\, n \cos nt\, dt$$

$$= w_{n,1}^{(r-1)} + w_{n,2}^{(r-1)} + w_{n,3}^{(r-1)*}.$$

In $w_{n,1}^{(r-1)}$ finden wir den Fall wieder, der schon beim Beweise des Satzes

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* Im Falle $r = 1$ fehlt das Glied $w_{n,1}^{(0)}$.

(a . 1) unter (8) vorgekommen ist. Daher ist :

$$R^{(1)}(w_{n,1}^{r-1}) = o(1). \tag{IV}$$

Es bleibt also noch die Glieder $w_{n,2}^{(r-1)}$ und $w_{n,3}^{(r-1)}$ zu behandeln ; in diesen Fällen tritt das Wesentliche der logarithmischen Summmierung hervor.

$$w_{n,2}^{(r-1)} = \frac{c_{r-1}^{r,r}}{r!\ \pi} \int_0^h G_r(t)\, \frac{\sin nt}{t}\, dt,$$

$$w_{n,3}^{(r-1)} = \frac{c_r^{r,r}}{r!\ \pi} \int_0^h G_r(t)\, n \cos nt\, dt.$$

$$R^{(1)}(w_{n,2}^{(r-1)}) = \frac{c_{r-1}^{r,r}}{r!\ \pi} \frac{1}{\log n} \int_0^h G_r(t) \left(\sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu t} \right) dt. \tag{15}$$

Nun berechnet man leicht :

$$\sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu t} = O\left(\frac{n}{1+nt}\right) \quad \text{für} \quad t \geqslant 0,$$

und somit :

$$\begin{aligned} R^{(1)}(w_{n,2}^{(r-1)}) &= O\left\{ \frac{1}{\log n} \int_0^h |G_r(t)|\, \frac{n}{1+nt}\, dt \right\} \\ &= O\left\{ \frac{1}{\log n}\, G^*(h)\, \frac{n}{1+nh} + \frac{1}{\log n} \int_0^h G^*(t)\, \frac{n^2}{(1+nt)^2}\, dt \right\} \\ &= o\left(\frac{1}{\log n}\right) + o\left(\frac{1}{\log n}\right) \left(\int_0^h \frac{nt}{1+nt}\, \frac{n}{1+nt}\, dt \right) \\ &= o(1) + o\left(\frac{1}{\log n}\right) O(\log n) = o(1). \end{aligned} \tag{V}$$

$$R^{(1)}(w_{n,3}^{(r-1)}) = \frac{c_r^{r,r}}{r!\ \pi} \frac{1}{\log n} \int_0^h G_r(t) \left(\sum_{\nu=1}^{n} \cos \nu t \right) dt. \tag{16}$$

Aus den Fejér'schen* Abschätzungen ergibt sich :

$$\sum_{\nu=1}^{n} \cos \nu t = O\left(\frac{n}{1+nt}\right) \quad \text{für} \quad 0 \leqslant t \leqslant \pi$$

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* L. Fejér, *loc. cit.*

und daher: $$R^{(1)}(w_{n,3}^{(r-1)}) = O\Big(\frac{1}{\log n}\int_0^h |G_r(t)|\,\frac{n}{1+nt}\,dt\Big)$$

$$= o(1). \qquad \text{(VI)}$$

Somit ist:

$$R^{(1)}(u_n^{(r-1)}) = \text{(I)}+\text{(II)}+\text{(III)}+\text{(IV)}+\text{(V)}+\text{(VI)} = o(1)^*. \quad \text{w.z.b.w.}$$

Ersetzt man die Bedingung (A . r) durch die Bedingung (B . r), so erhält man die Parallelsätze:

(a' . 1) $$C^{(r+1+\delta)}\{s_n(x)\} \to f(x)$$

(a' . 2) $$R^{(1)}[C^{(r)}\{s_n(x)\}] \to f(x) \qquad (r \geqslant 0).$$

Beweis der Sätze (b) *und* (c).

4. *Satz* (b . 1) $$C^{(r)}\{s_n(n)\} = o(\log n) \quad (r \geqslant 0).$$

Nach den früheren Ausführungen haben wir:

(17) $$C^{(r)}\{s_n(x)\} - f(x) = \int_0^h g(t)\,\phi_{r+1}(nt)\,dt + o(1).$$

Da es auf eine additive Konstante nicht ankommt, so ist die Behauptung des Satzes (b . 1) gleichwertig mit der folgenden:

(17 . 1) $$\int_0^h g(t)\,\phi_{r+1}(nt)\,dt = o(\log n).$$

Durch r-malige partielle Integration erhält man:

(18) $$\int_0^h g(t)\,\phi_{r+1}(nt)\,dt = \Big[g_1(t)\,\phi_{r+1}(nt)\Big]_0^h + \sum_{\nu-2}^{r}(-1)^{r-1}\Big[g_\nu(t)\,\phi_{r+1}^{(\nu-1)}(nt)\Big]_0^h$$

$$+(-1)^r\int_0^h g_r(t)\,\phi_{r+1}^{(r)}(nt)\,dt$$

$$= u_{n,1}^{(r)} + u_{n,2}^{(r)} + u_{n,3}^{(r)}\dagger.$$

Zufolge (6) ist: $$u_{n,1}^{(r)} = o(1), \quad u_{n,2}^{(r)} = o(1).$$

---

* Es sei an dieser Stelle bemerkt, daß die Giltigkeit des Satzes (a . 2) bereits aus der schwächeren Voraussetzung: $\int_a^h \frac{|G_r(t)|}{t}\,dt = o\Big(\log\frac{1}{a}\Big)$ hergeleitet werden kann.

† Im Falle $r = 0$ fehlen die Glieder $u_{n,1}^{(0)}$ und $u_{n,2}^{(0)}$, für $r = 1$ fehlt noch das Glied $u_{n,2}^{(1)}$.

Für das Integral $u_{n,3}^{(r)}$ bekommt man:

$$(19)\quad u_{n,3}^{(r)} = (-)^r \int_0^h g_r(t)\,\phi_{r+1}^{(r)}(nt)\,dt = \frac{1}{r!}\int_0^h G_r(t)\left(\sum_{\nu=0}^{r} c_\nu^{r+1,r}\,\phi_{r+1-\nu}(nt)\right)dt$$

$$= \frac{1}{r!}\sum_{\nu=0}^{r-1} c_\nu^{r+1,r}\int_0^h G_r(t)\,\phi_{r+1-\nu}(nt)\,dt + \frac{c_r^{r+1,r}}{r!\,\pi}\int_0^h G_r(t)\,\frac{\sin nt}{t}\,dt$$

$$= O\left(\int_0^h |G_r(t)|\,\frac{n}{(1+nt)^2}\,dt + \int_0^h |G_r(t)|\,\frac{n}{1+nt}\,dt\right)$$

$$= o(1)+o(\log n) = o(\log n). \qquad \text{w.z.b.w.}$$

5. *Satz* (b. 2) $\qquad C^{(r-\delta)}\{s_n(x)\} = o(n^\delta) \quad (r \geqslant 0,\ 0 < \delta < 1).$

Der Beweis des obigen Satzes zerfällt in zwei Teile:

$$\text{A. } r \geqslant 1, \quad \text{und} \quad \text{B. } r = 0.$$

A. $$C^{(r-\delta)}\{s_n(x)\} - f(x) = \int_0^h g(t)\,\phi_{r+1-\delta}(nt)\,dt + o(1).$$

Durch r-malige partielle Integration bekommt man:

$$(20)\quad \int_0^h g(t)\,\phi_{r+1-\delta}(nt)\,dt = g_1(t)\,\phi_{r+1-\delta}(nt)\int_0^h + \sum_{\nu=2}^{r}(-1)^{\nu-1} g_\nu(t)\,\phi_{r+1-\delta}^{(\nu-1)}(nt)\int_0^h$$

$$+(-)^r \int_0^h g_r(t)\,\phi_{r+1-\delta}^{(r)}(nt)\,dt$$

$$= v_{n,1}^{(r)} + v_{n,2}^{(r)} + v_{n,3}^{(r)}\,*.$$

Wegen (6) ist: $\qquad v_{n,1}^{(r)} = o(1), \quad v_{n,2}^{(r)} = o(1).$

Es bleibt also, unter Berücksichtigung der Abschätzungen (8) noch das Glied $v_{n,3}^{(r)}$ zu untersuchen:

$$(21)\quad v_{n,3}^{(r)} = (-1)^r \int_0^h g_r(t)\,\phi_{r+1-\delta}^{(r)}(nt)\,dt$$

$$= \frac{1}{r!}\int_0^h G_r(t)\left(\sum_{\nu=0}^{r} c_\nu^{r+1-\delta,\nu}\,\phi_{r+1-\delta-\nu}(nt)\right)dt$$

$$= O\left\{\sum_{\nu=0}^{r-1}\int_0^h |G_r(t)|\,|\phi_{r+1-\delta-\nu}(nt)|\,dt + \int_0^h |G_r(t)|\,|\phi_{1-\delta}(nt)|\,dt\right\}$$

$$= o(1)+O\left\{\int_0^h |G_r(t)|\,\frac{n}{(1+nt)^{1-\delta}}\,dt\right\}.$$

* Im Falle $r = 1$ fehlt das Glied $v_{n,2}^{(1)}$.

Partiell integriert, ergibt das letzte Integral:

$$(22)\quad \int_0^h |G_r(t)| \frac{n}{(1+nt)^{1-\delta}}\,dt$$

$$= \left[G_r^*(t)\frac{n}{(1+nt)^{1-\delta}}\right]_0^h + \int_0^h G_r^*(t)\frac{(1-\delta)n^2}{(1+nt)^{2-\delta}}\,dt$$

$$= \frac{G_r^*(h)}{h}\frac{nh}{1+nh}\frac{1}{(1+nh)^{-\delta}} + (1-\delta)\int_0^h \frac{G_r^*(t)}{t}\frac{nt}{1+nt}\frac{n}{(1+nt)^{1-\delta}}\,dt$$

$$= o(n^\delta) + o(1)\left(\int_0^h \frac{n\,dt}{(1+nt)^{1-\delta}}\right)$$

$$= o(n^\delta) + o(n^\delta) = o(n^\delta). \qquad \text{w.z.b.w.}$$

B. In diesem Falle ($r = 0$) versagt die bisher verwendete Methode; doch gelingt der Beweis, indem wir uns auf einen bekannten Satz von M. Riesz* über die Binomialreihe stützen. Er lautet:

Es sei $0 < \rho < 1$ *und*

$$\frac{1}{(1-z)^\rho} = \gamma_0 + \gamma_1 z + \ldots + \gamma_n z^n + \ldots,$$

dann ist

$$|\gamma_0 + \gamma_1 z + \ldots + \gamma_n z^n| < \frac{2}{|1-z|^\rho}$$

(für $n = 0,\ 1,\ 2,\ \ldots,\ |z| \leqslant 1,\ z \neq 1$).

Wir haben zu beweisen:

$$(23)\quad C^{(-\delta)}\{s_n(x)\} = \mathfrak{A}_0 + \sum_{\nu=1}^{\infty} \frac{A_{n-\nu}^{(-\delta)}}{A_n^{(-\delta)}}\mathfrak{A}_\nu(x) = o(n^\delta),$$

$$(24)\quad A_n^{(-\delta)} = \frac{\Gamma(n+1-\delta)}{\Gamma(n+1)\Gamma(1-\delta)} \sim \frac{n^{-\delta}}{\Gamma(1-\delta)}.$$

Die Behauptung (23) ist offenbar gleichwertig mit der nachstehenden:

$$(23.1)\quad C^{(-\delta)}\{s_n(x)\} - f(x) = \frac{1}{\pi}\int_0^\pi g(t)\left(\tfrac{1}{2} + \sum_{\nu=1}^{n}\frac{A_{n-\nu}^{(-\delta)}}{A_n^{(-\delta)}}\cos\delta t\right)dt = o(n^\delta).$$

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* M. Riesz, "Sur la sommation des séries de Fourier", *Acta Litterarum ac Scientiarum Univ Hungar.*, 1 (1923), 104-113. Auf den Beweis dieses Satzes verweisen wir: G. Szegö, "Bemerkung zu einer Arbeit von Herrn Fejér über die Legendre'schen Polynome", *Math. Zeitschrift*, 25 (1926), 172-187.

Bezeichnet man :

$$(25)\qquad \begin{cases} \Phi_n^{(-\delta)}(t) = \frac{1}{2}A_n^{(-\delta)} + \sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)} \cos \nu t, \\ \Psi_n^{(-\delta)}(t) = \dfrac{1}{A_n^{(-\delta)}}\, \Phi_n^{(-\delta)}(t), \end{cases}$$

so genügt es mit Hinsicht auf (24) bloβ zu beweisen, daβ :

$$(23.2)\qquad \int_0^{\pi} g(t)\, \Phi_n^{(-\delta)}(t)\, dt = o(1).$$

Es sei $h > 0$; teilt man das Integral (23 . 2) in zwei Teile :

$$\int_0^{\pi} = \int_0^{h} + \int_h^{\pi} = I_{n,1}^{(-\delta)} + I_{n,2}^{(-\delta)}$$

dann gilt zunächst :

$$(26)\qquad I_{n,2}^{(-\delta)} = \int_h^{\pi} g(t)\, \Phi_n^{(-\delta)}(t)\, dt = o(1).$$

Um (26) zu beweisen, genügt es nach einem Satze von H. Lebesgue* bloβ festzustellen, daβ :

(I) $\left|\Phi_n^{(-\delta)}(t)\right| < M(h)$ für alle n, $h \leqslant t \leqslant \pi$,

(II) $\lim\limits_{n\to\infty} \int_{\alpha}^{\beta} \Phi_n^{(-\delta)}(t)\, dt = 0$ für jedes Teilintervall $[\alpha, \beta]$ aus $[h, \pi]$.

(I)
$$\begin{aligned} \Phi_n^{(-\delta)}(t) &= \sum_{\nu=0}^{n} A_{n-\nu}^{(-\delta)} \cos \nu t - \tfrac{1}{2}A_n^{(-\delta)} \\ &= R\left\{ \sum_{\nu=0}^{n} A_{n-\nu}^{(-\delta)}\, e^{-i\nu t} \right\} - \tfrac{1}{2}A_n^{(-\delta)} \\ &= R\left\{ e^{-int} \sum_{\nu=0}^{n} A_{n-\nu}^{(-\delta)}\, e^{i(n-\nu)t} \right\} - \tfrac{1}{2}A_n^{(-\delta)} \\ &= R\left\{ e^{-int} \sum_{\nu=0}^{n} A_{\nu}^{(-\delta)}\, e^{i\nu t} \right\} - \tfrac{1}{2}A_n^{(-\delta)}. \end{aligned}$$

Nach dem Satze von M. Riesz ist :

$$(27)\qquad \left| \sum_{\nu=0}^{n} A_{\nu}^{(-\delta)}\, e^{i\nu t} \right| < \frac{2}{\left|1 - e^{it}\right|^{1-\delta}} = \frac{2}{\left\{2 \sin (t/2)\right\}^{1-\delta}} \quad (0 < t \leqslant 2\pi)$$

* H. Lebesgue, "Sur les intégrales singulières", *Annales de Toulouse* (3), 1 (1911), 25–117.

woraus folgt:

$$(28) \qquad |\Phi_{n(t)}^{(-\delta)}| < 1+\frac{2}{(2\sin\frac{1}{2}h)^{1-\delta}} = M(h) \quad \text{für} \quad h \leqslant t \leqslant \pi.$$

$$(II) \qquad \int_\alpha^\beta \Phi_n^{(-\delta)}(t)\,dt = \tfrac{1}{2}(\beta-\alpha)\,A_n^{(-\delta)} + \sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)}\left\{\frac{\sin\nu\alpha}{\nu}-\frac{\sin\nu\beta}{\nu}\right\}.$$

Nun ist:

$$\sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)}\frac{\sin\nu\alpha}{\nu} = \sum_{\nu=1}^{[n/2]} A_{n-\nu}^{(-\delta)}\frac{\sin\nu\alpha}{\nu} + \sum_{\nu=[n/2]+1}^{n} A_{n-\nu}^{(-\delta)}\frac{\sin\nu\alpha}{\nu}$$

$$= O\left(\frac{\log n}{n^\delta}\right) + O\left(\frac{1}{n}\right) \operatorname*{Max}_{[n/2]+1 \leqslant n' \leqslant n} \left| \sum_{\nu=[n/2]+1}^{n} A_{n-\nu}^{(-\delta)} \sin\nu\alpha \right|$$

Aus (27) und (28) ergibt sich:

$$\operatorname*{Max}_{[n/2]+1 \leqslant n' \leqslant n} \left| \sum_{\nu=[n/2]+1}^{n'} A_{n-\nu}^{(-\delta)} \sin\nu\alpha \right| < 2M(h)$$

und somit:

$$\int_\alpha^\beta \Phi_n^{(-\delta)}(t)\,dt = O\left(\frac{1}{n^\delta}\right) + O\left(\frac{\log n}{n}\right) + O\left(\frac{1}{n}\right) = o(1).$$

Jetzt haben wir noch zu zeigen, daβ aus der Voraussetzung (A . 0):

$$(29) \qquad I_{n,1}^{(-\delta)} = \int_0^h g(t)\,\Phi_n^{(-\delta)}(t)\,dt = o(1)$$

folgt. Nun ist:

$$|\Phi_n^{(-\delta)}(t)| \leqslant \frac{A_n^{(-\delta)}}{2} + \sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)} = A_n^{(1-\delta)} - \tfrac{1}{2}A_n^{(-\delta)}.$$

Daraus ergibt sich mit Berücksichtigung von (28):

$$(i) \qquad |\Phi_n^{(-\delta)}(t)| < B_1\, n^{1-\delta},$$

$$(ii) \qquad |\Phi_n^{(-\delta)}(t)| < B_2\, t^{\delta-1} \quad \text{für} \quad 0 \leqslant t \leqslant \pi.$$

Aus (i) und (ii) folgt aber:

$$(30) \qquad |\Phi_n^{(-\delta)}(t)| < B_3\,\frac{n^{1-\delta}}{(1+nt)^{1-\delta}} \quad \text{für} \quad 0 \leqslant t \leqslant \pi.$$

Daher erhalten wir, wie wir schon unter (22) bewiesen haben, für (29):

$$I_{n,1}^{(-\delta)} = O\left\{\int_0^h |g(t)| \frac{n^{1-\delta}}{(1+nt)^{1-\delta}}\, dt\right\} \tag{29.1}$$

$$= n^{-\delta} O\left\{\int_0^h |g(t)| \frac{n}{(1+nt)^{1-\delta}}\, dt\right\}$$

$$= n^{-\delta} o(n^{\delta}) = o(1). \qquad \text{w.z.b.w.}$$

6. *Satz* (c. 2): *Die nachstehende Reihe*

$$\sum_{\nu=1}^{\infty} \frac{\mathfrak{A}_\nu(x)}{\nu^{\delta}\phi_\nu} \tag{31}$$

ist unter der Voraussetzung (A . 0) $(C, -\delta)$ *summierbar; also fast überall.*

$\{1/\phi_\nu\}$ bedeutet in (31) eine konvexe Nullfolge, für die bekanntlich:

$$\Delta^{(1)}\left\{\frac{1}{\phi_\nu}\right\} = o\left(\frac{1}{\nu}\right).$$

Dem Beweise des obigen Satzes schicken wir einige bereits bekannte Lemmas voraus.

LEMMA I*. *Die Reihe* (31) *ist konvergent, wenn* (A . 0) *erfüllt ist.*

LEMMA II†. *Ist die unendliche Reihe* $\sum_{\nu=1}^{\infty} u_\nu$ (C, k) *summierbar, dann ist notwendig und hinreichend, damit die Reihe* $\sum_{\nu-1}^{\infty} u_\nu$ *auch* $(C, k-1)$ *summierbar sei, daß:*

$$\sum_{\nu=1}^{n} A_{n-\nu}^{(k-1)} u_\nu \nu = o(n^k).$$

LEMMA III‡. *Es sei:*

$$s_n^{(-\delta)} = \sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)} u_\nu, \quad s_{n,m}^{(-\delta)} = \sum_{\nu=1}^{m} A_{n-\nu}^{(-\delta)} u_\nu,$$

dann ist für $0 \leqslant \delta < 1$ *und* $1 \leqslant m \leqslant n$:

$$|s_{n,m}^{(-\delta)}| \leqslant \operatorname*{Max}_{1 \leqslant \nu \leqslant m} |s_\nu^{(-\delta)}|.$$

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* G. H. Hardy, *loc. cit.*

† G. H. Hardy, "Theorems relating to the summability and convergence of slowly oscillating series", *Proc. London Math. Soc.* (2), 8 (1910), 301–320.

‡ G. H. Hardy and M. Riesz, "The general theory of Dirichlet's series", *Cambridge Tracts*, 18 (Lemma 7, page 28).

Lemma IV. *Aus* (b . 12) *folgt, wenn* (A . 0) *erfüllt ist* :

$$s_n^{(-\delta)}(x) = \sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)} \mathfrak{A}_\nu(x) = O(1) \quad \textit{für alle } n,$$

und daher nach Lemma III.

$$s_{n,\,m}^{(-\delta)}(x) = \sum_{\nu=1}^{m} A_{n-\nu}^{(-\delta)} \mathfrak{A}_\nu(x) = O(1) \quad \textit{für alle } m \textit{ und } n.$$

Nach dem Lemma I ist die Reihe (31), $(C, 1-\delta)$ summierbar. Um nun von der $(C, 1-\delta)$ Summierbarkeit auf die $(C, -\delta)$ Summierbarkeit schliessen zu können, ist nach dem Lemma II notwendig und hinreichend, daβ :

$$\sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)} \frac{\mathfrak{A}_\nu(x)}{\nu^\delta \phi_\nu} \nu = o(n^{1-\delta}). \tag{32}$$

Gestützt auf das Lemma IV, erhalten wir aus (32) :

$$\sum_{\nu=1}^{n} A_{n,\nu}^{(-\delta)} \mathfrak{A}_\nu(x) \frac{\nu^{1-\delta}}{\phi_\nu} = \sum_{\nu=1}^{n-1} s_{n,\delta}^{(-\delta)}(x) \Delta^{(1)}\left(\frac{\nu^{1-\delta}}{\phi_\nu}\right) + s_n^{(-\delta)}(x) \frac{n^{1-\delta}}{\phi_n} \tag{32 . 1}$$

$$= \sum_{\nu=1}^{n-1} O(1)\, o(\nu^{-\delta}) + o(n^{1-\delta})$$

$$= \sum_{\nu=1}^{n-1} o(\nu^{-\delta}) + o(n^{1-\delta}) = o(n^{1-\delta}). \qquad \text{w.z.b.w.}$$

Bemerkung :

(i) Beim Beweise benutzten wir bloβ die Tatsache, daβ :

$$C^{(-\delta)}\{s_\nu(x)\} = O(n^\delta).$$

(ii) Der Faktor $1/\phi_\nu$ konnte nicht weggelassen werden, was offenbar damit zusammenhängt, daβ die Fourierreihe (1) unter der Voraussetzung (A . 0) noch divergieren kann.

(iii) Für quadratisch integrierbare Funktionen lässt sich der Faktor $1/\phi_\nu$ wegschaffen. Denn ist die Reihe :

$$\sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2)$$

konvergent, so gilt auch dasselbe von der nachstehenden Reihe :

$$\sum_{\nu=1}^{\infty} (a_\nu^2+b_\nu^2)\mu_\nu^2$$

wobei $\{\mu_\nu\}$ eine geeignet gewählte, monoton wachsende Zahlenfolge bedeutet, für die $\lim_{\nu\to\infty} \mu_\nu = \infty$; z.B.

$$\mu_\nu = r_\nu^{-\beta}, \quad \left(r_\nu = \sum_{\kappa=\nu+1}^{\infty} (a_\kappa^2+b_\kappa^2),\ 0<\beta<\tfrac{1}{2}\right).$$

Nun kann man aber zu der gegebenen Folge $\{\mu_\nu\}$ eine konkave Folge $\{\phi_\nu\}$ konstruieren, so daβ die Ungleichung $\phi_\nu < \mu_\nu$ für jedes n besteht und $\lim_{\nu\to\infty} \phi_\nu = \infty$.

Nach dem Riesz-Fischer'schen* Satze gibt es eine quadratisch integrierbare Funktion $f^*(x)$, für die

$$f^*(x) \sim \sum_{\nu=1}^{\infty} (a_\nu \phi_\nu \cos \nu x + b_\nu \phi_\nu \sin \nu x) \equiv \sum_{\nu=1}^{\infty} \mathfrak{A}_\nu^*(x).$$

Nach dem Satze (c . 12) ist also

$$\sum_{\nu=1}^{\infty} \frac{\mathfrak{A}_\nu^*(x)}{\nu^\delta \Phi_\nu} = \sum_{\nu-1}^{\infty} \frac{\mathfrak{A}_\nu(x)}{\nu^\delta} \tag{31 . 1}$$

fast überall $(C, -\delta)$ summierbar.

(iv) Für stetige Funktionen erhalten wir aus dem Bewiesenen folgende Corollare :

Cor. 1. Die Reihe (31 . 1) ist im ganzen Intervalle $[0, 2\pi]$ $(C, -\delta)$ beschränkt.

Cor. II. Die Reihe (31 . 1) ist im ganzen Intervalle $[0, \pi]$ für jedes $\gamma < \delta$ gleichmässig $(C, -\gamma)$ summierbar.

Cor. III. Nach einem Satze von H. Weyl† ist (31 . 1) die Fourierreihe einer stetigen Funktion, welche im ganzen Intervalle $[0, 2\pi]$ der Lipschitzbedingung von der Ordnung δ genügt.

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* F. Riesz, "Sur les systèmes orthogonaux de fonctions", *Comptes rendus*, 144 (1907), 615–619, 734–736. E. Fischer, "Sur la convergence en moyenne", *Comptes rendus*, 144 (1907), 1022–1024.

† H. Weyl, "Bemerkungen zum Begriffe des Differentialquotienten gebrochener Ordnung", *Vierteljahresschrift der naturf. Ges. in Zürich*, 62 (1917), 296–302.

Die Umkehrung dieser Corollare wurde von A. Zygmund* bewiesen. Sie lautet: 1st $f(\xi)$ eine stetige Funktion, welche in $[0, 2\pi]$ der Lipschitzbedingung von der Ordnung δ genügt, dann ist ihre Fourierreihe in $[0, 2\pi]$ gleichmässig $(C, -\gamma)$ summierbar für jedes $\gamma < \delta$ und dortselbst $(C, -\delta)$ beschränkt.

7. *Satz* (c . 1) $\sum_{\nu=2}^{\infty} \frac{\mathfrak{A}_\nu(x)}{\log \nu}$

ist (C, r) *summierbar.*

Nach dem Satze (a . 1) ist:

$$\sum_{\nu=2}^{\infty} \mathfrak{A}_\nu(x)$$

$(C, r+1)$ summierbar.

Nach einem Satze von Hardy† ist auch die folgende Reihe:

$$\sum_{\nu=2}^{\infty} \frac{\mathfrak{A}_\nu(x)}{\log \nu}, \tag{33}$$

$(C, r+1)$ summierbar. Um nun von $(C, r+1)$ auf die (C, r) Summierbarkeit der Reihe (33) schliessen zü können, ist nach Lemma II notwendig und hinreichend, daβ:

$$\sum_{\nu=2}^{n} A_{n-\nu}^{(r)} \mathfrak{A}_\nu(x) \frac{\nu}{\log \nu} = o(n^{r+1}).$$

Durch $(r+1)$-malige Abel'sche Umformung erhält man:

$$\sum_{\nu=2}^{n} A_{n-\nu}^{r} \mathfrak{A}_\nu(x) \frac{\nu}{\log \nu} = \sum_{\nu=2}^{n} \bar{C}_\nu^{(r)}(x) \frac{\nu^r}{r!} \Delta^{(r+1)} \left\{ A_{n-\nu}^{(r)} \frac{\nu}{\log \nu} \right\}.$$

Nach (b . 1) ist: $\bar{C}_\nu^{(r)}(x) = o(\log \nu).$

Eine Zwischenrechnung ergibt:

$$\nu^r \Delta^{(r+1)} \left\{ A_{n-\nu}^{(r)} \frac{\nu}{\log \nu} \right\} = n^r O\left(\frac{1}{\log n}\right)$$

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* A. Zygmund, *loc. cit.*

† G. H. Hardy, "Generalization of a theorem in the theory of divergent series", *Proc. London Math. Soc.* (2), 6 (1908), 255–264.

und daher :

$$\sum_{\nu=2}^{n} A_{n-\nu}^{(r)} \mathfrak{A}_\nu(x) \frac{\nu}{\log \nu} = \sum_{\nu=2}^{n} o(\log \nu)\, O\left(\frac{1}{\log \nu}\right) n^r = o(n^{r+1}),$$

w.z.b.w.

Es ist mir nicht gelungen, auf Grund des Theorems (b . 2), für $r \geqslant 1$ ein Analogon zu dem Satze (c . 2) herzuleiten; da aber das allgemeine Glied einer Fourierreihe gegen Null geht, so wird der Faktor $n^{-\delta}$ (bei geeigneter Wahl von δ) nach anderen Gesetzen konvergenzverbessernd wirken, wie aus den Untersuchungen von Hardy und Littlewood hervorgeht*.

8. Somit sind wir mit dem Beweise der angekündigten Sätze fertig. Ganz analog verlaufen die Überlegungen für die zu (1) konjugierte Reihe :

$$(34) \qquad \sum_{\nu=1}^{\infty} (-a_\nu \sin \nu x + b_\nu \cos \nu x) \equiv \sum_{\nu=1}^{\infty} \mathfrak{B}_\nu(x).$$

Im Falle $r = 0$ und $\delta = 0$ sind die Analoga zu den Sätzen (a), (b), und (c) bekannt†. Beschränkt man sich auf den Fall $r = 0$ und $0 < \delta < 1$, dann erhält man :

$$(\mathrm{b}'.12) \qquad C^{(-\delta)}\{\sigma_n(x)\} = o(n^\delta)$$

unter der Voraussetzung :

$$(\mathrm{A}'.0) \qquad \lim_{h=0} \frac{1}{h} \int_0^h |f(x+t) - f(x-t)|\, dt = o.$$

Die Bedingung (A' . 0) ist nach Lebesgue‡ fast überall erfüllt. Nach dem bereits erwähnten Satze von Hardy-Littlewood und Zygmund ist die Reihe (33) in einer mit $[0, 2\pi]$ massgleichen Menge M (C, α)

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* G. H. Hardy and J. E. Littlewood, "Contribution to the arithmetic theory of series", *Proc. London Math. Soc.* (2), 11 (1913), 411–478.

† G. H. Hardy and J. E. Littlewood, *loc. cit.* A. Zygmund, *loc. cit.* A. Plessner, "Zur Theorie der konjugierten trigonometrischen Reihen", *Dissert.* (Giessen, 1922). F. Lucács, "Zur Bestimmung des Sprunges einer Funktion aus ihrer Fourierreihe", *Journal für Math.*, 150 (1920), 107–112.

‡ H. Lebesgue, *Leçons sur les séries trigonométriques*, 13.

summierbar für jedes $\alpha > 0$. Daraus ergibt sich, genau wie im Falle des Satzes (c . 12), daß die nachstehende Reihe:

$$\sum_{\nu=1}^{\infty} \frac{\mathfrak{B}_\nu(x)}{\nu^\delta \Phi_\nu} \tag{c'. 12}$$

fast überall $(C, -\delta)$ summierbar ist.

Die Lebesgue'schen Konstanten.

9. Im Falle $(C, 0)$ betrachtet man die Lebesgue'schen* Konstanten:

$$\rho_n = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sin(n+\frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt \tag{35}$$

und zeigt, daß:

$$\rho_n < A \log n + B \tag{36}$$

wobei A und B von n unabhängige Zahlen sind. Für (C, δ), $0 < \delta < 1$, ist nach H. Cramér† für alle n:

$$\rho_n^{(\delta)} = \frac{1}{\pi} \int_0^{2\pi} \left| \tfrac{1}{2} + \sum_{\nu=1}^{\infty} \frac{A_{n-\nu}^{(\delta)}}{A_n^{(\delta)}} \cos \nu t \right| dt \leqslant K(\delta) \tag{36 . 1}$$

wobei $K(\delta)$ eine nur von δ abhängige Zahl bedeutet; für $\delta \geqslant 1$ ist $\rho_n^{(\delta)} = 1$ für jedes n.

Wir wollen jetzt für die Cesàro'schen Mittel negativer Ordnung die entsprechenden Lebesgue'schen Konstanten bestimmen, welche wir mit $\rho_n^{(-\delta)}$ bezeichnen.

$$\rho_n^{(-\delta)} = \frac{1}{\pi} \frac{1}{A_n^{(-\delta)}} \int_0^{2\pi} \left| \tfrac{1}{2} A_n^{(-\delta)} + \sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)} \cos \nu t \right| dt. \tag{37}$$

Für die konjugierte Reihe erhält man:

$$\rho_n^{(-\delta)} = \frac{1}{\pi} \frac{1}{A^{(-\delta)}} \int_0^{2\pi} \left| \sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)} \sin \nu t \right| dt. \tag{37 . 1}$$

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* H. Lebesgue, *loc. cit.*, 85–87.

† H. Cramér, "Etudes sur la sommation des séries de Fourier", *Arkiv for Mat.*, 13 (1919), No. 20, 1–21.

Für die beiden Integrale :

$$\int_0^{2\pi} \left| \tfrac{1}{2} A_n^{(-\delta)} + \sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)} \cos \nu t \right| dt,$$

$$\int_0^{2\pi} \left| \sum_{\nu=1}^{\infty} A_{n-\nu}^{(-\delta)} \sin \nu t \right| dt$$

bekommt man nach dem Satze von M. Riesz die folgende Majorante für alle n

$$1 + \int_0^{2\pi} \frac{2}{\{2 \sin \frac{1}{2}\phi\}^{1-\delta}} d\phi$$

und somit :

$$\rho_n^{(-\delta)} < \bar{K}(\delta) \cdot n^{\delta}, \quad \rho_n^{*(-\delta)} < \bar{K}(\delta) n^{\delta} \tag{36.2}$$

wobei $\bar{K}(\delta)$ eine nur von δ abhängige Zahl bedeutet.

10. Der Bestimmung der Lebesgue'schen Konstanten in der Theorie der Fourier'schen Reihen entspricht die Landau'sche obere Grenze in der Theorie der im Einheitskreise beschränkten Potenzreihen.

Ist (E) die Gesammtheit der Potenzreihen

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \tag{38}$$

welche absolut $\leqq 1$ für $|z| < 1$, dann ist für jeden Wert von n die obere Grenze G_n des Ausdruckes $|s_n| = |a_0 + a_1 + \ldots + a_n|$ gleich :

$$G_n = \sum_{\nu=0}^{n} (A_\nu^{(-\frac{1}{2})})^2 \sim \frac{1}{\pi} \log n. \tag{39}$$

Für (C, δ), $0 < \delta < 1$, hat Landau* bewiesen, daβ die obere Grenze $G_n^{(\delta)}$ von

$$|C_n^{(\delta)}| = \left| \sum_{\nu=0}^{n} \frac{A_{n-\nu}^{(\delta)}}{A_n^{(\delta)}} a_\nu \right|$$

für jedes n gleich ist :

$$G_n^{(\delta)} = \frac{1}{A_n^{(\delta)}} \sum_{\nu=0}^{\infty} (A^{-[(1-\delta)/2]})^2 \leqq L(\delta), \tag{39.1}$$

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* E. Landau, "Abschätzungen der Koeffizientensumme einer Potenzreihe", Abh. 1, *Archiv für Mathem.*, 21 (1913), 52–59; Abh. 2, *ibid.*, 21 (1913), 250–258; Abh. 3, *ibid.*, 24 (1916), 250–269. Vgl. auch O. Szász, "Ungleichungen für die Koeffizienten einer Potenzreihe", *Math. Zeitschrift*, 1 (1918), 163–183.

wobei $L(\delta)$ eine nur von δ abhängige Zahl ist, während im Falle $\delta \geqq 1$, $G_n^{(\delta)} = 1$ für jedes n.

Für die Cesàro'schen Mittel negativer Ordnung erhält man als obere Grenze $G_n^{(\delta)}$ des Ausdruckes:

$$|C_n^{(-\delta)}| = \sum_{\nu=0}^{n} \left| \frac{A_n^{(-\delta)}}{A_n^{(-\delta)}} a_\nu \right|,$$

$$(39.2) \quad G_n^{(-\delta)} = \frac{1}{A_n^{(-\delta)}} \sum_{\nu=0}^{n} (A_\nu^{[-(1+\delta)/2]})^2 \sim n^\delta \Gamma(1-\delta) \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{\{2 \sin \frac{1}{2}\phi\}^{1-\delta}}$$

$$(0 < \delta < 1).$$

Auf den Beweis von (39.2) gehen wir nicht mehr ein. Er verläuft genau so wie bei Landau. Es ergibt sich aus ihm, daβ für jedes n sich rationale Funktionen angeben lassen, für welche die obere Grenze $G_n^{(-\delta)}$ tatsächlich erreicht wird. Insbesondere lassen sich auf diesen Fall die Resultate von Bohr† and Neder‡ übertragen, wobei man sich auf den erwähnten Satz von M. Riesz über die Binomialreihe zu stützen hat.

Die Sätze dieser Arbeit lassen sich mit der hier verwendeten Methode auf die Fourier'schen Integrale übertragen. Wir verweisen diesbezüglich auf die Dissertation des Verfassers "Über die Cesàro'sche Summierbarkeit des Fourier'schen Integrales" [*Bulletin Internat. de l'Acad. de Cracovie*, Série A: Sc. math. (1926), 40–74].

Wien, Pfingsten, 1926.

[Zusatzbemerkung während der Korrektur, 10 June, 1927.]

1. Die Giltigkeit des Satzes (a . 1) bleibt erhalten, wenn die Voraussetzung (A . r) durch die folgende, allgemeinere:

$$(A^* . r) \qquad \int_0^h G_r(t)\,dt = o(h), \quad \int_0^h |G_r(t)|\,dt = O(h) \quad (r \geqq 0)$$

ersetzt wird§. In dieser Hinsicht bin ich in der Lage den nachstehenden

† H. Bohr, "Über die Koeffizientensumme einer beschränkten Potenzreihe", Abh. 1, *Göttinger Nachrichten*, Math. nat. Kl. (1916), 276–291; Abh. 2, *ibid.* (1917), 119–128.

‡ L. Neder, "Über die Koeffizientensumme einer beschränkten Potenzreihe", *Math. Zeitschrift*, 11 (1922), 115–123.

§ G. H. Hardy and J. E. Littlewood, "The Fourier series of a positive function", *Journal London Math. Soc.*, 1 (1926), 134–138; S. Pollard, "Direct proof of a new criterion of summability of a Fourier series", *ibid.*, 1 (1926), 233–235.

Approximationssatz zu beweisen :

Wenn die folgenden Bedingungen :

$$\text{(i)} \int_0^h G_r(t)\,dt = o(h^{1+\alpha}), \qquad \text{(ii)} \int_0^h |G_r(t)|\,dt = O(h^{1+\alpha})$$

$$(r \geqslant 0,\ o \leqslant \alpha < \delta \leqslant 1)$$

erfüllt sind, so gilt der Satz :

$$\text{(a*. 1)} \qquad \lim_{n=\infty} n^{\alpha}\left[C^{(r+\delta)}\{s_n(x)\} - f(x)\right] = 0 \quad (o \leqslant \alpha < \delta \leqslant 1).$$

Ist überdies für $0 \leqslant \delta \leqslant 1$:

$$\text{(iii)} \int_0^h |G_r(t)|\,dt = o(h^{1+\delta})$$

so gilt :

$$\text{(a*. 11)} \qquad \lim_{n=\infty} \frac{n^{\delta}}{\log n}\left[C_{(r+\delta)}\{s_n(x)\} - f(x)\right] = 0 \quad (r \geqslant 0).$$

Im Sätze (a* . 1) sind für $r = 0$ einige neuere Sätze von Szász† und Alexitis‡ unter allgemeineren Bedingungen enthalten; für $\alpha = 0$ und beliebiges $r \geqslant 0$ bekommt man die Sätze von Hardy und Littlewood und von Pollard. Auf den Beweis der obigen Sätze werde ich bei einer anderen Gelegenheit zurückkommen. Es sei aber sogleich bemerkt, daβ man beim Beweise des Satzes (a* . 1) auf die elegante Methode des Herrn Fejér verzichten muβ. Im Gegenteil, wir können den Satz (a* . 1) erst durch die feinere Einteilung des Integrales

$$\int_0^h \quad \text{in} \quad \int_0^{1/n} + \int_{1/n}^{P/n} + \int_{P/n}^h$$

gewinnen.

2. Der Satz (a . 2) lässt sich auf arithmetischen Wege aus den Sätzen (a . 1) und (b . 1) in leichter Weise herleiten§.

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† O. Szász, "Über die arithmetischen Mittel Fourierscher Reihen", *Acta Math.*, 48 (1926), 355–362; "Über die Cesàro'schen Mittel Fourierscher Reihen," *Acta Litt. ac Scient. Univ. Hung.*, 3 (1927), 38–48.

‡ G. Alexitis, "Über den Grad der Annäherung durch die Cesàro'schen Mittel der Fourierreihe", *Acta Litt. ac Scient. Univ. Hung.*, 3 (1927), 32–37.

§ Vgl. meine demnächst erscheinende Note: "Beitrag zur Darstellung der Ableitungen einer Funktion durch ihre Fouriersche Reihe", im *Bull. Intern. de l'Acad. de Cracovie*, Série A: Scien. mathém., 1927.

ELECTRIC AND MAGNETIC DISPLACEMENT CURRENTS.

By E. M. WELLISH.

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[THROUGHOUT this paper Heaviside rational units have been employed and the formulae are dimensional, so that they are valid when all the quantities entering are expressed either in the electrostatic or in the electromagnetic system of units. ϵ and μ denote respectively the electric and magnetic inductivities of free aether, and $c = (\epsilon\mu)^{-\frac{1}{2}}$ represents the velocity with which it transmits electromagnetic disturbances; the measure of c is the "ratio of the units".]

1. *Introductory.*

One of Faraday's fundamental ideas was that all electric currents are circuital irrespective of the conditions of production. Maxwell introduced the notion of electric displacement in developing his mathematical theory of electricity in conformity with this view. He pictured the aethereal medium both in free space and in dielectric material as being permeated by an electric substance; on the establishment of an electric field the particles of this substance are displaced from their positions of normal equilibrium, the strain or displacement being proportional to the intensity of the applied field. Mathematically, the electric displacement $\mathbf{D}$ at a given point is a vector function of the time such that the quantity of electric substance which has crossed a small (vector) area $d\mathbf{f}$ up to that time is

$$(\mathbf{D} \,.\, d\mathbf{f}).$$

In terms of the electric field intensity

$$\mathbf{D} = \epsilon\mathbf{E},$$

and from this equation it can be deduced that

$$\operatorname{div} \mathbf{D} = \rho,$$

where ρ is the volume density of the true electrification at the point. This relation shows that the true electricity and Maxwell's hypothetical substance behave jointly as an incompressible fluid. Any local alteration in the electric displacement can consequently be depicted as a current, and we are thus led to the idea of an electric displacement current whose density reckoned per unit area is represented by $\partial \mathbf{D}/\partial t$. If to the true current resulting from the motion of electric charges we add the current of electric displacement we obtain a total current vector which, in accordance with the idea of Faraday, is always circuital. Mathematically,

$$\operatorname{div} (\rho \mathbf{v} + \partial \mathbf{D}/\partial t) = 0,$$

where $\mathbf{v}$ is the velocity of the electric charge at the point considered.

Corresponding to the electric displacement $\mathbf{D}$ we have the magnetic induction $\mathbf{B} = \mu \mathbf{H}$, for which $\operatorname{div} \mathbf{B} = 0$ in the absence of free magnetism. The vector $\mathbf{B}$ is usually defined in terms of magnetic lines or tubes, the number crossing a small (vector) area $d\mathbf{f}$ being $(\mathbf{B} \,.\, d\mathbf{f})$. However, $\mathbf{B}$ can be depicted as a displacement of the particles of a magnetic fluid, and, by analogy with the electric case, $\partial \mathbf{B}/\partial t$ will denote the density of the magnetic displacement current.

The actual nature of the processes involved in the displacements $\mathbf{D}$ and $\mathbf{B}$ is as yet unknown; on this account it is customary and convenient to represent them as linear displacements of hypothetical particles. A similar procedure is followed in the representation of undulatory oscillations; in a plane-polarized electric wave the vectors $\mathbf{D}$ and $\mathbf{B}$ are situated in the wave front and are at right angles to one another.

Quite independently of the physical interpretation of $\partial \mathbf{D}/\partial t$ and $\partial \mathbf{B}/\partial t$ it is a matter of importance to determine these vectors in certain cases, especially when they arise from the motion of discrete electric charges; the development in recent years of the electron theory of matter renders this task all the more necessary. In the present paper, after an investigation of the currents associated with the uniform motion of a Lorentz electron, expressions are obtained for the displacement current vectors arising from the (practically) unrestricted motion of an electric charge. The treatment is based throughout on the classical equations of electromagnetic theory; even though the progress of the quantum theory of radiation should lead to a revision of these fundamental equations, it was felt advisable to make this natural development of a theory

which has proved so fruitful. Incidentally, the classical theory of a linear oscillator has been extended in several respects.

Certain special cases of displacement currents arising from the slow motion of electric charges have already been treated by Ames*; in the present paper a brief reference is made to a few special cases after the general expression has been obtained.

To complete these introductory remarks we now give a mechanical representation of displacement currents:

Let F be a point fixed in the aether; suppose that, under the influence of an applied field $\mathbf{R}$ an electric (or magnetic) particle originally at F is displaced to a position P. Then FP will represent the direction of $\mathbf{R}$ and may also be regarded as representing $\mathbf{R}$ in magnitude. If now the particle is further displaced from P to a neighbouring position Q, FQ will represent the new field $\mathbf{R}+d\mathbf{R}$. The change of field is represented by PQ, and, if this requires a time dt, the displacement current per unit area will be proportional to the velocity with which the particle traverses the distance PQ.

2. *Methods of procedure.*

The most convenient method of determining the displacement current density will depend, of course, on the particular circumstances of the problem under consideration. We may classify the various methods under the three following heads:

Method A. Direct differentiation with respect to the time of the displacement at a point fixed in the aether.

Method B. Utilization of the theorem

$$\partial\mathbf{R}/\partial t = (\partial\mathbf{R}/\partial t)_m - (\mathbf{v}\nabla)\mathbf{R}$$

where $\mathbf{R}$ is a vector point function, $(\partial\mathbf{R}/\partial t)_m$ denotes its rate of change at a point fixed in relation to a reference system moving with velocity $\mathbf{v}$, and the operation $(\mathbf{v}\nabla)\mathbf{R}$ refers to a definite instant of time.

Method C. Application of the fundamental equations

$$\partial\mathbf{D}/\partial t + \rho\mathbf{v} = \operatorname{curl}\mathbf{H}$$

$$\partial\mathbf{B}/\partial t = -\operatorname{curl}\mathbf{E}.$$

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* *Journ. Frank Inst.*, 195 (1923), 787.

3. *Investigation of currents at the surface of a Lorentz electron in uniform motion.*

We have seen in the introductory section that the fundamental equations of electromagnetic theory lead to the equation

$$\operatorname{div}(\partial\mathbf{D}/\partial t+\rho\mathbf{v}) = 0.$$

The total current is thus circuital everywhere, and at the boundary of a moving charge the solenoidal condition must be satisfied. In this section we shall investigate the currents at the surface of a Lorentz electron in uniform motion, and verify, incidentally, that the solenoidal condition is satisfied. It is convenient to regard the charge of the electron as distributed uniformly with density ρ throughout the space between two oblate spheroids with their shorter axes in the direction of motion; the axes of the inner and outer spheroids (Heaviside ellipsoids) are taken to be $(a\kappa, a, a)$ and $(b\kappa, b, b)$ respectively where

$$\kappa^2 = 1-\beta^2 = 1-v^2/c^2.$$

By choosing a successively to be zero and to be equal to b we arrive at the special cases of a Lorentz electron, first with uniform volume distribution of charge, and secondly with superficial charge.

Choose the centre of the electron as origin and the direction of motion as x-axis. We know that the convection potential at any point $P(x, y, z)$ is given by

$$\psi = \kappa\Phi,$$

where Φ is the electrostatic potential at the point $P_0(x/k, y, z)$ due to the same charge supposed uniformly distributed throughout the space between the two concentric spheres

$$R = a, \quad R = b.$$

The electric density in the static distribution will obviously be

$$\rho_0 = \kappa\rho.$$

Now

$$\Phi = \tfrac{1}{3}\epsilon^{-1}\rho_0(\tfrac{3}{2}b^2-\tfrac{1}{2}R^2-a^3R^{-1}), \quad \text{where} \quad a < R < b.$$

For the point P_0, $R = (x^2/\kappa^2+y^2+z^2)^{\frac{1}{2}}$, and we obtain for the convection potential at a point P within the region occupied by the electronic charge

$$\psi = \tfrac{1}{3}\epsilon^{-1}\rho\left[\tfrac{3}{2}\kappa^2b^2-\tfrac{1}{2}\{x^2+\kappa^2(y^2+z^2)\}-\kappa^3a^3\{x^2+\kappa^2(y^2+z^2)\}^{-\frac{1}{2}}\right].$$

If the test-charge at P has a velocity $\mathbf{v}$, the force per unit charge on it is

$$\mathbf{F} = -\nabla\psi = -(\mathbf{i}\partial\psi/\partial x+\mathbf{j}\partial\psi/\partial y+\mathbf{k}\partial\psi/\partial z)$$

$$= \tfrac{1}{3}\epsilon^{-1}\rho(1-\kappa^3a^3s^{-3})(x\mathbf{i}+\kappa^2y\mathbf{j}+\kappa^2z\mathbf{k})$$

where $$s = \{x^2+\kappa^2(y^2+z^2)\}^{\frac{1}{2}}.$$

Now $$\mathbf{F} = \mathbf{E}+\mu[\mathbf{vH}], \quad \text{where} \quad \mathbf{H} = \epsilon[\mathbf{vE}];$$

hence $$\mathbf{F} = \kappa^2\mathbf{E}+c^{-2}(\mathbf{vE})\mathbf{v}.$$

But, since $\mathbf{v} = v\mathbf{i}$, we have

$$(\mathbf{vE}) = (\mathbf{vF}) = \tfrac{1}{3}\epsilon^{-1}\rho vx\,(1-\kappa^3a^3s^{-3});$$

consequently, after reduction,

$$\mathbf{E} = \tfrac{1}{3}\epsilon^{-1}\rho(1-\kappa^3a^3s^{-3})\mathbf{R},$$

where $\mathbf{R}$ is the vector distance from O to P, both points being considered at the same instant t.

The electric displacement current density at P is given by

$$\partial\mathbf{D}/\partial t = -v\partial\mathbf{D}/\partial x, \quad \text{where} \quad \mathbf{D} = \epsilon\mathbf{E}.$$

We deduce that

$$\partial\mathbf{D}/\partial t = -\tfrac{1}{3}\rho v[\mathbf{i}+\kappa^3a^3s^{-5}\{(3x^2-s^2)\mathbf{i}+3xy\mathbf{j}+3xz\mathbf{k}\}].$$

When $a = 0$, *i.e.* when the volume distribution of charge throughout the interior of the bounding ellipsoid is uniform, the internal displacement current density is constant and equal to $-\frac{1}{3}\rho v\mathbf{i}$

If the point P is situated just inside the outer boundary, whose equation is $s = \kappa b$, we have

$$\partial\mathbf{D}/\partial t = -\tfrac{1}{3}\rho v\left[\mathbf{i}+\frac{a^3}{\kappa^2b^5}\{(3x^2-\kappa^2b^2)\mathbf{i}+3xy\mathbf{j}+3xz\mathbf{k}\}\right].$$

Since the convection current density at this point is $\rho v\mathbf{i}$, the total current density is given by

$$\tfrac{2}{3}\rho v\mathbf{i}-\frac{\rho va^3}{3\kappa^2b^5}\{(3x^2-\kappa^2b^2)\mathbf{i}+3xy\mathbf{j}+3xz\mathbf{k}\}.$$

The electric displacement at an external point is given by

$$\mathbf{D} = \tfrac{1}{3}\rho\kappa^3(b^3-a^3)s^{-3}\mathbf{R},$$

and the displacement current density at a point just outside the outer surface at P is similarly found to be

$$\tfrac{1}{3}\rho v\ \frac{b^3-a^3}{\kappa^2 b^5}\ \{(3x^2-\kappa^2 b^2)\mathbf{i}+3xy\mathbf{j}+3xz\mathbf{k}\}.$$

The vector difference of these current densities is

$$\frac{\rho v}{\kappa^2 b^2}\ \{(x^2-\kappa^2 b^2)\mathbf{i}+xy\mathbf{j}+xz\mathbf{k}\}.$$

Now the tangent line at P which intersects the line of motion has direction cosines proportional to

$$x^2-\kappa^2 b^2 : xy : xz.$$

Thus the vector difference of the current densities at neighbouring points just outside and just inside the surface of the Lorentz electron is directed along that tangent line to the surface which lies in the meridian plane through the points. The solenoidal condition is therefore satisfied at the surface of the electron; as regards the current just inside the surface, its normal component is required for crossing the boundary, and thus connecting with the outer displacement current, while its tangential component completes the circuit between the two terminals of a displacement current tube.

4. *General expressions for the electric and magnetic displacement current densities due to the (practically) unrestricted motion of a point-charge.*

We proceed from the expressions given by Liénard*, Abraham†, and Schott‡ for the electric and magnetic fields due to the non-uniform motion

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* *L'éclairage électrique* (July, 1898).

† *Die Theorie der Strahlung* (Leipzig, 1908).

‡ *Ann. der Physik*, 24 (1907), 637. Cf. also *Electromagnetic radiation* (Cambridge, 1912), where complete references are given.

of a point-charge, viz.

$$4\pi\epsilon e^{-1}\mathbf{E} = -\frac{\dot{\mathbf{v}}r}{c^2 s^2} + \left(1-\frac{\mathbf{v}^2}{c^2}+\frac{(\dot{\mathbf{v}}\mathbf{r})}{c^2}\right)\frac{r}{s^3}\left(\mathbf{r}_1-\frac{\mathbf{v}}{c}\right), \qquad (\alpha)$$

$$4\pi e^{-1}\mathbf{H} = \frac{1}{cs^2}[\dot{\mathbf{v}}\mathbf{r}]+\left(1-\frac{\mathbf{v}^2}{c^2}+\frac{(\dot{\mathbf{v}}\mathbf{r})}{c^2}\right)\frac{1}{s^3}[\mathbf{v}\mathbf{r}], \qquad (\beta)$$

$$\mathbf{H} = \sqrt{\frac{\epsilon}{\mu}}\,[\mathbf{r}_1\mathbf{E}], \qquad (\beta')$$

where $\mathbf{E}$ and $\mathbf{H}$ refer to a field-point P at time t. In these expressions r denotes the distance of P from the position Q' of the moving charge at a previous time τ such that $\tau+r/c = t$. If Q is the virtual position of the point-charge at time t, on the supposition that it remains unaccelerated between the times τ and t, and if F is the foot of the perpendicular from Q to $Q'P$, then we denote QP, FP, by R, s respectively. Here, and subsequently, dots denote differentiation with regard to τ; thus $\dot{\mathbf{v}}$ denotes the acceleration of the point-charge, since its velocity is $\mathbf{v}$ at time τ.

The availability of these formulae is subject to the following limitations:

(i) $|\mathbf{v}|$ must be less than c.

(ii) r must be large compared with the radius b of a sphere which just encloses the point-charge.

(iii) $2b|\dot{\mathbf{v}}|/\{c(c-|\mathbf{v}|)\}$ must be small compared with unity.

For the sake of generality we regard $\mathbf{v}$ and $\dot{\mathbf{v}}$ as having any directions; let the co-ordinates of the field-point P be (x, y, z) referred to any set of fixed axes, and let those of the charge at time τ be (ξ, η, ζ), so that

$$r^2 = (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2.$$

Before proceeding to the derivation of the expressions for the displacement current densities it is convenient to tabulate here several results which will subsequently be found useful; indications of proof are occasionally given, the verification being in general quite simple. The field-point P is regarded as fixed in all differential coefficients with respect

to the time, whereas in all space derivatives the time t is kept constant. The quantities L, M, N are defined as follows:

$$c^2L = c^2-\mathbf{v}^2+(\dot{\mathbf{v}}\mathbf{r}),$$

$$c^2M = c\mathbf{v}_r-\mathbf{v}^2+(\dot{\mathbf{v}}\mathbf{r}),$$

$$c^2N = \mathbf{v}_r^2-\mathbf{v}^2+(\dot{\mathbf{v}}\mathbf{r}).$$

$$\tau+\frac{r}{c}=t, \quad (1) \qquad \mathbf{R}=\mathbf{r}-\frac{\mathbf{v}r}{c}, \quad (2) \qquad s=r\left(1-\frac{\mathbf{v}_r}{c}\right)=(\mathbf{R}\mathbf{r}_1), \quad (3)$$

$$\frac{\partial\tau}{\partial t}=\frac{r}{s}, \quad (4) \qquad \dot{r}=-\mathbf{v}_r, \quad (5) \qquad \frac{\partial r}{\partial t}=-\frac{r\mathbf{v}_r}{s}, \quad (6)$$

$$\dot{s}=-cM, \quad (7) \qquad \frac{\partial s}{\partial t}=-\frac{rcM}{s}, \quad \left[\text{Put } s=r-\frac{(\mathbf{v}\mathbf{r})}{c}.\right] \quad (8)$$

$$\dot{\mathbf{r}}=-\mathbf{v}, \quad (9) \qquad \frac{\partial\mathbf{r}}{\partial t}=-\frac{\mathbf{v}r}{s}, \quad (10) \qquad \frac{\partial\mathbf{r}_1}{\partial t}=s^{-1}(\mathbf{v}_r\mathbf{r}_1-\mathbf{v}), \left[\text{Put } \mathbf{r}_1=\frac{\mathbf{r}}{r}.\right] \quad (11)$$

$$\frac{\partial\mathbf{R}}{\partial t}=-\mathbf{v}-\frac{r^2}{cs}\dot{\mathbf{v}}, \quad [\text{Use (2).}] \quad (12)$$

$$\frac{\partial r}{\partial x}=\frac{x-\xi}{s}, \quad \left[\text{Put } \frac{\partial\xi}{\partial x}=-\frac{\mathbf{v}_x}{c}\frac{\partial r}{\partial x}, \text{ etc.}\right] \quad (13)$$

$$\frac{\partial\xi}{\partial x}=-\frac{\mathbf{v}_x(x-\xi)}{cs}, \quad \frac{\partial\xi}{\partial y}=-\frac{\mathbf{v}_x(y-\eta)}{cs}, \quad \frac{\partial\xi}{\partial x}=-\frac{\mathbf{v}_z(x-\xi)}{cs}, \text{ etc.}$$

[Use (1) and (13).] (14)

$$\frac{\partial\mathbf{r}_1}{\partial x}=\frac{1}{r}\left(\mathbf{i}-\frac{x-\xi}{rs}\mathbf{R}\right), \quad (15) \qquad \frac{\partial\mathbf{r}}{\partial x}=\mathbf{i}+\frac{\mathbf{v}(x-\xi)}{cs}, \quad [\text{Use (14).}] \quad (16)$$

$$\frac{\partial s}{\partial x}=-\frac{\mathbf{v}_x}{c}+\frac{x-\xi}{s}L, \text{ etc.} \quad (17)$$

$$\operatorname{div}\mathbf{r}_1=\frac{\partial x}{\partial}\frac{x-\xi}{r}+\frac{\partial}{\partial y}\frac{y-\eta}{r}+\frac{\partial}{\partial z}\frac{z-\zeta}{r}=\frac{2}{r}, \quad (18)$$

$$\operatorname{div}\mathbf{r}=3+\frac{(\mathbf{v}\mathbf{r})}{cs}, \quad (19) \qquad \nabla r=\frac{\mathbf{r}}{s}, \quad (20) \qquad \nabla r^{-1}=-\frac{\mathbf{r}}{r^2s}, \quad (21)$$

$$\nabla r^{-2}=-\frac{2\mathbf{r}}{r^3s}, \quad (22) \qquad \nabla s=-\frac{\mathbf{v}}{c}+\frac{L\mathbf{r}}{s}, \quad (23) \qquad \frac{\partial v}{\partial t}=\frac{\dot{v}r}{s}, \quad (24)$$

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\dot{\mathbf{v}} r}{s}, \quad (25) \qquad \frac{\partial \dot{\mathbf{v}}}{\partial t} = \frac{\ddot{\mathbf{v}} r}{s}, \quad (26) \qquad \frac{\partial \dot{v}}{\partial x} = -\frac{\dot{v}(x-\xi)}{cs}, \text{ etc.} \quad (27)$$

$$\frac{\partial \mathbf{v}}{\partial x} = -\frac{\dot{\mathbf{v}}(x-\xi)}{cs}, \text{ etc.} \quad (28) \qquad \frac{\partial \dot{\mathbf{v}}}{\partial x} = -\frac{\ddot{\mathbf{v}}(x-\xi)}{cs}, \text{ etc.} \quad (29)$$

$$\frac{\partial \mathbf{v}_x}{\partial x} = -\frac{(\dot{\mathbf{v}}\mathbf{i})(x-\xi)}{cs}, \quad \frac{\partial \mathbf{v}_z}{\partial y} = -\frac{(\dot{\mathbf{v}}\mathbf{k})(y-\eta)}{cs}, \text{ etc.} \quad (30)$$

$$\frac{\partial (\dot{\mathbf{v}}\mathbf{i})}{\partial x} = -\frac{(\ddot{\mathbf{v}}\mathbf{i})(x-\xi)}{cs}, \quad \frac{\partial (\dot{\mathbf{v}}\mathbf{k})}{\partial y} = -\frac{(\ddot{\mathbf{v}}\mathbf{k})(y-\eta)}{cs}, \text{ etc.} \quad (31)$$

$$\frac{\partial v_r}{\partial t} = s^{-1}c^2 N, \text{ [Put } v_r = (\mathbf{v}\mathbf{r}_1).] \quad (32) \qquad \frac{\partial \mathbf{v}_r}{\partial x} = \frac{\mathbf{v}_x}{r} - \frac{x-\xi}{rs} cM, \text{ etc.} \quad (33)$$

$$\nabla v = -\frac{v\mathbf{r}}{cs}, \quad (34) \qquad \nabla v^2 = \nabla \mathbf{v}^2 = -\frac{2(\dot{\mathbf{v}}\mathbf{v})\, r}{cs}, \quad (35)$$

$$\frac{\partial^2 \tau}{\partial t^2} = s^{-3} r^2 cN, \quad (36) \qquad \frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial t}\right) = \frac{r}{cs^2}\left(\mathbf{v}_x - \frac{x-\xi}{s} cM\right), \text{ etc.} \quad (37)$$

$$\nabla \frac{\partial \tau}{\partial t} = \frac{r}{cs^2}\left(\mathbf{v} - \frac{\mathbf{r}}{s} cM\right), \quad (38)$$

$$\operatorname{curl} \mathbf{r}_1 = \frac{1}{cs}[\mathbf{r}_1 \mathbf{v}], \quad (39) \qquad \operatorname{curl} \mathbf{r} = \frac{1}{cs}[\mathbf{r}\mathbf{v}], \quad (40)$$

$$\operatorname{curl} \mathbf{v} = \frac{1}{cs}[\dot{\mathbf{v}}\mathbf{r}], \quad (41) \qquad \operatorname{curl} \dot{\mathbf{v}} = \frac{1}{cs}[\ddot{\mathbf{v}}\mathbf{r}], \quad (42)$$

$$c^2 \dot{L} = \frac{c^2 \partial L}{\partial \tau} = (\ddot{\mathbf{v}}\mathbf{r}) - 3(\mathbf{v}\dot{\mathbf{v}}), \quad (43)$$

$$\nabla L = \frac{\dot{\mathbf{v}}}{c^2} - \frac{\mathbf{r}}{cs}\frac{\partial L}{\partial \tau}, \quad \text{[use (16), (28), (29), (43).]} \quad (44)$$

$$L+N = \frac{s^2}{r^2} + 2M, \quad (45) \qquad L-M = \frac{s}{r}, \quad (46)$$

Electric displacement current.

Method A. In differentiating the expression (a) with regard to t at the fixed field-point P the first term yields, after using (6), (8), and (26),

$$-\frac{r^2}{c^2 s^3}\{\ddot{\mathbf{v}} + \dot{\mathbf{v}} s^{-1} c(M+N)\}.$$

From the second term we obtain

$$\mathbf{R}\frac{r}{s^4}\left[3s^{-1}cLM+c^{-2}\{(\ddot{\mathbf{v}}\mathbf{r})-3(\mathbf{v}\dot{\mathbf{v}})\}\right]-L\frac{\mathbf{v}}{s^3}-L\frac{r^2}{cs^4}\dot{\mathbf{v}}$$

by the use of (6), (8), (10), (11), (25), and (26).

Thus the electric displacement current density is given by

$$\frac{4\pi}{e}\frac{\partial\mathbf{D}}{\partial t}=\mathbf{R}\frac{r}{s^4}\left[c^{-2}\{(\ddot{\mathbf{v}}\mathbf{r})-3(\mathbf{v}\dot{\mathbf{v}})\}+3s^{-1}cLM\right]$$
$$-\mathbf{v}\frac{L}{s^3}-\dot{\mathbf{v}}\frac{r^2}{cs^4}(L+M+N)-\ddot{\mathbf{v}}\frac{r^2}{c^2s^3}. \quad (\gamma)$$

In consequence of (43) and (45) this may be written

$$\frac{4\pi}{e}\frac{\partial\mathbf{D}}{\partial t}=\mathbf{R}\frac{r}{s^4}\{\dot{L}+3s^{-1}cLM\}-\mathbf{v}\frac{L}{s^3}-\dot{\mathbf{v}}\frac{1}{cs^2}(1+3s^{-2}r^2M)-\ddot{\mathbf{v}}\frac{r^2}{c^2s^3} \quad (\gamma')$$

Method B. We choose a second set of axes parallel to the fixed set, but having a velocity $\mathbf{v}$ at time τ.

Then $$4\pi e^{-1}\frac{\partial\mathbf{D}}{\partial t}=4\pi e^{-1}\left(\frac{\partial\mathbf{D}}{\partial t}\right)_m-4\pi e^{-1}(\mathbf{v}\nabla)\mathbf{D},$$

where the suffix m indicates that the differentiation is to be effected at a point fixed in the moving system, and the operation $(\mathbf{v}\nabla)\mathbf{D}$ refers to the fixed time t.

Suppose that the two sets of axes coincide at time t; then at time $t+dt$ the field-points in the two systems will be separated by a distance $\mathbf{v}dt$.

In the fixed system disturbances which arrive at the field-point at successive instants t, $t+dt$ will leave the point-charge at instants

$$\tau, \quad \tau+d\tau,$$

where $$\tau+\frac{r}{c}=t.$$

In the moving system the disturbances will leave the charge at instants τ, $\tau+dt$, since r remains constant.

Consequently $$\left(\frac{\partial\mathbf{v}}{\partial t}\right)_m=\dot{\mathbf{v}}, \quad \left(\frac{\partial\dot{\mathbf{v}}}{\partial t}\right)_m=\ddot{\mathbf{v}}.$$

Also $(\partial v_r/\partial t)_m = (\mathbf{r}_1\dot{\mathbf{v}})$, and from (3) we deduce that

$$\left(\frac{\partial s}{\partial t}\right)_m = -c^{-1}(\mathbf{r}\dot{\mathbf{v}}).$$

On substituting these values, we obtain

$$4\pi e^{-1}\left(\frac{\partial \mathbf{D}}{\partial t}\right)_m = \mathbf{R}\,\frac{1}{cs^3}\left\{\frac{1}{c}(\ddot{\mathbf{v}}\mathbf{r}) - \frac{2}{c}(\mathbf{v}\dot{\mathbf{v}}) + \frac{3}{s}(\dot{\mathbf{v}}\mathbf{r})\,L\right\}$$
$$-\dot{\mathbf{v}}\,\frac{r}{cs^3}\left(1-\frac{\mathbf{v}^2}{c^2}+\frac{3(\dot{\mathbf{v}}\mathbf{r})}{c^2}\right) - \ddot{\mathbf{v}}\,\frac{r}{c^2 s^2}.$$

Using (13), (15), (16), (17), (28), (29) we deduce that

$$4\pi e^{-1}(\dot{\mathbf{v}}\nabla)\,\mathbf{LD} = \mathbf{R}\,\frac{1}{cs^3}\left\{\frac{1}{c}\left(\frac{3r}{s}-2\right)(\mathbf{v}\dot{\mathbf{v}}) - \frac{1}{c}\left(\frac{r}{s}-1\right)(\ddot{\mathbf{v}}\mathbf{r})\right.$$
$$\left. + \frac{3}{s}(\dot{\mathbf{v}}\mathbf{r})\,L - 3\,\frac{r}{s^2}\,cLM\right\}$$
$$+\mathbf{v}\,\frac{L}{s^3} + \dot{\mathbf{v}}\,\frac{r}{cs^3}\left\{\left(1-\frac{s}{r}\right)\left(3\,\frac{r}{s}L-1\right) - \frac{2v^2}{c^2}\right\} + \ddot{\mathbf{v}}\,\frac{r}{c^2 s^2}\left(\frac{r}{s}-1\right).$$

By combining these results we obtain at once the expressions (γ), (γ') already found for the displacement current density.

Method C. $$\frac{\partial \mathbf{D}}{\partial t} = \operatorname{curl}\mathbf{H} = \epsilon c\,\operatorname{curl}[\mathbf{r}_1\mathbf{E}].$$

Also, since $\operatorname{div}\mathbf{E} = 0$,

$$\operatorname{curl}[\mathbf{r}_1\mathbf{E}] = -\mathbf{E}\operatorname{div}\mathbf{r}_1 + (\mathbf{E}\nabla)\,\mathbf{r}_1 - (\mathbf{r}_1\nabla)\,\mathbf{E};$$

thus

$$4\pi e^{-1}\,\frac{\partial \mathbf{D}}{\partial t} = 4\pi\epsilon c e^{-1}\{-\mathbf{E}\operatorname{div}\mathbf{r}_1 + (\mathbf{E}\nabla)\,\mathbf{r}_1\} - (\mathbf{r}_1\nabla)\,4\pi\epsilon c e^{-1}\,\mathbf{E}.$$

From (2), (13), and (14),

$$(\mathbf{E}\nabla)\,\mathbf{r}_1 = r^{-1}(\mathbf{E}-\mathbf{E}_r s^{-1}\mathbf{R});$$

from (α) and (3)

$$4\pi\epsilon e^{-1}\mathbf{E}_r = 1-\frac{\mathbf{v}^2}{c^2}s^{-2};$$

hence

$$4\pi\epsilon c e^{-1}\{-\mathbf{E}\operatorname{div}\mathbf{r}_1 + (\mathbf{E}\nabla)\,\mathbf{r}_1\} = -\mathbf{R}\,\frac{c}{rs^3}\left(L+1-\frac{\mathbf{v}^2}{c^2}\right) + \dot{\mathbf{v}}\,\frac{1}{cs^2}.$$

Also, by using (13), (15), (16), (17), (28) and (29), we obtain

$$(\mathbf{r}_1\nabla)4\pi\epsilon c e^{-1}\mathbf{E} = -\mathbf{R}\left[\frac{c}{rs^3}\left(L+1-\frac{\mathbf{v}^2}{c^2}\right)+\frac{r}{c^2s^4}\{(\ddot{\mathbf{v}}\mathbf{r})-3(\mathbf{v}\dot{\mathbf{v}})\}+3\frac{r}{s^5}cLM\right]$$

$$+\mathbf{v}\frac{L}{s^3}+\dot{\mathbf{v}}\frac{1}{cs^2}(2+3s^{-2}r^2M)+\ddot{\mathbf{v}}\frac{r^2}{c^2s^3}.$$

The expressions (γ), (γ') follow readily from the foregoing.

Magnetic displacement current.

By analogy with the electrical problem, the density is $\mu\partial\mathbf{H}/\partial t$.

Method A. $$\frac{4\pi}{e}\mathbf{H} = \frac{1}{cs^2}[\dot{\mathbf{v}}\mathbf{r}]+\frac{L}{s^3}[\mathbf{v}\mathbf{r}].$$

Using (8), (10), (24), (25) and (46), we obtain

$$\frac{4\pi}{e}\frac{\partial\mathbf{H}}{\partial t} = [\mathbf{v}\mathbf{r}]\frac{r}{s^4}(\dot{L}+3s^{-1}cLM)$$

$$+[\dot{\mathbf{v}}\mathbf{r}]\frac{1}{s^3}(1+3s^{-1}rM)+[\ddot{\mathbf{v}}\mathbf{r}]\frac{r}{cs^3}+[\mathbf{v}\dot{\mathbf{v}}]\frac{r}{cs^3}, \quad (\delta)$$

where $\dot{L}$ is given in (43).

Method C. $$\frac{\partial\mathbf{B}}{\partial t} = -\operatorname{curl}\mathbf{E},$$

therefore
$$4\pi e^{-1}\frac{\partial\mathbf{H}}{\partial t} = -c^2\operatorname{curl}4\pi\epsilon e^{-1}\mathbf{E}$$

$$= \operatorname{curl} rs^{-2}\dot{\mathbf{v}}-c^2\operatorname{curl}s^{-3}rL\left(\mathbf{r}_1-\frac{\mathbf{v}}{c}\right).$$

Since $\operatorname{curl}\phi\mathbf{A} = \phi\operatorname{curl}\mathbf{A}+[\nabla\phi, \mathbf{A}]$ we have, using (20), (23) and (42),

$$\operatorname{curl}s^{-2}r\dot{\mathbf{v}} = \frac{1}{s^3}(1+2s^{-1}rM)[\dot{\mathbf{v}}\mathbf{r}]-\frac{r}{cs^3}[\ddot{\mathbf{v}}\mathbf{r}]+\frac{2r}{sc^3}[\mathbf{v}\dot{\mathbf{v}}]$$

and, using (20), (23), (39), (41), (43), and (44),

$$c^2\operatorname{curl}s^{-3}rL\left(\mathbf{r}_1-\frac{\mathbf{v}}{c}\right) = -\frac{r}{s^4}(\dot{L}+3s^{-1}cLM)[\mathbf{v}\mathbf{r}]-\frac{r}{s^4}M[\dot{\mathbf{v}}\mathbf{r}]+\frac{r}{cs^3}[\mathbf{v}\dot{\mathbf{v}}],$$

whence $4\pi e^{-1}\partial\mathbf{H}/\partial t$ is given by the expression (δ).

5. Formulae appropriate to the wave-zone.

If we retain only terms of order r^{-1}, the expressions (α), (β), (γ), (δ) become

$$4\pi\epsilon e^{-1}\mathbf{E} = \frac{1}{c^2 s^3}\{(\dot{\mathbf{v}}\mathbf{r})\mathbf{R} - rs\dot{\mathbf{v}}\}$$

$$= \frac{1}{c^2 s^3}\{\mathbf{r}[\mathbf{R}\dot{\mathbf{v}}]\}, \qquad (\epsilon)$$

since

$$rs = (\mathbf{R}\mathbf{r})$$

[we may include here the expression given by Heaviside*, viz. $c^{-2}\partial^2\mathbf{r}_1/\partial t^2$; its equivalence with the above follows from differentiating (11) and excluding irrelevant terms];

$$4\pi e^{-1}\mathbf{H} = \frac{1}{cs^2}[\dot{\mathbf{v}}\mathbf{r}]; \qquad (\zeta)$$

$$\frac{4\pi}{e}\frac{\partial\mathbf{D}}{\partial t} = \mathbf{R}\frac{r}{c^2 s^4}\left\{(\ddot{\mathbf{v}}\mathbf{r}) + \frac{3}{cs}(\dot{\mathbf{v}}\mathbf{r})^2\right\} - \mathbf{v}\frac{3r^2}{c^3 s^4}(\dot{\mathbf{v}}\mathbf{r}) - \ddot{\mathbf{v}}\frac{r^2}{c^2 s^3}$$

$$= \frac{r}{c^2 s^4}\{\mathbf{r}[\mathbf{R}\ddot{\mathbf{v}}]\} + \frac{3r(\dot{\mathbf{v}}\mathbf{r})}{c^3 s^5}\{r[\mathbf{R}\dot{\mathbf{v}}]\}; \qquad (\eta)$$

$$\frac{4\pi}{e}\frac{\partial\mathbf{H}}{\partial t} = [\mathbf{v}\mathbf{r}]\frac{r}{c^2 s^4}\left\{(\ddot{\mathbf{v}}\mathbf{r}) + \frac{3}{cs}(\dot{\mathbf{v}}\mathbf{r})^2\right\} + [\dot{\mathbf{v}}\mathbf{r}]\frac{3r}{c^2 s^4}(\dot{\mathbf{v}}\mathbf{r}) + [\ddot{\mathbf{v}}\mathbf{r}]\frac{r}{cs^3}. \qquad (\theta)$$

The expressions (η) and (θ) can also be obtained by differentiating (ϵ) and (ζ) and retaining only terms of order r^{-1}.

Using the result

$$c[\mathbf{r}_1\mathbf{R}] = [\mathbf{v}\mathbf{r}],$$

we obtain readily

$$\frac{\partial\mathbf{H}}{\partial t} = \sqrt{\left(\frac{\epsilon}{\mu}\right)}\left[\mathbf{r}_1\frac{\partial\mathbf{E}}{\partial t}\right]. \qquad (\iota)$$

It is important to notice that this formula is valid only in the wave-zone, whereas the corresponding formula (β') is valid at all points in the field.

If the motion of the point-charge is uniformly accelerated, so that $\ddot{\mathbf{v}} = 0$, it follows, from (ϵ) and (η), that

$$\frac{\partial\mathbf{E}}{\partial t} = \frac{3r}{cs^2}(\dot{\mathbf{v}}\mathbf{r})\mathbf{E}. \qquad (\kappa)$$

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* *Electromagnetic theory*, 3 (London, 1912), 438 (34).

Also, from (ζ) and (θ),

$$\frac{\partial \mathbf{H}}{\partial t} = \frac{3r}{cs^2}(\dot{\mathbf{v}}\mathbf{r})\mathbf{H}. \qquad (\lambda)$$

Thus, for uniformly accelerated motion of the point-charge, $\partial\mathbf{E}/\partial t$ and $\partial\mathbf{H}/\partial t$ have the same directions as $\mathbf{E}$ and $\mathbf{H}$ respectively at any field-point situated in the wave-zone.

6. *Linear oscillator.*

Suppose that the point-charge executes simple harmonic vibrations defined by the equation

$$\xi = a \sin \omega\tau.$$

Choose the line of oscillation as x-axis and choose the y-axis so that the field-point P is given by (x, y); with the notation of § 4 let α, θ, $\theta+\chi$ be the angles which $OP(\mathbf{r}_0)$, $Q'P(\mathbf{r})$, and $QP(\mathbf{R})$ make respectively with OX.

(i) *Various expressions for the electric displacement.*

From expression (ϵ) we have for the electric displacement at time t,

$$4\pi\mathbf{D} = \frac{e}{c^2 s^3}\{(\dot{\mathbf{v}}\mathbf{r})\mathbf{R} - rs\dot{\mathbf{v}}\}.$$

If $(\dot{\mathbf{v}}\mathbf{r})\mathbf{R} - rs\dot{\mathbf{v}}$ be resolved along $\mathbf{r}$, we obtain

$$\dot{v}rR\cos\theta\cos\chi - rs\dot{v}\cos\theta,$$

which vanishes, since $s = R\cos\chi$.

Thus the electric displacement at P is perpendicular to $Q'P$.

Let $\mathbf{t}_1$ be a unit vector transverse to $\mathbf{r}$ and in the sense of increasing θ. Resolving along $\mathbf{t}_1$, we obtain

$$4\pi\mathbf{D} = \frac{eyr\dot{v}}{c^2 s^3}\mathbf{t}_1. \qquad (\mu)$$

When ξ is positive, $\dot{v}$ is negative and $\mathbf{D}$ has the sense of decreasing θ.

The above expression (μ) is, of course, valid only in the wave-zone, and is subject to the limitations already imposed in § 4; with these reservations, however, it is strictly accurate. Since the oscillations are

relatively small, it is evident that we may in general replace r by r_0 without appreciable error; but it is important to notice, as Heaviside* observes, that this substitution is not justified in the case of the product $\omega\tau$. The difference between

$$\omega\tau = \omega\left(t-\frac{r}{c}\right)$$

and

$$\omega\tau_0 = \omega\left(t-\frac{r_0}{c}\right)$$

is, in general, of the order $\omega a/c$, *i.e.* of the ratio of the maximum velocity of the charge to the velocity of light; the error involved remains of this order even when the field-point tends to an infinite distance.

Schott† has given an expression of an entirely different character for the electric displacement; we shall now show that it can be derived from the expression already given in (μ).

Since $r = r_0-\xi\cos\alpha$ to the first power of ξ/r,

$$\tau = \tau_0+c^{-1}a\cos\alpha\sin\omega\tau. \qquad \text{(i)}$$

On differentiating this twice and using (3) we find that

$$\frac{\partial^2\tau}{\partial\tau_0^2} = -\frac{r^3}{s^3}\cdot\frac{\omega^2\xi\cos\alpha}{c}.$$

If we put $y = r_0\sin\alpha = r\sin\alpha$, approximately, the expression (μ) becomes

$$4\pi\mathbf{D} = \frac{e\tan\alpha}{cr}\frac{\partial^2\tau}{\partial\tau_0}\,\mathbf{t}_1. \qquad \text{(ii)}$$

Now from (i) $\qquad \omega\tau = \omega\tau_0+\beta\cos\alpha\sin\omega\tau,$

where $\beta = \omega a/c = v_m/c$, v_m being the maximum velocity of the charge. Thus $\omega\tau_0$ and $\omega\tau$ can be regarded as the mean and eccentric anomalies in a planetary orbit of eccentricity $\beta\cos\alpha$.

From the expansion given by Bessel we have

$$\omega\tau = \omega\tau_0+2\sum_{\sigma=1}^{\infty}\sigma^{-1}J_\sigma(\sigma\beta\cos\alpha)\sin(\sigma\omega\tau_0),$$

whence

$$\frac{\partial^2\tau}{\partial\tau_0^2} = -2\omega\sum_{\sigma=1}^{\infty}\sigma J_\sigma(\sigma\beta\cos\alpha)\sin(\sigma\omega\tau_0).$$

* *Loc. cit.*, 441.

† *Loc. cit.*, Art. 93 (152).

Substituting in (ii) we obtain

$$4\pi\mathbf{D} = -\frac{2e\beta\tan\alpha}{ar}\sum_{\sigma=1}^{\infty}\sigma J_\sigma(\sigma\beta\cos\alpha)\sin(\sigma\omega\tau_0)\,\mathbf{t}_1,$$

and this is equivalent to the expression given by Schott.

In this form the electric displacement at the field-point is analysed into a series of harmonic constituents of frequencies

$$\frac{\sigma\omega}{2\pi} \quad (\sigma = 1, 2, 3, \ldots),$$

the wave-length of the principal vibration being given by

$$\lambda = 2\pi c\omega^{-1}.$$

Various expressions for the electric displacement at a distant point in the field of a linear oscillator can be derived from (μ) and are given below. The symbol $\mathbf{p}$ represents the moment of the oscillator at time τ, so that

$$p = \epsilon\xi = ea\sin\omega\tau.$$

The suffix zero implies that the quantity affected is to be taken at time $\tau_0 = t - r_0/c$; thus

$$\dot{v}_0 = -\omega^2\xi_0 = -\omega^2 a\sin\omega\tau_0.$$

Imagine a sphere of very large radius with centre at the field-point and contracting with the velocity of light; if t is the time when the contraction is complete, then τ_0, τ will denote respectively the times at which the sphere passed through the origin and the point-charge.

$$4\pi\mathbf{D} = -\frac{\omega^2 yr}{c^2 s^3}\,p\mathbf{t}_1 = \frac{eyr\dot{v}}{c^2 s^3}\,\mathbf{t}_1 \qquad (\mu_1)$$

$$= \left\{-\frac{\omega^2 yr}{c^2 s^3}\,p_0 + \frac{e(x-\xi)\,yv\dot{v}}{c^2 s^3} + \ldots\right\}\mathbf{t}_1$$

$$= \frac{eyr}{c^2 s^2}\{\dot{v}_0 + c^{-1}v\dot{v}\cos\theta + \ldots\}\,\mathbf{t}_1 \qquad (\mu_2)$$

$$= -\frac{\omega^2 yea}{c^2 r^2}\{\sin(\omega\tau_0) + 2\beta\cos\alpha\sin(2\omega\tau_0) + \ldots\}\,\mathbf{t}_1$$

$$= -\frac{2e\beta\tan\alpha}{ar}\sum_{\sigma=1}^{\infty}\sigma J_\sigma(\sigma\beta\cos\alpha)\sin(\sigma\omega\tau_0)\,\mathbf{t}_1. \qquad (\mu_3)$$

The actual displacement at a field-point at time t consists, as we have seen, of an infinite series of harmonics, of which the fundamental is

$$-\frac{\omega^2 yea}{4\pi c^2 r_0^2} \sin \omega \left(t - \frac{r_0}{c}\right) \mathbf{t}_1.$$

In (μ_1) the displacement is given by expressions of harmonic form, but with slightly variable amplitudes and epochs; in the remaining forms it is expressed as a series, the second term in each case having a ratio of order β to the first term. In (μ_2) the principal term has a slightly variable amplitude, while in (μ_3), on replacing r by r_0, we have a series of true harmonics.

(ii) *Electric displacement current density.*

From formula (μ_1) we have

$$4\pi \mathbf{D} = -\frac{\omega^2 yr}{c^2 s^3} p \mathbf{t}_1.$$

It is easy to show that

$$\frac{\partial \mathbf{t}_1}{\partial t} = -\frac{vy}{rs} \mathbf{r}_1;$$

this is of the order r^{-1} and leads, therefore, to a term of order r^{-2} in $\partial \mathbf{D}/\partial t$; we consequently omit it from any formula appropriate to the wave-zone. Hence

$$4\pi \frac{\partial \mathbf{D}}{\partial t} = 4\pi \left(\frac{\partial \mathbf{D}}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial \mathbf{D}}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial \mathbf{D}}{\partial s} \frac{\partial s}{\partial t}\right).$$

The second term arises from variation in r where it occurs explicitly in the expression for $\mathbf{D}$; it is equal to

$$\frac{\omega^2 yr}{c^2 s^4} p v_r \mathbf{t}_1,$$

and, being of the order r^{-2}, is therefore neglected.

Using (4) and (8), and retaining only terms of order r^{-1}, we deduce that

$$4\pi \frac{\partial \mathbf{D}}{\partial t} = -\frac{\omega^2 yr^2}{c^2 s^4} \left\{\dot{p} + \frac{3p(\dot{\mathbf{v}}\mathbf{r})}{cs}\right\} \mathbf{t}_1. \qquad (\nu_1)$$

or, since $(\dot{\mathbf{v}}\mathbf{r}) = \dot{v}(x-\xi)$ and $\omega^2 p = -e\dot{v}$,

$$4\pi \frac{\partial \mathbf{D}}{\partial t} = -\frac{yr^2}{c^2 s^4}\left\{\omega^2 \dot{p} - \frac{3e(x-\xi)\dot{v}^2}{cs}\right\}\mathbf{t}_1. \qquad (\nu_1')$$

These forms of the electric displacement current density are valid only in the wave-zone and are subject to the limitations already specified; with these reservations they are strictly accurate.

It should be noticed that the second term in (ν_1) arise solely through variation in the magnitude s; s may not alter appreciably during the oscillation, but, if the magnitude of $\dot{\mathbf{v}}$ is sufficient, the small variation in s may occur at an enormous rate in a very small interval of time.

The following series forms, which may be readily verified, correspond to those already given for $4\pi\mathbf{D}$;

$$4\pi \frac{\partial \mathbf{D}}{\partial t} = -\frac{yr^2}{c^2 s^4}\left\{\omega^2 \dot{p}_0 - \frac{e(x-\xi)\dot{v}_0^2}{cr}\left(1+\frac{3r}{s}\right)+\ldots\right\}\mathbf{t}_1 \qquad (\nu_2)$$

$$= -\frac{\omega^3 yea}{c^2 r^2}\left\{\cos(\omega\tau_0)+4\beta\cos\alpha\cos(2\omega\tau_0)+\ldots\right\}\mathbf{t}_1$$

$$= -\frac{2e\beta\tan\alpha}{ar}\sum_{\sigma=1}^{\infty}\sigma^2 J_\sigma(\sigma\beta\cos\alpha)\cos(\sigma\omega\tau_0)\mathbf{t}_1 \qquad (\nu_3)$$

In these forms the second term has a ratio of order β to the first term. In (ν_2) the principal term has a slightly variable amplitude, while in (ν_3), on replacing r by r_0, we have a series of true harmonics. The forms (ν_3) can be obtained directly from (μ_3) on using the result

$$\frac{\partial}{\partial\tau_0} = \frac{\partial}{\partial t},$$

which follows from $\tau_0 = t - \frac{r_0}{c}$.

(iii) *Phase relations between oscillator and oscillations.*

From the expression (μ_1) we see that, for any field-point not on the line of oscillation, $\mathbf{D}$ vanishes only with p or $\dot{v}$, *i.e.* for values of τ such that $\sin\omega\tau = 0$.

Since $r = r_0$ for these values of τ, we have

$$\tau = t - \frac{r_0}{c}.$$

It follows that the time t at which the electric displacement vanishes is the same for all particles situated at the same distance from O; such particles will pass simultaneously through their equilibrium positions.

Let any value of τ which corresponds to a maximum displacement at the field-point be denoted by τ_m; then τ_m can be obtained by equating to zero either of the expressions (ν).

On substituting for $\underline{p}$, $\dot{p}$, and $\dot{v}$ in terms of τ and using (3) in the form

$$s = r(1 - \gamma \cos \omega\tau)$$

we obtain
$$2\gamma \cos^2 \omega\tau_m + \cos \omega\tau_m - 3\gamma = 0, \qquad (\pi)$$

where
$$\gamma = c^{-1}\omega a \cos\theta = \beta \cos\theta.$$

Replacing $\cos\theta$ by $\cos\alpha$, the difference being of order ξ/r, and therefore negligible, the values of τ_m are given by

$$\cos \omega\tau_m = \frac{-1 + \sqrt{(1 + 24\gamma^2)}}{4\gamma}, \qquad (\rho_1)$$

the negative sign being rejected from the square root, since γ has values less than unity.

For small values of γ this equation takes the form

$$\cos \omega\tau_m = 3\gamma = 3\beta \cos\alpha. \qquad (\rho_2)$$

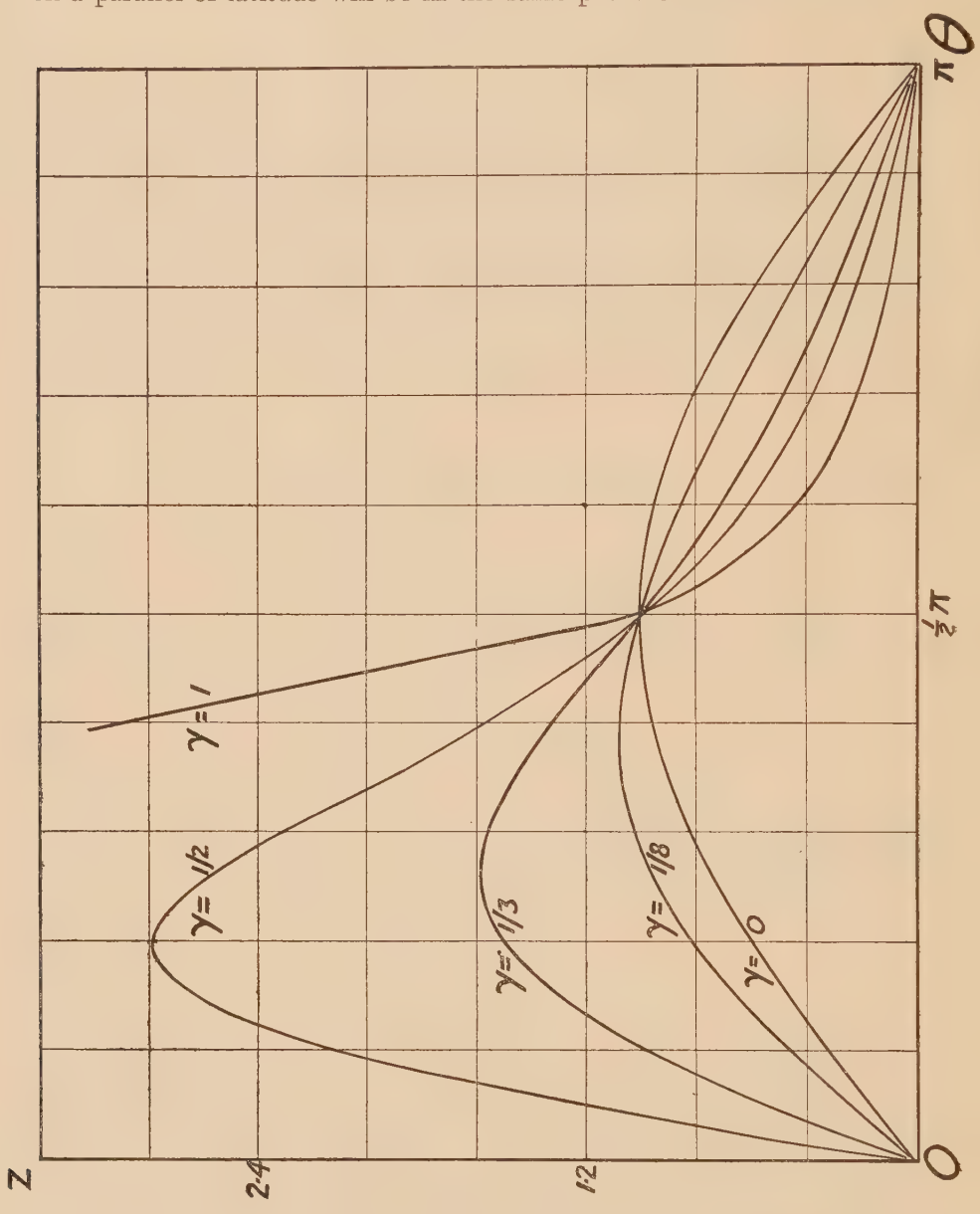
It follows, therefore, that, in general, there will be a phase difference in the oscillation of the various particles which at any instant are situated on a sphere described round O. The difference between the phase of an oscillation at time t and that of the oscillator at the effective time τ we shall call the *virtual phase difference*. If we consider maximum displacements at the various field-points we deduce, from equation (ρ_1), that

for $\alpha = \pm\frac{1}{2}\pi$, there is virtual phase equality;

for $\alpha = 0, \pi$, the oscillation has respectively a maximum advance and lag of virtual phase.

If the axis of the oscillator is regarded as defining the polar axis of

a spherical surface, the various particles which at any instant are situated on a parallel of latitude will be in the same phase of their oscillations.



From the equations (μ_1) we deduce readily that the electric displacement at points *equidistant from the oscillator* is proportional to

$$\frac{\sin\alpha \sin\omega\tau}{(1-c^{-1}v_m \cos\alpha \cos\omega\tau)^3}. \qquad (\sigma)$$

In the diagram the function

$$z = \frac{\sin\theta}{(1-\gamma\cos\theta)^3}$$

is exhibited for values of θ from 0 to π. The five curves correspond to the following values of γ : 0, $\frac{1}{8}$, $\frac{1}{3}$, $\frac{1}{2}$, 1 ; each curve can readily be extended, from the consideration that it is symmetrical with regard to the points where it crosses the θ-axis. These curves represent for $\theta = \omega\tau$ the variation with the time of the electric displacement at a given field-point ; since α and $\omega\tau$ are interchangeable in the expression (σ) they represent also for $\theta = \alpha$ the variation with the direction of the displacement at a given instant of time, provided that we replace γ by $\gamma' = c^{-1}v_m\cos\omega\tau$. Each curve has a point of inflexion where it crosses the axis ; and, if γ lies between $\frac{1}{8}$ and 1, the corresponding curve has an additional point of inflexion between successive crossings of the axis.

It is evident from the graphs that the to-and-fro vibrations which constitute a complete oscillation are described in general in different times, although the total time of oscillation is the same at all field-points. The rapid vibration always occurs when the displaced particle is moving towards the axis of the oscillator. In particular, particles which are situated on the equator of the spherical surface specified above will describe their to-and-fro vibrations in equal times ; these particles have always the same virtual phase as the oscillator.

We have proved that the maximum displacement at a field-point, situated in any assigned direction α, occurs for values of τ given by the equation (π), viz.,

$$2\beta\cos\alpha\cos^2\omega\tau + \cos\omega\tau - 3\beta\cos\alpha = 0.$$

If now α_M specifies the direction to the point at which the maximum displacement has its greatest value, and if τ_M denotes the corresponding value of τ, then

$$2\beta\cos\alpha_M\cos^2\omega\tau_M + \cos\omega\tau_M - 3\beta\cos\alpha_M = 0. \qquad (\tau_1)$$

Now the displacement at points equidistant from the oscillator is proportional to the expression (σ) ; since α and $\omega\tau$ are interchangeable in this expression we can deduce similarly that

$$2\beta\cos\omega\tau_M\cos^2\alpha_M + \cos\alpha_M - 3\beta\cos\omega\tau_M = 0. \qquad (\tau_2)$$

It is evident that $\omega\tau_M = \alpha_M$. (υ)

Substituting in either of the equations (τ), we obtain

$$\cos a_M(2\beta \cos^2 a_M + 1 - 3\beta) = 0,$$

whence either $$a_M = \tfrac{1}{2}\pi \qquad (\phi_1)$$

or $$2\cos^2 a_M = 3 - \beta^{-1}. \qquad (\phi_2)$$

The value $\frac{1}{2}\pi$ is to be chosen when the maximum velocity of the oscillator is in the range defined by $0 < \beta \leqslant \frac{1}{3}$; for the range $\frac{1}{3} \leqslant \beta < 1$, a_M is to be determined by the equation (ϕ_2).

In order that the foregoing results should be valid it is necessary that the three conditions already specified in § 4 should be fulfilled. The first and second conditions have been observed throughout the mathematical development; in the third condition let us give $|\mathbf{v}|$ and $|\dot{\mathbf{v}}|$ their maximum values, viz. v_M and $4\pi^2 c^2 a\lambda^{-2}$ respectively, so that this condition will be satisfied *a fortiori* provided that $a\lambda^{-2}$ is small compared with

$$\frac{(1-\beta)}{8\pi^2 b}. \qquad (\chi)$$

As a practical illustration let us suppose that the fundamental radiation emitted by the linear oscillator has a wave-length $\lambda = 10^{-8}$ cm. This is, roughly, the wave-length of the K-radiation from bromine; in order to find approximate values of a and β, let us assume that this radiation is due to an electron moving with uniform speed v in a circular orbit of radius a round the nucleus. This assumption is, of course, contrary to modern ideas in regard to radiation from moving electrons; still it is of interest to ascertain whether results deduced from Maxwell's fundamental equations are theoretically valid under such circumstances.

If $-e$ and m denote respectively the charge and mass of the electron, the charge on the bromine nucleus is $35e$, and we have approximately

$$mv^2 a^{-1} = 35e^2 a^{-2}.$$

Also $$\lambda = 2\pi a c v^{-1}.$$

Taking

$$e = 4{\cdot}77 \times 10^{-10}\ \text{E.S.U.}, \quad m = 9 \times 10^{-28}\ \text{gram},$$

we obtain

$$a = 2{\cdot}9 \times 10^{-10}\ \text{cm.}, \qquad v = 5{\cdot}5 \times 10^{9}\ \text{cm./sec.}$$

If we now regard these values as being respectively the amplitude and maximum velocity of the linear oscillator we obtain

$$a\lambda^{-2} = 2{\cdot}9 \times 10^{6},$$

$$\frac{(1-\beta)}{8\pi^2 b} = 10^{11},$$

taking $b = 10^{-13}$ cm. for the electronic radius.

Thus the relation (χ) is satisfied.

We conclude with the following tabulation of numerical values, in which the field-points are regarded as being equidistant from the oscillator, and are specified by values of α selected at intervals of 15° between 0 and 90°. For assigned values of α and β, $\omega\tau_m$ is determined from the equation (ρ_1); on substituting in the expression (σ) we obtain a measure of the maximum displacement (D_m) at the specified field-point; for the field-point $\alpha = \frac{1}{2}\pi$ it will be observed that D_m is expressed by unity for all values of β.

The values of $\omega\tau_M$ and α_M, which correspond to the greatest value of D_m are obtained from equations (υ) and (ϕ); if we replace τ by τ_M in the expression (σ) we obtain a measure of the displacement (D) at the specified field-point (α) at the instant when the displacement at the field-point (α_M) is greatest.

T denotes $\omega\tau_m/\pi$ and represents, therefore, the fraction of the time of a complete oscillation which is occupied in the vibration at the field-point; we have already seen that, in the quicker vibration, the displaced particle moves towards the axis of the oscillator.

The tabulation for values of α from 90° to 180° can be completed by utilizing the following results, which are readily deducible from the formulae quoted:

$$(\omega\tau_m)_{\pi-\alpha} + (\omega\tau_m)_\alpha = \pi,$$

$$(D_m)_{\pi-\alpha} = (D_m)_\alpha.$$

The former may be written

$$T_{\pi-\alpha} + T_\alpha = 1,$$

and it conforms with the rule just quoted concerning the direction of the rapid vibration at a field-point.

The tabulations for $\beta = 0$ and $\beta = 1$ contain respectively the limiting values for small and large velocities of the oscillator.

$\beta = 0$ $(\omega\tau_M = \alpha_M = \frac{1}{2}\pi)$.

α	$\omega\tau_m$	D_m	D	T
0	90°	0	0	0·5000
15°	90°	0·2588	0·2588	0·5000
30°	90°	0·5000	0·5000	0·5000
45°	90°	0·7071	0·7071	0·5000
60°	90°	0·8660	0·8660	0·5000
75°	90°	0·9659	0·9659	0·5000
90°	90°	1·0000	1·0000	0·5000

$\beta = 0{\cdot}1$ $(\omega\tau_M = \alpha_M = \frac{1}{2}\pi)$.

α	$\omega\tau_m$	D_m	D	T
0	73° 30′	0	0	$0{\cdot}408_3$
15°	74° 2′	$0{\cdot}269_8$	0·2588	$0{\cdot}411_3$
30°	75° 34′	$0{\cdot}516_9$	0·5000	$0{\cdot}419_8$
45°	78° 6′	$0{\cdot}723_1$	0·7071	$0{\cdot}433_9$
60°	81° 30′	$0{\cdot}875_8$	0·8660	$0{\cdot}452_8$
75°	85° 34′	$0{\cdot}968_8$	0·9659	$0{\cdot}475_4$
90°	90°	1·0000	1·0000	0·5000

$\beta = \frac{1}{3}$ $(\omega\tau_M = \alpha_M = \frac{1}{2}\pi)$.

α	$\omega\tau_m$	D_m	D	T
0	46° 41′	0	0	0·259
15°	47° 39′	$0{\cdot}398_3$	0·2588	$0{\cdot}264_7$
30°	50° 40′	$0{\cdot}709_4$	0·5000	$0{\cdot}281_5$
45°	55° 59′	$0{\cdot}895_8$	0·7071	$0{\cdot}311_0$
60°	64° 7′	$0{\cdot}977_5$	0·8660	$0{\cdot}356_2$
75°	75° 38′	$0{\cdot}988_4$	0·9659	$0{\cdot}420_2$
90°	90°	1·0000	1·0000	0·5000

$\beta = 0{\cdot}5$ $(\omega\tau_M = \alpha_M = \frac{1}{4}\pi)$.

α	$\omega\tau_m$	D_m	D	T
0	34° 38′	0	0	$0{\cdot}192_4$
15°	35° 42′	$0{\cdot}672_8$	$0{\cdot}640_9$	$0{\cdot}198_3$
30°	39° 2′	$1{\cdot}077_2$	$1{\cdot}058_6$	$0{\cdot}216_8$
45°	45°	$1{\cdot}185_2$	$1{\cdot}185_2$	0·2500
60°	54° 28′	$1{\cdot}128_5$	$1{\cdot}097_7$	$0{\cdot}302_6$
75°	69° 10′	$1{\cdot}039_8$	$0{\cdot}910_9$	$0{\cdot}384_4$
90°	90°	1·0000	$0{\cdot}707_1$	0·5000

$\beta = 0{\cdot}9$ $(\omega\tau_M = \alpha_M = 13°\,38')$.

α	$\omega\tau_m$	D_m	D	T
0	12°	0	0	$0{\cdot}066_7$
13° 38′	13° 38′	$16{\cdot}460_9$	$16{\cdot}460_9$	$0{\cdot}075_7$
15°	13° 57′	$16{\cdot}334_5$	$16{\cdot}331_5$	$0{\cdot}077_5$
30°	19° 2′	$8{\cdot}943_1$	$8{\cdot}260_3$	$0{\cdot}105_7$
45°	26° 26′	$3{\cdot}954_5$	$3{\cdot}000_9$	$0{\cdot}146_8$
60°	37° 53′	$1{\cdot}983_5$	$1{\cdot}145_9$	$0{\cdot}210_5$
75°	56° 16′	$1{\cdot}217_1$	$0{\cdot}491_7$	$0{\cdot}312_6$
90°	90°	1·0000	$0{\cdot}235_7$	0·5000

$\beta = 1$ $(\omega\tau_M = \alpha_M = 0)$.

α	$\omega\tau_m$	D_m	D	T
0	0	∞	∞	0
15°	6° 48′	448·57	0	$0{\cdot}037_8$
30°	14° 9′	$29{\cdot}68_5$	0	$0{\cdot}078_6$
45°	22° 53′	$6{\cdot}489_6$	0	$0{\cdot}127_1$
60°	34° 37′	$2{\cdot}413_2$	0	$0{\cdot}192_3$
75°	53° 34′	$1{\cdot}282_1$	0	$0{\cdot}297_6$
90°	90°	1·0000	0	0·5000

A CLASS OF NULL SERIES

By G. TEMPLE.

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BY way of introduction we recall briefly the main results which have been obtained in this subject. First let us consider series of the type studied by Sturm and Liouville. Let $u_n(x)$ denote a solution of the equation

$$\frac{d}{dx}\left\{p(x)\frac{du(x)}{dx}\right\}+\{q(x)+\rho_n^2\sigma(x)\}u(x)=0$$

valid in the interval $a \leqslant x \leqslant b$, and satisfying at the end-points homogeneous mixed boundary conditions of the type

$$\cos\alpha\,.\,u(a)=\sin\alpha\,.\,p(a)\left(\frac{du(x)}{dx}\right)_{x=a},$$

$$\cos\beta\,.\,u(b)=\sin\beta\,.\,p(b)\left(\frac{du(x)}{dx}\right)_{x=b}.$$

Then it is known* that a series formed with the functions $u_n(x)$, *i.e.* $\sum_{n=1}^{\infty} u_n(x)$, which is uniformly convergent in the interval $a \leqslant x \leqslant b$, cannot converge to the sum zero at all points of the interval, unless each term of the series is separately zero. Furthermore, it has been shown by Riemann and others† that in the case of trigonometrical series, and by Watson that in the case of Fourier-Bessel and Bessel-Dini series‡,

* A recent and interesting discussion is given in the paper "Neue Herleitung der Sturm-Liouville'schen Reihenentwicklung stetiger Funktionen", by H. Prüfer, *Math. Ann.*, 95 (1926), 499–518.

† See the account given by E. W. Hobson in *The theory of functions of a real variable.*

‡ G. N. Watson, *The theory of Bessel functions.*

there exists no series of this type which converges to zero at all points of a certain interval, with the exception of a certain reducible set of points.

On the other hand, numerous null series are known which do not belong to the Sturm-Liouville type. For example, the series

$$\sum_{n=1}^{\infty} \cos nx \,.\, \cos^n x$$

converges uniformly to zero at all points within the interval

$$0 < \delta \leqslant x \leqslant \pi - \delta.$$

Series of the type studied by Schlömilch are prolific in providing null series uniformly convergent throughout suitable intervals which usually exclude points in the neighbourhood of the end-points of the interval wherein normal Schlömilch series converge*. For example, the series

$$\frac{\frac{1}{2}}{\Gamma(\nu+1)} + \sum_{n=1}^{\infty} \frac{(-)^n J_\nu(nx)}{(\frac{1}{2}nx)^\nu}$$

and

$$\sum_{n=1}^{\infty} \frac{(-)^n J_\nu(nx)}{(\frac{1}{2}nx)^{\nu-2m}}$$

converge to zero respectively in the intervals

$$-\pi+\delta \leqslant x \leqslant -\delta, \quad \delta \leqslant x \leqslant \pi-\delta \quad \text{and} \quad -\pi+\delta \leqslant x \leqslant \pi-\delta$$

(m being a positive integer and ν lying between $\pm\frac{1}{2}$).

In this note we propose to investigate the existence of series which are uniformly summable $(C, 1)$ to zero at all points of a certain closed interval with the possible exception of one point, the terms of the series being solutions of differential equations of the Sturm-Liouville type. The introduction of the concept of uniform summability permits great simplifications to be made in the analysis.

On the basis of a number of theorems due to Hobson and others it is shown that there are no null series of the type defined above which are uniformly convergent throughout the whole domain of definition of the Sturm-Liouville functions. Null series can exist only when they possess at least one singular point; at a singular point the series may be summable to some sum other than zero or may not be summable at all. But the existence of such a singular point permits the use of null series in extending the methods by which Chapman has proved Fejér's

* See C. Fox, "A class of null series", *Proc. London Math. Soc.*, 2, 24 (1925), 479-493.

theorem for Fourier series, Laplace's series, and Legendre's series, and gives an interesting interpretation of Hobson's general convergence theorem*.

1. *Definitions.*

Let $l(z)$ be a function of the real variable z, which is continuous and of limited total fluctuation in the interior of the interval $(0, \pi)$. Let $L(u)$ denote the differential expression

$$\frac{d^2u(z)}{dz^2} - l(z)u(z).$$

Let "A" and "B" denote two linear boundary conditions relating to the behaviour of solutions of a differential equation, the one referring to their behaviour in the neighbourhood of the point $z = 0$, the other referring to their behaviour in the neighbourhood of the point $z = \pi$. Furthermore, let those conditions be inconsistent with the equation $L(u) = 0$, so that there exists no function $u(z)$ continuous together with its first and second derivatives $u'(z)$ and $u''(z)$ in the interval $0 \leqslant z \leqslant \pi$, and therein satisfying the equation $L(u) = 0$, while at the points $z = 0$, $z = \pi$, $u(z)$ satisfies the boundary conditions "A" and "B".

Then these conditions are sufficient for the validity of a theorem due to Kneser†, subsequently to be cited as Theorem A.

THEOREM A. *There exists one and only one function $K(z, \zeta)$ which is a continuous function of z in the interval $0 \leqslant z \leqslant \pi$, whose derivatives with respect to z, namely $K'(z, \zeta)$ and $K''(z, \zeta)$, are continuous functions of z in the intervals of z,*

$$0 < \delta \leqslant z \leqslant \zeta - \delta \quad \text{and} \quad \zeta + \delta \leqslant z \leqslant \pi - \delta;$$

which satisfies the differential equation $L(u) = 0$ in each of these intervals, and satisfies the boundary conditions "A" and "B"; and which satisfies in addition the following equation:

$$\lim_{\delta \to 0} \{K'(\zeta - \delta, \zeta) - K'(\zeta + \delta, \zeta)\} = 1 \quad (0 < \zeta < \pi).$$

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* As far as possible we adopt the notation of E. W. Hobson, *Proc. London Math. Soc.* (2), 6 (1908), 349-395, in his paper "On a general convergence theorem."

† A. Kneser, "Die Theorie der Integralgleichungen . . . ", *Math. Annalen*, 63 (1907), 477–524.

This function, which is a Green's function for the equation $L(u) = 0$, is called the "nucleus".

The differential equation which is satisfied by the functions forming the successive terms of the Sturm-Liouville series which we consider is

$$L(u)+\rho^2 u(z) = 0, \tag{1}$$

and concerning the solutions of this equation we have the following theorems :

THEOREM B. *The necessary and sufficient condition that the differential equation* (1) *should possess a solution* $u(z)$ *continuous in the interval* $0 \leqslant z \leqslant \pi$, *possessing continuous derivatives* $u'(z)$ *and* $u''(z)$ *in the interval* $0 < z < \pi$, *and satisfying the boundary conditions* "A" *and* "B", *is that the parameter* ρ *should be equal to one of the terms of a certain real increasing sequence*

$$\rho_1 < \rho_2 < \rho_3 < \dots < \rho_n < \rho_{n+1} < \dots$$

in which no term is zero, and $1/\rho_n = O(1/n)$. *The numbers* ρ_n *may be called the "characteristic numbers" for the nucleus* $K(z, \zeta)$. *(The usual definition makes the characteristic numbers equal to* ρ_n^2*).*

THEOREM C. *When* ρ *is equal to a characteristic number* ρ_n, *the equation* (1) *possesses one and only one solution* $\phi_n(z)$ *satisfying the conditions enunciated in Theorem* B, *and subject to the additional limitation that*

$$\int_0^\pi [\phi_n(z)]^2 dz = 1.$$

These functions are known as the characteristic functions of the nucleus $K(z, \zeta)$.

THEOREM D. *The characteristic function* $\phi_n(z)$ *satisfies an integral equation of the form*

$$\phi_n(z) = a_n \cos \rho_n(z-z_0)+\frac{1}{\rho_n}\int_{z_0}^{z} l(\xi)\,\phi_n(\xi) \sin \rho_n(z-\xi)\,d\xi, \tag{2}$$

wherein a_n *and* z_0 *are constants,* $0 < z_0$, $z < \pi$, *and* a_n *is a bounded function of* ρ_n.

Let c_1, c_2, ..., c_n, ... denote a number of constant coefficients, independent of z and bounded functions of ρ_n. Then the Sturm-Liouville

null series associated with equation (1) is defined to be a series of the type $\sum_{n=1}^{\infty} c_n \phi_n(z)$ which is uniformly summable by Cesàro's method of arithmetic means to the sum-function zero, throughout both the intervals $0 \leqslant z \leqslant \zeta - \delta$, $\zeta + \delta \leqslant z \leqslant \pi$. At the point $z = \zeta$ the series may be summable to some other value than zero, or it may be not summable at all. The apparently arbitrary character of this definition will disappear when we view the applications of null series to general theorems on the convergence of Sturm-Liouville functions. Since the null series may have a discontinuity at the point $z = \zeta$, it may be described as a null series with parameter ζ.

2. *The existence theorem for null series.*

This section is devoted to a proof of the following three theorems:

THEOREM 1. *If the series* $\sum_{n=1}^{\infty} c_n \phi_n(z)$ *is uniformly summable* $(C, 1)$ *with respect to* z *in any interval with sum-function zero, then in that interval* $F(z)$, *the sum of the associated uniformly convergent series*

$$\sum_{n=1}^{\infty} \frac{c_n \phi_n(z)}{\rho_n^2},$$

has continuous derivatives $F'(z)$ *and* $F''(z)$, *and* $F(z)$ *satisfies the equation* $L(F) = 0$.

THEOREM 2. *There is no series* $\sum_{n=1}^{\infty} c_n \phi_n(z)$ *uniformly summable with sum-function zero throughout the whole interval* $0 \leqslant z \leqslant \pi$.

THEOREM 3. *The only null series with parameter* ζ *are numerical multiples of the series* $\sum_{n=1}^{\infty} \phi_n(\zeta) \phi_n(z)$.

Theorem 1.

The proof of Theorem 1 depends upon two theorems due respectively to Chapman* and Hardy†.

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* S. Chapman, "On the general theory of summability . . . ", *Quart. J. of Math.*, 43 (1911), 1–52.

† Whittaker and Watson, *A course of modern analysis* (1920), 156–158.

THEOREM E. *If the series* $\sum_{n=1}^{\infty} u_n(z)$ *is uniformly summable* $(C,1)$ *with respect to* z *over the range* $\alpha \leqslant z \leqslant \beta$ *with sum-function* $S(z)$, *then the series* $\sum_{n=1}^{\infty} v_n(z)$, *where*

$$v_n(z) = \int_{z_1}^{z} u_n(\xi)\, d\xi \quad (\alpha \leqslant z,\ z_1 \leqslant \beta),$$

is also uniformly summable $(C,1)$ *with respect to* z *over the same range and has the sum-function* $\int_{z_1}^{z} S(\xi)\, d\xi$.

THEOREM F. *If the series* $\sum_{n=1}^{\infty} u_n(z)$ *is uniformly summable* $(C,1)$ *with respect to* z *over the range* $\alpha \leqslant z \leqslant \beta$, *and if* $|u_n(z)| < Kn^{-1}$ *where* K *is independent of* z *and* n, *then the series* $\sum_{n=1}^{\infty} u_n(z)$ *converges uniformly over the range* $\alpha \leqslant z \leqslant \beta$.

Proof. Since the functions $l(z)$ and $\phi_n(z)$ are bounded in the interior of the interval $(0, \pi)$ (see Definitions) for all values of n, and since, by Theorem B, $1/\rho_n = O(1/n)$, therefore the series $\sum_{n=1}^{\infty} c_n \phi_n(z)\, l(z)\, \rho_n^{-2}$ converges uniformly in the interval $0 < \delta \leqslant z \leqslant \pi - \delta$ and *a fortiori* is uniformly summable $(C,1)$ in this interval. Hence the series

$$\sum_{n=1}^{\infty} c_n \phi_n(z) \{1 - l(z)\, \rho_n^{-2}\} \tag{3}$$

is uniformly summable $(C,1)$ in any closed and connected interval I which excludes the three points $z = 0,\ \zeta,\ \pi$.

By utilizing the expression for $\phi_n(z)$ given by equation (2) we may integrate the series (3) term by term, and thus obtain the series

$$\sum_{n=1}^{\infty} c_n \psi_n(z),$$

where $$\psi_n(z) = \frac{a_n \sin \rho_n(z - z_0)}{\rho_n} - \frac{1}{\rho_n^2} \int_{z_0}^{z} l(\xi)\, \phi_n(\xi) \cos \rho_n(z - \xi)\, d\xi. \tag{4}$$

By Theorem E, the series thus obtained is uniformly summable $(C,1)$ provided that both z and z_0 belong to the interval I.

Since c_n and the integral which occurs in equation (4) are both bounded functions of ρ_n and z and since $1/\rho_n = O(1/n)$, it follows that

$$|c_n \psi_n(z)| < Kn^{-1},$$

where K is a constant independent of ρ_n and z. Hence, by Theorem F, the series $\sum_{n=1}^{\infty} c_n \psi_n(z)$ is uniformly convergent when z and z_0 are in the interval I.

A second term-by-term integration yields the series

$$\sum_{n=1}^{\infty} \left\{ -\frac{c_n \phi_n(z)}{\rho_n^2} + \frac{a_n c_n}{\rho_n^2} \right\}$$

which is absolutely and uniformly convergent in the interval $0 \leqslant z \leqslant \pi$.

Since c_n and a_n are bounded functions of n, we deduce that both the series $\sum_{n=1}^{\infty} c_n \phi_n(z) \rho_n^{-2}$ and $\sum_{n=1}^{\infty} a_n c_n \rho_n^{-2}$ are uniformly convergent in this interval. Let $F(z)$ denote the sum of the series $\sum_{n=1}^{\infty} c_n \phi_n(z) \rho_n^{-2}$. (This is effectively Riemann's associated function.) Then $F(z)$ is continuous in the interval $0 \leqslant z \leqslant \pi$.

From Theorem E we may now deduce that the series $\sum_{n=1}^{\infty} c_n \psi_n(z)$ converges to the sum $-F'(z)$ uniformly, provided that z and z_0 belong to the interval I. Also the series

$$\sum_{n=1}^{\infty} c_n \phi_n(z) \{1 - l(z) \rho_n^{-2}\}$$

is uniformly summable with sum $-F''(z)$ when z and z_0 both belong to the interval I. Hence, in this interval, $F''(z)$ is continuous and equals $+l(z) F(z)$, *i.e.* $F(z)$ satisfies the equation $L(F) = 0$ [since $\sum_{n=1}^{\infty} c_n \phi_n(z)$ has the sum-function zero].

Theorem 2.

If the series $\sum_{n=1}^{\infty} c_n \phi_n(z)$ is uniformly summable $(C, 1)$ with sum-function zero throughout the interval $0 \leqslant z \leqslant \pi$, then the function $F(z)$ and its two derivatives $F'(z)$ and $F''(z)$ are also continuous in this interval, and everywhere they satisfy the equation $L(F) = 0$. Moreover, since the functions $\phi_n(z)$ all satisfy the boundary conditions "A" and "B"

at the points $z = 0$ and $z = \pi$, so also must the function $F(z)$. But by our definitions the only solution of the equation $L(u) = 0$ which satisfies all these conditions is zero. Hence $F(z) \equiv 0$. The series defining $F(z)$ is uniformly convergent in the interval $0 \leqslant z \leqslant \pi$. Hence each term in the series vanishes. Therefore there does not exist a Sturm-Liouville null series free from a singularity whereat the series is not summable to zero.

Theorem 3.

The proof of this theorem depends upon three theorems due to Hobson (*loc. cit., supra*).

THEOREM G. *The series* $\sum_{n=1}^{\infty} \frac{\phi_n(z)\,\phi_n(\zeta)}{\rho_n^2}$ *converges uniformly in the interval* $0 \leqslant z \leqslant \pi$ *to the sum* $K(z, \zeta)$, *the nucleus of equation* (1).

THEOREM H. *There exist bounded functions of* z *and of* n, $a_1(z, n)$, $a_2(z)$, $a_3(z, n)$ *such that, when* $0 < z < \pi$,

$$\sqrt{\left(\frac{\pi}{2}\right)}\,\phi_n(z) = \cos nz\left\{1 + \frac{a_1(z,n)}{n^2}\right\} + \sin nz\left\{\frac{a_2(z)}{n} + \frac{a_3(z,n)}{n^2}\right\}.$$

THEOREM J. *The series* $\sum_{n=1}^{\infty} \int_{\alpha}^{\beta} \phi_n(z)\,\phi_n(\zeta)\,d\zeta$ *converges uniformly to zero if* z *is in either of the intervals* $0 \leqslant z \leqslant \alpha - \delta$, $\beta + \delta \leqslant z \leqslant \pi$.

Proof. It has been proved in Theorem 1 that, associated with any null series $\sum_{n=1}^{\infty} c_n \phi_n(z)$ with parameter ζ, there is a uniformly convergent series $\sum_{n=1}^{\infty} c_n\,\phi_n(z)\,\rho_n^{-2}$ whose sum $F(z)$ is continuous throughout the interval $0 \leqslant z \leqslant \pi$, while its derivatives are continuous throughout the intervals $0 < \delta \leqslant z \leqslant \zeta - \delta$ and $\zeta + \delta \leqslant z \leqslant \pi - \delta$. Furthermore, $F(z)$ satisfies the equation $L(F) = 0$ in these intervals, and also satisfies the boundary conditions "A" and "B" when $z = 0$ and $z = \pi$. Hence, by Theorem A, $F(z)$ can be none other than a numerical multiple of the nucleus $K(z, \zeta)$. It then follows, from Theorem G, that the coefficients c_n are constant multiples of the functions $\phi_n(\zeta)$.

Hence, if there is a null series of the type which we have specified, it can only be a multiple of the series

$$\sum_{n=1}^{\infty} \phi_n(\zeta)\,\phi_n(z). \tag{5}$$

By Theorem H the general term of this series may be written in the form

$$\phi_n(\xi)\,\phi_n(z) = \frac{2}{\pi}\left\{\cos nz\cos n\xi + \frac{\beta_1}{n}\cos nz\sin n\xi + \frac{\beta_2}{n}\sin nz\cos n\xi + \frac{\beta_3}{n^2}\right\},$$

where $|\beta_1|$, $|\beta_2|$, $|\beta_3|$ are less than some fixed number independent of z, ξ, and n; and this representation is valid when

$$0 \leqslant z \leqslant \xi - \delta \quad \text{or} \quad \xi + \delta \leqslant z \leqslant \pi.$$

The series $\sum_{n=1}^{\infty} \beta_3 n^{-2}$ is uniformly convergent in the interval $0 \leqslant z \leqslant \pi$, and the series

$$\sum_{n=1}^{\infty} \cos nz\cos n\xi, \qquad \sum_{n=1}^{\infty} \frac{\beta_1}{n}\cos nz\sin n\xi, \qquad \sum_{n=1}^{\infty} \frac{\beta_2}{n}\cos n\xi\sin nz$$

are easily seen to be uniformly summable $(C, 1)$ in the intervals $0 \leqslant z \leqslant \xi - \delta$, $\xi + \delta \leqslant z \leqslant \pi$.* Hence the series (5) is uniformly summable $(C, 1)$ in these intervals. Let its sum be $f(z, \zeta)$.

Integrate these series from α to β where z is outside the range $(\alpha - \delta, \beta + \delta)$. Then, by Theorems E and J, we find that $f(z, \zeta) = 0$. Hence the series (5) is a null series with parameter ζ.

Summary. The series $\sum_{n=1}^{\infty} \phi_n(\xi)\,\phi_n(z)$ is uniformly summable $(C, 1)$ with sum-function zero when $0 \leqslant z \leqslant \zeta - \delta$ or $\zeta + \delta \leqslant z \leqslant \pi$, and it is effectively the only null series of characteristic functions $\phi_n(z)$ which has one and only one parametric singularity ζ.

3. *Examples of null series.*

The results of the preceding section may be put into a more familiar form if the differential equation (1) is transformed by the substitutions indicated by the equations

$$\frac{dz}{dx} = \sqrt{\left(\frac{\sigma(x)}{p(x)}\right)},$$

$$u_n(z) = f_n(x)\left\{\sigma(x)\,p(x)\right\}^{\frac{1}{4}}$$

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* See the paper by Chapman, *loc. cit.*

to an equation of the form

$$\frac{d}{dx}\left\{p(x)\frac{df_n(x)}{dx}\right\}+\{q(x)+\rho_n^2\sigma(x)\}f_n(x)=0.$$

If the functions $f_n(x)$ form a system of normal and orthogonal functions for this equation over the interval of values of x, $a \leqslant x \leqslant b$, corresponding to the range $0 \leqslant z \leqslant \pi$, then the series

$$\sum_{n=1}^{\infty} f_n(\xi)f_n(x)$$

will be a null series uniformly summable $(C, 1)$ to zero in each of the intervals $a \leqslant x \leqslant \xi-\delta$, $\xi+\delta \leqslant x \leqslant b$.

The simplest examples of null series are the series of trigonometrical functions and Legendre polynomials, whose general terms are

$$\frac{2}{\pi}\cos nx \cos n\xi, \quad \frac{2}{\pi}\sin nx \sin n\xi, \quad \text{and} \quad \frac{2n+1}{2}P_n(x)P_n(\xi);$$

in these expressions n is a positive integer *or zero*, and the intervals of uniform summability are $0 \leqslant x \leqslant \xi-\delta$, $\xi+\delta \leqslant x \leqslant \pi$ for the first two series, and $-1 \leqslant x \leqslant \xi-\delta$, $\xi+\delta \leqslant x \leqslant +1$ for the last series. These are the series which play such an important part in the researches of Fejér and Chapman (*loc. cit.*), and a reference to the paper quoted will show that these series are also uniformly summable (C, K) throughout the same interval where K may be any number greater than zero for the first two series, and greater than $\frac{1}{2}$ for the last series.

The analysis of Theorem 1 may also be utilized to deduce the following theorem :

THEOREM 1′. *If $A_n(x)$ denotes a solution of the differential equation*

$$A(u) \equiv \frac{d^2u}{dx^2}+P(x)\frac{du}{dx}+Q(x)u=\rho_n^2R(x)u,$$

wherein the parameters ρ_n may be ranged in an increasing sequence

$$0=\rho_0<\rho_1<\rho_2<\ldots<\rho_n<\rho_{n+1}<\ldots,$$

so that $1/\rho_n = O(1/n)$, then the series $\sum_{n=0}^{\infty} A_n(x)$ can only be a null series for an interval of values of x wherein the sum of the associated series

$$\sum_{n=1}^{\infty}\frac{A_n(x)}{\rho_n^2}$$

satisfies the equation $A(u) = R(x)A_0(x)$.

The general term of a Schlömilch series of type ν is

$$a_n(nx)^{-\nu}J_\nu(nx),$$

where n is a positive integer. These functions satisfy the equation

$$\frac{d^2u}{dx^2}+\frac{2\nu+1}{x}\frac{du}{dx}=-n^2u.$$

Hence, if the series $\quad \frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\frac{J_\nu(nx)}{(nx)^\nu}$

is a null series throughout any interval I, the associated series

$$\sum_{n=1}^{\infty}\frac{a_n}{n^2}\frac{J_\nu(nx)}{(nx)^\nu}$$

must have a solution of the equation

$$\frac{d^2u}{dx^2}+\frac{2\nu+1}{x}\frac{du}{dx}=\tfrac{1}{2}a_0$$

as its sum, *i.e.* the sum of the associated series must be of the form

$$\frac{a_0x^2}{8(\nu+1)}+A+Bx^{-2\nu} \quad \text{if} \quad \nu\neq 0,$$

or of the form $\quad \frac{a_0x^2}{8}+A+B\log x \quad \text{if} \quad \nu=0.$

It may now be verified, as in Watson's *Theory of Bessel functions*, that the only null series of Schlömilch type in the interval

$$0<\delta\leqslant x\leqslant\pi-\delta$$

are multiples of $\quad \frac{\frac{1}{2}}{\Gamma(\nu+1)}+\sum_{n=1}^{\infty}\frac{(-)^nJ_\nu(nx)}{(\frac{1}{2}nx)^\nu}.$

4. *The generalization of Fejér's theorem.*

If a function $\psi(x)$ is expansible in a series of normal orthogonal functions $f_n(x)$ uniformly convergent in the interval $a\leqslant x\leqslant b$, the coefficients in the series will be given by the equation

$$a_n=\int_a^b\psi(x)f_n(x)\,dx.$$

Whatever be the character of the function $\psi(x)$, subject to the limitation that it is absolutely integrable with respect to x over the closed interval $a \leqslant x \leqslant b$, we shall call the series

$$\sum_{n=1}^{\infty} g_n(\xi) \quad \text{where} \quad g_n(\xi) = f_n(\xi) \int_a^b \psi(x) f_n(x)\, dx$$

the Sturm-Liouville series associated with the function $\psi(x)$. In this section we shall prove the following theorem concerning this series.

THEOREM 4. *If the function $\psi(x)$ is absolutely integrable over the interval $a \leqslant x \leqslant b$, and has limited total fluctuation in the range*

$$\xi - \delta \leqslant x \leqslant \xi + \delta \quad (0 < \delta),$$

the associated Sturm-Liouville series $\sum_{n=1}^{\infty} g_n(\xi)$ is summable $(C, 1)$ with sum-function $\frac{1}{2} \lim_{\delta \to 0} \{\psi(\xi-\delta)+\psi(\xi+\delta)\}$ provided that $a < \xi < b$.

(The modification of the theorem when ξ is equal to a or b is obvious.)

The proof of this theorem depends upon two theorems due to Chapman and Hobson (*loc. cit.*).

THEOREM K. *If the series $\sum_{n=1}^{\infty} u_n(x)$ is a series of functions which are continuous functions of x in the closed interval $a \leqslant x \leqslant b$, and is uniformly summable $(C, 1)$ over any interval $a < \alpha \leqslant x \leqslant \beta < b$ with sum-function $S(x)$, while the series*

$$\sum_{n=1}^{\infty} \int_a^{\beta} u_n(x)\, dx$$

is summable $(C, 1)$ to the sum $K + \int_a^{\beta} S(x)\, dx$; and if the function $\psi(x)$ is absolutely integrable with respect to x over the closed interval $a \leqslant x \leqslant b$, and has limited total fluctuation in the interval $a \leqslant x \leqslant \alpha$, while $\psi(x)$ tends to a unique limit $\psi(a+0)$ as x tends to a, then the series*

$$\sum_{n=1}^{\infty} \int_a^{\beta} \psi(x)\, u_n(x)\, dx$$

is summable $(C, 1)$ with the sum

$$K\psi(a+0) + \int_a^{\beta} \psi(x)\, S(x)\, dx.$$

* When $x = a$, we define $S(a)$ to mean $\lim_{x \to a} S(x)$.

THEOREM L. *The series*

$$\sum_{n=1}^{\infty}\int_{\xi}^{\xi+\delta} f_n(\xi)f_n(x)\,dx \quad \text{and} \quad \sum_{n=1}^{\infty}\int_{\xi-\delta}^{\xi} f_n(\xi)f_n(x)\,dx$$

both converge uniformly to the sum $\frac{1}{2}$ *in the interval* $0<\delta\leqq\xi\leqq\pi-\delta$.

Proof. The conditions of Theorem K are satisfied if we write

$$u_n(x)=f_n(x)f_n(\xi)$$

and choose a, α, β, and b suitably; and since the series $\sum_{n=1}^{\infty} u_n(x)$ is now a null series, $S(x)=0$.

Hence the Sturm-Liouville series associated with the function $\psi(x)$,

viz. $$\sum_{n=1}^{\infty}\int_{a}^{\xi} f_n(x)f_n(\xi)\,\psi(x)\,dx+\sum_{n=1}^{\infty}\int_{\xi}^{b} f_n(x)f_n(\xi)\,\psi(x)\,dx,$$

is summable $(C,1)$ with sum-function

$$\psi(\xi-0)\sum_{n=1}^{\infty}\int_{\xi-\delta}^{\xi} f_n(x)f_n(\xi)\,\psi(x)\,dx+\psi(\xi+0)\sum_{n=1}^{\infty}\int_{\xi}^{\xi+\delta} f_n(x)f_n(\xi)\,\psi(x)\,dx.$$

From Theorem L we find at once that this sum-function is

$$\tfrac{1}{2}\{\psi(\xi-0)+\psi(\xi+0)\}.$$

Corollary. If we utilize the representation of the characteristic functions $\phi_n(z)$ given by Theorem H, the Riemann-Lebesgue lemma, and Hardy's Theorem F, we may deduce that, at any point ξ in the interior of an interval (α,β) such that $a<\alpha<\beta<b$ within which $\psi(x)$ has limited total fluctuation, the Sturm-Liouville series is convergent, and its sum is

$$\tfrac{1}{2}\{\psi(\xi-0)+\psi(\xi+0)\}.$$

SOME PROPERTIES OF ENUMERATION IN THE THEORY OF MODULAR SYSTEMS

By F. S. MACAULAY.

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I. *The general polynomial-ideal.*

The object of this note is to discover the limiting relations which must exist between the terms of the series $D_0, D_1, \ldots, D_l, \ldots$ where D_l is the number of linearly independent homogeneous polynomials of degree l (or of degrees less than or equal to l in the case of non-homogeneous polynomials) belonging to some actual modular system, the system being either perfectly general or else general of its kind. A modular system (or in modern nomenclature a *polynomial-ideal*) is defined to be any aggregate of polynomials in n variables such that the sum of any two, and also the product of any one by a constant* or any of the variables $x_1, x_2, \ldots, x_n$, belongs to the ideal or aggregate.

The converse and more important question is that of finding the actual values of $D_0, D_1, D_2, \ldots$ for a given polynomial-ideal, that is, an ideal defined by the stated conditions which its members individually and collectively have to satisfy. But with few exceptions, some of which will be noted later, no general answer can be found to this converse question. In a classical memoir, Hilbert† has shown that *when l is large enough D_l* becomes a polynomial in l; and Ostrowski‡ has employed Hilbert's process

* The domain of the coefficients is supposed to consist of all elements of a corpus, but restricted in general to rational integral functions of any parameters that may be included. A constant means any element of this domain.

† D. Hilbert, "Über die Theorie der algebraischen Formen", *Math. Annalen*, 36 (1890), 473 *ff.* He shows that the number H_l complementary to D_l is expressed, when l is large enough, by a polynomial in l of degree equal to the number of dimensions of the spread of the ideal. This polynomial $\chi(l)$ will be called the Hilbert function.

‡ A. Ostrowski, "Über ein algebraisches Übertragungsprinzip", *Abh. Math. Seminar d. Hamburgischen Universität*, 1 (1922), Hefte 3 and 4, 281 *ff.*

for expressing in a neat form the generating function $D_0+D_1x+D_2x^2+\ldots$, which, if known, gives the value of D_l for *all*, not merely large, values of l.

Polynomial-ideals are of two kinds: (i) H-ideals, or ideals of homogeneous polynomials, and (ii) non-H-ideals, or ideals of non-homogeneous polynomials. Under (i) we shall specially consider a p.p.-ideal*, that is, an ideal of power products or an ideal of which no polynomial is a member unless each of its terms separately is a member. An H-ideal M is a more general kind of ideal than the non-H-ideal $M(x_n=1)$, owing to the fact that the most important point of the H-ideal, viz. the point (0, 0, ..., 0), is obliterated† by putting $x_n=1$. If, however, M and $M(x_n=1)$ have the same D series $D_0, D_1, D_2, \ldots$ they are said to be *equivalent*. To every non-H-ideal in n variables there is an equivalent H-ideal in $n+1$ variables (but not *vice versa*) got simply by making all the members of the non-H-ideal homogeneous by the insertion of an additional variable x_0.

Again, the members of an ideal are of two kinds: (i) *principal* and (ii) *derived*. In an H-ideal, if all the members (principal and derived) of degree l are multiplied by $x_1, x_2, \ldots, x_n$ we get all the derived members of degree $l+1$ (which can be reduced to a linearly independent set); and any other set of members of degree $l+1$, linearly independent of one another and of the derived members, may be taken as principal members of degree $l+1$. By Hilbert's theorem (*loc. cit.*) the total number of linearly independent principal members is finite, *i.e.* any polynomial-ideal M defined as above has a *finite basis* $(F_1, F_2, \ldots, F_k)$ in terms of which any member F of M is linearly expressible, viz. $F=A_1F_1+A_2F_2+\ldots+A_kF_k$, where $A_1, A_2, \ldots, A_k$ are polynomials.

The Hilbert number H_l, complementary to D_l, is the number of independent linear relations satisfied by the coefficients of the general member of the ideal of degree l. The linear relations themselves are called the *ideal-equations for degree* l of the ideal.

It is not seldom convenient to express results in terms of $H_0, H_1, H_2, \ldots$ rather than $D_0, D_1, D_2, \ldots$. This can always be done, since D_l+H_l is the number of p.p.'s of degree l in the case of an H-ideal and the number of p.p.'s of degrees $\leqslant l$ in the case of a non-H-ideal. The generating function of the series $D_0+H_0, D_1+H_1, \ldots$ for an H-ideal is $(1-x)^{-n}$, viz.

$$D_l+H_l=\frac{(l+1)(l+2)\ldots(l+n-1)}{1, 2, \ldots, (n-1)}.$$

* The contraction p.p. for power product will be used throughout the paper.

† On this account the theory of H-ideals is completely mutilated if we pay regard only to the ratios of the variables, thus contrasting with the case of homogeneous coordinates in Geometry.

We shall denote* this number D_l+H_l by $(l+1)_{n-1}$, and regard it as a function of l with n a constant. It is to be noticed that $(l)_n$ is zero when $l=0$, and is 1 when $l=1$; also that $(l)_n$ can be expressed as a binomial coefficient $\binom{l+n-1}{n}$, though this notation is clearly inconvenient since it does not dissociate l from n. We have $D_l+H_l=(l+1)_n$ for a non-H-ideal.

The p.p.'s of degree l can be written in a definite order (which we shall call their *ascending* order) according to the rule that $x_1^{p_1}x_2^{p_2}\dots x_n^{p_n}$ comes before $x_1^{q_1}x_2^{q_2}\dots x_n^{q_n}$ if the first of the indices $p_1, p_2, \dots, p_n$ which differs from the corresponding index in $q_1, q_2, \dots, q_n$ is greater than it. Taking $(x_1, x_2, \dots, x_n)^l$ to stand for all p.p.'s of degree l expanded in ascending order, we have

$$(x_1, \dots, x_n)^l = x_1^l, x_1^{l-1}(x_2, \dots, x_n), \dots, x_1^p(x_2, \dots, x_n)^{l-p}, \dots, (x_2, \dots, x_n)^l,$$

of which the part as far as $x_1^p(x_2, \dots, x_n)^{l-p}$ comprises the p.p.'s $x_1^p(x_1, x_2, \dots, x_n)^{l-p}$. The p.p. $x_1^{p_1}x_2^{p_2}\dots x_r^{p_r}x_n^{p_n}$ $(p_r>0)$ is the last p.p. in $x_1^{p_1}\dots x_r^{p_r}(x_{r+1}, \dots, x_n)^{p_n}$ and is followed by the first p.p. in $x_1^{p_1}\dots x_r^{p_r-1}(x_{r+1}, \dots, x_n)^{p_n+1}$, viz. $x_1^{p_1}\dots x_r^{p_r-1}x_{r+1}^{p_n+1}$. Again $(x_0, x_1, \dots, x_n)^l$ or $(1, x_1, \dots, x_n)^l$ is equal to

$$1, \quad (x_1, \dots, x_n), \quad (x_1, \dots, x_n)^2, \quad \dots, \quad (x_1, x_2, \dots, x_n)^l.$$

Corresponding to any given H-ideal M, after subjecting the variables to a general homogeneous linear substitution, we can deduce two corresponding p.p.-ideals P, P', each of which has the same D series $D_0, D_1, D_2, \dots$ as M. The first, P, is the ideal whose members of any degree l consist of the first D_l p.p.'s in $(x_1, x_2, \dots, x_n)^l$. It will be shown immediately that all the p.p.'s of degree $l+1$ which can be *derived* from the first N p.p.'s in $(x_1, x_2, \dots, x_n)^l$ consist of the first $Q(N)$ p.p.'s in $(x_1, x_2, \dots, x_n)^{l+1}$, where $Q(N)$ is a certain function of N. Hence, in order to prove that the aggregate of p.p.'s P as described above constitutes an ideal, it must be shown that the relation $D_{l+1} \geqslant Q(D_l)$ holds for any H-ideal M. The proof is given in § II of the paper, and its truth will be assumed here.

The second p.p.-ideal, P', is obtained thus: write the D_l members of the H-ideal M of degree l so that their terms are in ascending order, and modify them linearly by means of one another so that no two members begin with the same term. The p.p.'s with which they begin are then the

* We prefer $(l+1)_{n-1}$ to $(l)_{n-1}$ because, in the latter case, $(l)_n$ could not be zero unless l took a negative value. Later on we call l in the notation $(l)_{n-1}$ a digit, and we find it convenient to exclude the digit 0.

D_l p.p.'s of P' of degree l. These D_l p.p.'s evidently satisfy the test for an ideal, viz. that the D_{l+1} members include all those that can be *derived* from the D_l members. *This proves the existence of a p.p.-ideal having the same D series* $D_0, D_1, D_2, \ldots$ *as that of any given H-ideal* M. It was only in order to demonstrate this that we have introduced P' here. The relation $D_{l+1} \geqslant Q(D_l)$, which we wish to prove for any H-ideal, has now only to be proved for any p.p.-ideal. In what follows we shall only consider P, until we come to the proof of $D_{l+1} \geqslant Q(D_l)$ in § II.

The general H-ideal. Let M be any H-ideal, and $D_0, D_1, D_2, \ldots$ its D series. In place of M we take the corresponding p.p.-ideal P having the same D series. Let $x_1^{p_1} x_2^{p_2} \ldots x_r^{p_r} x_n^{p_n}$ be the last of the D_l p.p.'s of P of degree l, where $r \leqslant n-1$ and $p_r \geqslant 1$, and any (or all) of $p_1, p_2, \ldots, p_{r-1}, p_n$ may be zero. Then the D_l p.p.'s of P are

$$x_1^{p_1+1}(x_1, \ldots, x_n)^{l-p_1-1},$$

$$x_1^{p_1} x_2^{p_2+1}(x_2, \ldots, x_n)^{l-p_1-p_2-1}, \ldots, x_1^{p_1} \ldots x_r^{p_r}(x_r, \ldots, x_n)^{l-p_1-\ldots-p_r}$$

where the last set differs from the others in having p_r in place of p_1+1, p_2+1, ..., in the 1st, 2nd, ..., sets. Also the derived p.p.'s of degree $l+1$, obtained by multiplying the D_l p.p.'s by $(x_1, \ldots, x_n)$, are

$$x_1^{p_1+1}(x_1, \ldots, x_n)^{l-p_1},$$

$$x_1^{p_1} x_2^{p_2+1}(x_2, \ldots, x_n)^{l-p_1-p_2}, \ldots, x_1^{p_1} \ldots x_r^{p_r}(x_r, \ldots, x_n)^{l-p_1\ldots-p_r+1},$$

which consist of all the p.p.'s of $(x_1, x_2, \ldots, x_n)^{l+1}$ from the first x_1^{l+1} up to $x_1^{p_1} \ldots x_r^{p_r} x_n^{p_n+1}$. Hence, assuming that D_l has neither of its extreme values 0 and $(l+1)_{n-1}$, and putting

$$l-p_1 = l_1, \quad l-p_1-p_2 = l_2, \ldots, l-p_1-\ldots-p_r = l_r-1,$$

we have the result:

If $\quad D_l = (l_1)_{n-1}+(l_2)_{n-2}+\ldots+(l_r)_{n-r}, \quad l \geqslant l_1 \ldots \geqslant l_r \geqslant 1, \quad n > r > 0,$

then $\quad (l+2)_{n-1} \geqslant D_{l+1} \geqslant (l_1+1)_{n-1}+(l_2+1)_{n-2}+\ldots+(l_r+1)_{n-r}.$

The excluded cases, which present no difficulty, are: (i) if $D_l = 0$, then $(l+2)_{n-1} \geqslant D_{l+1} \geqslant 0$; and (ii) if $D_l = (l+1)_{n-1}$, then $D_{l+1} = (l+2)_{n-1}$. It can be easily proved that the above form for D_l is unique, *i.e.* if D_l is

known, and $0 < D_l < (l+1)_{n-1}$, the positive integers (or digits) $l_1, l_2, \ldots, l_r$ have unique values.

We denote the number $(a_1)_{n-1}+(a_2)_{n-2}+\ldots+(a_r)_{n-r}$, $a_1 \geqslant a_2 \ldots \geqslant a_r \geqslant 1$, $n > r > 0$, by $(a_1, a_2, \ldots, a_r)_{n-1}$. Also, if $(a_1, a_2, \ldots, a_r)_{n-1} = N$, we denote

$$(a_1+1, a_2+1, \ldots. a_r+1)_{n-1} \text{ by } Q(N), \text{ or } Q_{n-1}(N),$$

and $$(a_1-1, a_2-1, \ldots, a_r-1)_{n-1} \text{ by } Q^{-1}(N), \text{ or } Q^{-1}_{n-1}(N),$$

in which it is to be understood that if $a_r-1, a_{r-1}-1, \ldots$ are zeros *they are not to be written.* Thus $Q(N)$ and $Q^{-1}(N)$ are functions of N; and, if $Q^{-1}(N) = N_1$, N is only equal to $Q(N_1)$ if the digits $a_1, a_2, \ldots, a_r$ of N contain no units; in fact, when $Q^{-1}(N) = N_1$, then N is equal to $Q(N_1)$ increased by the number of units in the digits of N.

What we have proved is that (without exception) there is an H-ideal having $D_0, D_1, D_2, \ldots$ as its D series, provided that

$$(l+2)_{n-1} \geqslant D_{l+1} \geqslant Q(D_l)$$

is true for all values of l, defining $Q(0)$ to be equal to 0; and that there is no such H-ideal unless these relations are satisfied. In the case of the p.p.-ideal P the D_{l+1} p.p.'s consist of the first $Q(D_l)$ p.p.'s in $(x_1, x_2, \ldots, x_n)^{l+1}$, which are *derived* members, followed by the next $D_{l+1}-Q(D_l)$ p.p.'s in the same, which are *principal* members. *P is determinate if we know its D series.* It is to be noticed that the limits of D_{l+1} depend only on the values of l and D_l, and not on the values of $D_0, D_1, \ldots, D_{l-1}$. This might have been foreseen independently.

The general non-H-ideal. The D_l members of a non-H-ideal in n variables consist of D_{l-1} members of degree $\leqslant l-1$, and D_l-D_{l-1} (or ΔD_l) members of actual degree l which are linearly independent, not only in respect to their terms as a whole, but in respect to their terms of highest degree l. If we reject all terms of degree $< l$ in the ΔD_l members ($l = 0, 1, 2, \ldots$) we obtain the members of an H-ideal whose D series is $\Delta D_0, \Delta D_1, \Delta D_2, \ldots$. Conversely, if we take any H-ideal, and merely change the origin, it becomes a non-H-ideal such that the D_l of the H-ideal is equal to the ΔD_l of the non-H-ideal. Hence the conditions for a non-H-ideal are the same as those for an H-ideal if D_l is replaced by ΔD_l; *i.e.* the necessary and sufficient conditions that $D_0, D_1, D_2, \ldots$, may be the D series of some actual non-H-ideal are that the relations

$$(l+2)_{n-1} \geqslant \Delta D_{l+1} \geqslant Q(\Delta D_l)$$

should be true for all values of l. The higher limit of D_{l+1} depends only on the values of l and D_l, and the lower limit, viz. $D_l+Q(D_l-D_{l-1})$, on the values of D_l, D_{l-1}.

The Hilbert function. From the above we can find an expression for the Hilbert function $\chi(l)$ of the general H-ideal, together with the restrictions to which its coefficients must be subject, which neither Hilbert nor Ostrowski finds. $\chi(l)$ is the polynomial in l to which H_l becomes equal when l is large enough. Hilbert expresses it in the form

$$\chi(l) = a\binom{l}{n-r-1} + b\binom{l}{n-r-2} + \ldots + k,$$

in terms of binomial coefficients, while Ostrowski adopts the preferable form

$$\chi(l) = a(l+1)_{n-r-1} + b(l+1)_{n-r-2} + \ldots + k.$$

In both forms a (the *order* of the ideal) has the same positive integral value, and b, c, ..., k are integers, positive or negative; r is the *rank* of the ideal, and $n-r-1$ the *dimensions*.*

Since, by Hilbert's theorem, any polynomial-ideal has a finite basis, *i.e.* only a finite number of principal members, it follows that the p.p.-ideal P corresponding to any given H-ideal M has a *last* principal member. Let this be $x_r^{a_r} x_{r+1}^{a_{r+1}} \ldots x_s^{a_s} x_n^{a_n}$, where r, a_r, a_s are all greater than zero, and s is generally equal to $n-1$, but may be less (or the last principal member might possibly be $x_n^{a_n}$). Then the last p.p. of P of degree

$$l \geqslant a_r + \ldots + a_s + a_n \quad \text{is} \quad x_r^{a_r} \ldots x_r^{a_s} x_n^{l-a_r-\ldots-a_s},$$

and we have (from above, where D_l was first calculated),

$$D_l = (l, l, \ldots, \overline{r-1} \text{ times}, l-a_r, l-a_r-a_{r+1}, \ldots, l-a_r-\ldots-a_s+1)_{n-1}$$

$$= (l, l, \ldots, \overline{r-1} \text{ times}, l-a, l-b, \ldots, l-k)_{n-1}, \quad 1 \leqslant a \leqslant b \ldots \leqslant k.$$

Also $H_l + D_l = (l+1)_{n-1} = (l, l, \ldots, r-1 \text{ times})_{n-1} + (l+1)_{n-r}$.

By subtraction, when l is large enough (viz. $l-k > a_n$),

$$\chi(l) = (l+1)_{n-r} - (l-a, l-b, \ldots, l-k)_{n-r}, \quad 1 \leqslant a \leqslant b \ldots \leqslant k,\dagger$$

* It is usual to say that an H-ideal of rank r has $n-r-1$, not $n-r$, dimensions, thus excluding the origin $(0, 0, \ldots, 0)$, or treating it as a vertex of projection outside the $(n-1)$-space corresponding to the ratios of the variables.

† With the exception that $\chi(l) = (l+1)_{n-r} - (l-a+1)_{n-r}$ when the last principal member of P is $x_r^{a_r} x_n^{a_n}$. a (or a_r) is the order as before.

which is a polynomial in l of degree $n-r-1$. This seems to be the simplest form in which to leave $\chi(l)$, and shows its restrictions. It can be brought easily to the Ostrowski form, but not easily to the Hilbert form.

The *rank* of a polynomial-ideal is defined as the least number r such that all members of the ideal can be made to vanish by making r of the variables functions of the remaining $n-r$ (regarded as arbitrary parameters). The p.p.-ideal P above has $x_1^\lambda, x_2^\lambda, \ldots, x_r^\lambda$ $(\lambda = a_r + \ldots + a_s + a_n)$ as members, and all its members vanish when $x_1, x_2, \ldots, x_r$ vanish. Hence the rank of P is r, and its spread is $(x_1, x_2, \ldots, x_r)$. There is no ambiguity about the rank of an H-ideal as there is about the dimensions.

II. *Proof of the main theorem.*

Note.—This proof of the theorem which has been assumed earlier is given only to place it on record. It is too long and complicated to provide any but the most tedious reading.

It is required to prove that the relation $D_{l+1} \geqslant Q_{n-1}(D_l)$ holds for any H-ideal in n variables; and, as we have already seen (p. 534), it is sufficient to prove it for any p.p.-ideal. Any given integer $N > 0$ can be expressed uniquely in the form

$$N = (a_1)_{n-1} + (a_2)_{n-2} + \ldots + (a_r)_{n-r} = (a_1, a_2, \ldots, a_r)_{n-1}$$

in the *scale* of $n-1$ with not more than $n-1$ *digits* $a_1 \geqslant a_2 \ldots \geqslant a_r \geqslant 1$.

$$Q_{n-1}(N) = (a_1+1, \ldots, a_r+1)_{n-1}, \quad \text{and} \quad Q_{n-1}^{-1}(N) = (a_1-1, \ldots, a_r-1)_{n-1}$$

if $a_r > 1$, while $Q_{n-1}^{-1}(N) = (a_1-1, \ldots, a_p-1)_{n-1}$ if $a_p > 1$ and $a_{p+1}, \ldots, a_r$ are units (the notation excludes a zero digit). $Q(N)$ and $Q^{-1}(N)$ are both defined as 0 if $N = 0$; also $Q^{-1}(N)$ is defined as 0 if it has no digits, as is the case with $Q_{n-1}^{-1}(N)$ when $N < n$. $Q(N)$, $Q^{-1}(N)$ will be understood to be $Q_{n-1}(N)$, $Q_{n-1}^{-1}(N)$ unless a scale different from $n-1$ is indicated by the context.

If $N = (a_1, a_2, \ldots, a_r)_{n-1}$, we have

$$N + Q^{-1}(N) + Q^{-2}(N) + \ldots = (a_1, a_2, \ldots, a_r)_n,$$

and $N - Q^{-1}(N) = (a_1, a_2, \ldots, a_r)_{n-2}$, though this is not in its proper form if $r = n-1$.

Before proceeding to the proof proper we must prove some properties of $Q(N)$ and $Q^{-1}(N)$, especially such as relate to the increase of $Q(N)$ as N

increases. *The increase in $Q(N)$ is never less than the increase in N, and the increase $Q(N+1)-Q(N)$ in $Q(N)$ due to an increase of 1 in N is $n-r(\geqslant 1)$, where r is the number of digits in N.* There are two cases: (i) $r < n-1$, in which case

$$N+1 = (a_1, a_2, \ldots, a_r, 1)_{n-1} \text{ and } Q(N+1)-Q(N) = (2)_{n-r-1} = n-r;$$

and (ii) $r = n-1$ and the last p digits of N equal, in which case

$$N = (a_1, \ldots, a_{n-p-1}, a_{n-p}^{p})_{n-1}. \quad N+1 = (a_1, \ldots, a_{n-p-1}, a_{n-p}+1)_{n-1},$$

and $$Q(N+1)-Q(N) = (a_{n-p}+2)_p - \{(a_{n-p}+1)^p\}_p = 1.$$

LEMMA I.—*The increase $Q(N+I)-Q(N)$ in $Q(N)$ for a given increase I in N has the following properties*: (i) *if $N = 0$ it is greater than for any value of $N > 0$*; (ii) *if $N+I = (a)_{n-1}$ it is as small as for any less value of N*; (iii) *if $N = (a)_{n-1}$ it is as great as for any greater value of N.*

All this is true in scale 1; for, when $N = 0$, $Q_1(N+I)-Q_1(N) = I+1$; and, when $N > 0$, $Q_1(N+I)-Q_1(N) = I$. We assume it true for all scales up to $n-2$ and prove it true for scale $n-1$.

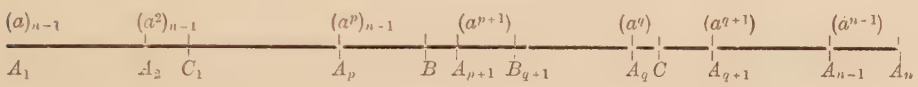
Suppose first that N and $N+I$ are both in the limited range $(a)_{n-1}$ to $(a+1)_{n-1}$. Divide this range into $n-1$ smaller ones of which the p^{th} is $(a^p)_{n-1}$ to $(a^{p+1})_{n-1}$, and the last one $(a^{n-1})_{n-1}$ to $(a^n)_{n-1}$ is a single step, $(a^n)_{n-1}$ being an improper form of $(a+1)_{n-1}$. $Q(N)$ increases along the range $N = (a^p)_{n-1}$ to $(a^{p+1})_{n-1}$ of extent $(a)_{n-p-1}$ in exactly the same way as along the range $N = (a^{p-1}, a-1)_{n-1}$ to $(a^p)_{n-1}$, that is, a range of the same extent at the end of the previous range $(a^{p-1})_{n-1}$ to $(a^p)_{n-1}$, and therefore also of any preceding range. The reason is that any number N in the range $(a^p)_{n-1}$ to $(a^{p+1})_{n-1}$ short of the end number is $(a^p, b, \ldots)_{n-1}$, where $b \leqslant a-1$, and to this corresponds the number $(a^{p-1}, a-1, b, \ldots)_{n-1}$ in the previous range with the same number of digits, and at the same distance from the end of the range. We express this fact by the symbol $\sim$, meaning "is equivalent as regards increase in $Q(N)$". Thus

$$\begin{aligned}(a^p)_{n-1} \text{ to } (a^{p+1})_{n-1} &\sim (a^{p-1}, a-1)_{n-1} \text{ to } (a^p)_{n-1} \\ &\sim \{a^{p-2}, (a-1)^2\}_{n-1} \text{ to } (a^{p-1})_{n-1} \sim \text{etc.} \\ &\sim \{a, (a-1)^{p-1}\}_{n-1} \text{ to } (a^2)_{n-1}.\end{aligned}$$

Again $(a^p)_{n-1}$ to $(a^{p+1})_{n-1}$ in scale $n-1 \sim 0$ to $(a)_{n-p-1}$ in scale $n-p-1$.

Hence, for an interval $I < (a)_{n-p-1}$ in the range $(a^p)_{n-1}$ to $(a^{p+1})_{n-1}$, $Q(N+I)-Q(N)$ is greater when $N = (a^p)_{n-1}$ than anywhere else, and as small when $N+I = (a^{p+1})_{n-1}$ as anywhere else; since this is true of the range 0 to $(a)_{n-p-1}$ in scale $n-p-1$.

Let the interval I (or BC in the figure) be placed anywhere on the range $(a)_{n-1}$ to $(a+1)_{n-1}$, B (or N) being between A_p, A_{p+1} {or $(a^p)_{n-1}$, $(a^{p+1})_{n-1}$}, and C (or $N+I$) between A_q, A_{q+1}. When the interval I, or BC, moves up or down the range, let B be at B_{q+1}, ..., B_n when C is at A_{q+1}, ..., A_n; and let C be at C_p, ..., C_1 when B is at A_p, ..., A_1. We have to compare



the values of $Q(N+I)-Q(N)$ for the several I intervals

$$A_1C_1,\ A_2C_2,\ \ldots,\ A_pC_p,\ BC,\ B_{q+1}A_{q+1},\ \ldots,\ B_nA_n,$$

which values may be denoted by

$$R(A_1C_1)\ \ldots,\ R(BC),\ \ldots,\ R(B_nA_n).$$

Now $$R(BC)-R(B_{q+1}A_{q+1}) = R(BB_{q+1})-R(CA_{q+1}).$$

But, if B_{q+1} is in A_pA_{p+1}, $R(BB_{q+1})$ is as great as the R of an interval equal to BB_{q+1} ending at A_{p+1}, or at A_{q+1}, *i.e.* $R(BB_{q+1}) \geqslant R(CA_{q+1})$. If, however, B_{q+1} is beyond A_{p+1}, then

$$R(BB_{q+1}) = R(BA_{p+1})+R(A_{p+1}B_{q+1}),$$

and $R(BA_{p+1})$ is equal to the R of an interval equal to BA_{p+1} at the end of CA_{q+1}, and $R(A_{p+1}B_{q+1})$ is greater than the R of an interval equal to $A_{p+1}B_{q+1}$ at the beginning of CA_{q+1}, *i.e.* $R(BB_{q+1}) > R(CA_{q+1})$. In any case $R(BC) \geqslant R(B_{q+1}A_{q+1})$.

Similarly we can prove that

$$R(BC) \geqslant R(B_{q+1}A_{q+1}) \geqslant \ldots \geqslant R(B_nA_n);$$

and by the same reasoning we have

$$R(A_1C_1) > R(A_2C_2) > \ldots > R(A_pC_p) > R(BC).$$

We can proceed now to the unlimited range 0 to $(a)_{n-1}$, where a is as high as we please. Since

$$(a)_n \text{ to } (a,\ a)_n \text{ in scale } n \sim 0 \text{ to } (a)_{n-1} \text{ in scale } n-1,$$

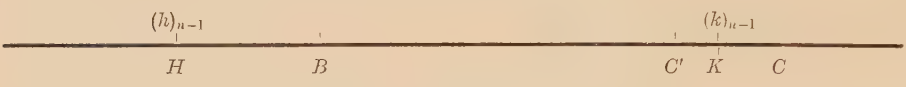
and, for the range in scale n, $Q(N+I)-Q(N)$ is greater when $N=(a)_n$ than anywhere else, and as small when $N+I=(a,\ a)_n$ as anywhere else, so in the range 0 to $(a)_{n-1}$ in scale $n-1$, $Q(N+I)-Q(N)$ is greater when $N=0$ than for any value of $N>0$, and is as small when $N+I=(a)_{n-1}$ as for any less value of N. This proves (i) and (ii).

To prove (iii) we have to repeat some of the reasoning. Let B (or N) be between $(h)_{n-1}$, $(h+1)_{n-1}$, and C (or $N+I$) between $(k)_{n-1}$, $(k+1)_{n-1}$. Take C' so that $HC'=BC=I$. We have to prove first that

$$R(HC')\geqslant R(BC),$$

and this has been proved already when C is in the range $(h)_{n-1}$ to $(h+1)_{n-1}$. So we assume C beyond $(h+1)_{n-1}$, *i.e.* $k\geqslant h+1$. Then

$$R(HC')-R(BC)=R(HB)-R(C'C).$$



If C' is in HK, $R(C'C)=R(C'K)+R(KC)$, and $R(KC)$ is equal to the R of an interval equal to KC beginning at H, since

$$(h)_{n-1}\text{ to }(h+1)_{n-1}\sim(k)_{n-1}\text{ to }(k,\ h+1)_{n-1},$$

and $R(C'K)\leqslant$ the R of the remaining part of the interval HB, by (ii). Hence $R(C'C)\leqslant R(HB)$, *i.e.* $R(HC')\geqslant R(BC)$. If C' is beyond K, $R(HB)$ is equal to the R of an interval equal to HB or $C'C$ beginning at K (by the equivalence above), which is greater than $R(C'C)$; *i.e.* $R(HC')>R(BC)$. In any case $R(HC')\geqslant R(BC)$. Similarly, by taking N at $(h-1)_{n-1}$, $(h-2)_{n-1}$, ..., $(a)_{n-1}$ successively, the value of $Q(N+I)-Q(N)$ is as great when N is at $(a)_{n-1}$ as when N is at B. This proves (iii).

There must clearly be many properties of a similar kind. By (i) we have $Q(I)-Q(0)>Q(N+I)-Q(N)$, or $Q(N_1+N_2)<Q(N_1)+Q(N_2)$ when neither N_1 nor N_2 is zero.

LEMMA II.—(i) *If* $N_1<(a+1)_{n-1}$, $N_2\leqslant(a+1)_{n-1}$,

and $$N_1+N_2=(a+1)_{n-1}+R,\ \textit{where}\ R\geqslant 0,$$

then $$Q(N_1)+Q(N_2)\geqslant Q(a+1)_{n-1}+Q(R).$$

(ii) *If* $N_1 < (a+1)_{n-1}$, $Q^{-1}(N_2) = (a, a_1, \ldots, a_p)_{n-1}$, $N_1 > (a_1, \ldots, a_p)_{n-1}$,

and $$N_1+N_2 = (a+1)_{n-1}+R, \text{ where } R \geqslant 0,$$

then $$Q(N_1)+Q(N_2) \geqslant Q(a+1)_{n-1}+Q(R).$$

The theorem is divided into two parts according as $N_2 \leqslant (a+1)_{n-1}$ or $(a+1)_{n-1} \leqslant N_2 \leqslant (a+2)_{n-1}$, because we shall need to use them in this way; but both might be combined more clearly under one heading, viz.

$$N_1 = (b_1, b_2, \ldots, b_p)_{n-1} < (a+1)_{n-1}, \qquad Q^{-1}(N_2) \leqslant (a, b_1, \ldots, b_p)_{n-1}.$$

The other condition imposed, $N_1+N_2 \geqslant (a+1)_{n-1}$, is not essential. If $N_1+N_2 < (a+1)_{n-1}$ we should take $N_1+N_2 = R$, and obtain

$$Q(N_1)+Q(N_2) \geqslant Q(N_1+N_2),$$

which has just been proved above.

(i) Since $N_2 \leqslant (a+1)_{n-1}$, $(a+1)_{n-1}-N_2 = N_1-R$, where both sides are positive; also $N_1 < (a+1)_{n-1}$. Hence, by I (ii),

$$Q(a+1)_{n-1}-Q(N_2) \leqslant Q(N_1)-Q(R),$$

or $$Q(N_1)+Q(N_2) \geqslant Q(a+1)_{n-1}+Q(R).$$

(ii) The property is true for scale 1 (since $N_2 = a+1$ and $N_1 = R$) and can be proved by induction for scale $n-1$. Imagine N_2, or $(a+1, a_1+1, \ldots, a_p+1, 1^q)_{n-1}$, to remain fixed while N_1, R increase together. We have to prove that

$$Q(R)-Q(N_1) \leqslant Q(N_2)-Q(a+1)_{n-1}.$$

When $N_1 = (a_1+1)_{n-1}$, we have

$$R = (a_1+1)_{n-1}+(a_1+1, \ldots, a_p+1, 1^q)_{n-2} = (a_1+1, a_1+1, \ldots, a_p+1, 1^q)_{n-1},$$

$$Q(R)-Q(N_1) = Q_{n-2}(a_1+1, \ldots, a_p+1, 1^q)_{n-2} = Q(N_2)-Q(a+1)_{n-1}.$$

And as $Q(R)-Q(N_1)$ is as great when $N_1 = (a_1+1)_{n-1}$ as for any greater value of N_1 {I (iii)},

$$Q(R)-Q(N_1) \leqslant Q(N_2)-Q(a+1)_{n-1} \text{ when } N_1 \geqslant (a_1+1)_{n-1}.$$

It remains to prove the same when $N_1 < (a_1+1)_{n-1}$. Put

$$N_1 = (a_1)_{n-1}+N_1',$$

$$N_2 = (a+1)_{n-1}+N_2',$$

$$N_1'+N_2' = (a_1+1)_{n-2}+R'.$$

Then $N_1' > (a_2, \ldots, a_p)_{n-2}$, $Q_{n-2}^{-1}(N_2') = (a_1, a_2, \ldots, a_p)_{n-2}$,

$$N_1' < (a_1+1)_{n-2}, \quad \text{and} \quad R' \geqslant 0;$$

so that the conditions are satisfied for applying the theorem to N_1', N_2', R' in scale $n-2$; *i.e.* we have

$$Q_{n-2}(R')-Q_{n-2}(N_1') \leqslant Q_{n-2}(N_2')-Q_{n-2}(a_1+1)_{n-2}.$$

But, by adding the three equations above, we have

$$N_1+N_2-(a+1)_{n-1} = (a_1)_{n-1}+(a_1+1)_{n-2}+R',$$

or $$R = (a_1+1)_{n-1}+R', \quad \text{where} \quad R' < N_2' < (a_1+2)_{n-2};$$

hence $$Q(R) = (a_1+2)_{n-1}+Q_{n-2}(R');$$

similarly for $Q(N_1)$, $Q(N_2)$. Hence

$$\begin{aligned} Q(R)-Q(N_1) &= (a_1+2)_{n-1}-(a_1+1)_{n-1}+Q_{n-2}(R')-Q_{n-2}(N_1') \\ &\leqslant (a_1+2)_{n-2}+Q_{n-2}(N_2')-Q_{n-2}(a_1+1)_{n-2} \\ &\leqslant Q_{n-2}(N_2') \leqslant Q(N_2)-Q(a+1)_{n-1}. \end{aligned}$$ Q.E.D.

LEMMA III.—*If there are s numbers $N_1, N_2, \ldots, N_s$ such that*

$$N_1 < (a+1)_{n-1}, \quad Q^{-1}(N_2) \leqslant N_1, \ \ldots, \ Q^{-1}(N_s) \leqslant N_{s-1},$$

and $$N_1+N_2+\ldots+N_s = (a+1)_{n-1}+\ldots+(a+t_s-1)_{n-1}+R_s,$$

where $$0 \leqslant R_s < (a+t_s)_{n-1},$$

then

$$Q(N_1)+Q(N_2)+\ldots+Q(N_s) \geqslant Q(a+1)_{n-1}+\ldots+Q(a+t_s-1)_{n-1}+Q(R_s).$$

This has been proved for two numbers in Lemma II with less restrictive conditions as regards N_1, N_2. We assume the property for p numbers

and prove it for $p+1$. Put

$$N_1+N_2+\ldots+N_p=(a+1)_{n-1}+\ldots+(a+t_p-1)_{n-1}+R_p,$$

where $$0 \leqslant R_p < (a+t_p)_{n-1}.$$

Now $$N_1 < (a+1)_{n-1}, \quad Q^{-1}(N_2) \leqslant N_1 < (a+1)_{n-1},$$

therefore $$N_2 < (a+2)_{n-1}, \quad N_3 < (a+3)_{n-1}, \text{ etc.}$$

Hence $p > t_p-1$, *i.e.* $p \geqslant t_p$. Also $N_p < (a+t_p)_{n-1}$; for if

$$N_p \geqslant (a+t_p)_{n-1}, \text{ then } N_p > R_p, \text{ and}$$

$$N_{p-1} \geqslant Q^{-1}(N_p) \geqslant (a+t_p-1)_{n-1}, \quad N_{p-2} \geqslant (a+t_p-2)_{n-1}, \text{ etc.},$$

which is not consistent with

$$N_p+N_{p-1}+\ldots = R_p+(a+t_p-1)_{n-1}+\ldots, \quad p \geqslant t_p.$$

Now $Q^{-1}(N_{p+1}) \leqslant N_p < (a+t_p)_{n-1}$, therefore $N_{p+1} < (a+t_p+1)_{n-1}$.

There is no difficulty in extending the theorem so as to include the number N_{p+1} if $N_{p+1} \leqslant (a+t_p)_{n-1}$ by applying either I (i) or else II (i). So take $N_{p+1} > (a+t_p)_{n-1}$. In this case

$$R_p+N_{p+1}=(a+t_p)_{n-1}+R_{p+1}, \quad (a+t_p+1)_{n-1} > N_{p+1} > (a+t_p)_{n-1}.$$

Put $$Q^{-1}(N_{p+1})=(a+t_p-1, a_1, a_2, \ldots)_{n-1}.$$

Then $$N_1+N_2+\ldots+N_p \geqslant Q^{-p}(N_{p+1})+Q^{-p+1}(N_{p+1})+\ldots+Q^{-1}(N_{p+1}),$$

i.e. $$(a+t_p-1)_n-(a)_n+R_p \geqslant (a+t_p-1, a_1, a_2, \ldots)_n - (a+t_p-p-1, a_1-p, \ldots)_n,$$

i.e. $$R_p > (a_1, a_2, \ldots)_{n-1},$$

since $$(a)_n > (a+t_p-p-1, a_1-p, \ldots)_n, \quad (t_p \leqslant p).$$

Thus all the conditions in II (ii) as regards N_1, N_2, $a+1$, R are satisfied for R_p, N_{p+1}, $a+t_p$, R_{p+1}, and we have

$$Q(R_p)+Q(N_{p+1}) \geqslant Q(a+t_p)_{n-1}+Q(R_{p+1}).$$

By adding this to the relation

$$Q(N_1)+Q(N_2)+\ldots+Q(N_p) \geqslant Q(a+1)_{n-1}+\ldots+Q(a+t_p-1)_{n-1}+Q(R_p),$$

Q.E.D.

the property is extended from p to $p+1$ numbers.

We can extend the limit of N_{p+1} to $(a+t_p)_{n-1}$ if this should be higher than the limit given by $Q^{-1}(N_{p+1}) \leqslant N_p$.

We have now to prove that whatever N p.p.'s of degree l in n variables are taken the derived p.p.'s of degree $l+1$ cannot be less in number than $Q_{n-1}(N)$. If $n=2$ the N p.p.'s would have to be chosen successively in order that the number of derived p.p.'s might be as small as possible, and the number would then be $N+1$, which is $Q_1(N)$. We assume the theorem for n variables $x_1, x_2, \ldots, x_n$ and proceed to prove it true for $n+1$ variables $x_0, x_1, x_2, \ldots, x_n$. Let the N p.p.'s consist of N_a p.p.'s $x_0^{l-a}S_a$, N_{a+1} p.p.'s $x_0^{l-a-1}S_{a+1}$, and so on, up to N_b p.p.'s $x_0^{l-b}S_b$, where S_p denotes a collection of N_p p.p.'s of degree p in the n variables $x_1, x_2, \ldots, x_n$ and $N = N_a+N_{a+1}+\ldots+N_b = \Sigma N_p$.

The derived p.p.'s consist of

$$x_0^{l-a+1}S_a,\ x_0^{l-a}S_{a+1},\ \ldots,\ x_0^{l-b+1}S_b,\ \text{and}\ x_0^{l-a}OS_a,\ \ldots,\ x_0^{l-b}OS_b,$$

where OS_p denotes the collection of p.p.'s derived from S_p by multiplying S_p by O or $(x_1, x_2, \ldots, x_n)$. Arranged according to powers of x_0 they are

$$\begin{array}{cccccc} S_a & S_{a+1} & S_{a+2} & \ldots & S_b & \\ & OS_a & OS_{a+1} & \ldots & OS_{b-1} & OS_b. \end{array}$$

All these p.p.'s are to be counted, except that those which are repeated in any pair S_{p+1}, OS_p are to be counted only once. The first step in reducing the total number of derived p.p.'s to a minimum is therefore to choose the p.p.'s S_p so that the number of p.p.'s in OS_p may be a minimum, and at the same time may coincide with the p.p.'s in S_{p+1} as far as possible. Both these conditions are fulfilled by choosing the p.p.'s S_p $(p=a, \ldots, b)$ to be the first N_p p.p.'s in $(x_1, x_2, \ldots, x_n)^p$, for then the number of p.p.'s in OS_p is a minimum (assuming the theorem for n variables) and equal to $Q(N_p)$, and the greater of OS_p, S_{p+1} includes the less. Hence, so long as the numbers $N_a, \ldots, N_b$ are fixed the least possible total number of derived p.p.'s is the sum of the higher numbers in the $b-a+2$ pairs

$$\begin{array}{cccccc} N_a & N_{a+1} & N_{a+2} & \ldots & N_b & 0 \\ 0 & Q(N_a) & Q(N_{a+1}) & \ldots & Q(N_{b-1}) & Q(N_b). \end{array}$$

We have now to consider how we may change the numbers $N_a, \ldots, N_b$

(keeping ΣN_p or N unchanged) so that the *total derived number* (or sum of higher numbers in the $b-a+2$ pairs) may not increase at any time; and prove that it can be brought by such a process to $Q_n(N)$. This amounts to proving that the original N p.p.'s have at least $Q_n(N)$ derived p.p.'s of the next higher degree.

If $N_{b-1} < Q^{-1}(N_b)$, increase N_{b-1} up to the old $Q^{-1}(N_b)$, and diminish N_b by a like amount to give the new N_b. This changes the three end pairs; the higher of the first pair N_{b-1}, $Q(N_{b-2})$ is increased *at most* by the increase in N_{b-1}, for $Q(N_{b-2})$ has not been changed; the higher of the next pair has not been increased, for N_b has been decreased and the new $Q(N_{b-1})$ is not greater than the old N_b; and the higher of the last pair has been decreased by *at least* the decrease in N_b. Hence this change does not increase the total derived number. Going downwards from N_{b-1}, let N_p be the next number such that $N_p < Q^{-1}(N_{p+1})$. Increase N_p up to $Q^{-1}(N_{p+1})$ and diminish N_b by a like amount (or N_b to zero, and then N_{b-1} by the remaining amount). This again does not increase the total derived number, and leaves all the relations $N_q \geqslant Q^{-1}(N_{q+1})$, $q > p$, intact. Hence we may assume that $N_p \geqslant Q^{-1}(N_{p+1})$ throughout. This reduces the system of numbers to some sort of ordering.

Take a_1 so that $\quad (a_1+1)_{n-1} > N_a \geqslant (a_1)_{n-1}$.

Then, since $\quad Q^{-1}(N_{a+1}) \leqslant N_a < (a_1+1)_{n-1}$,

we have $N_{a+1} < (a_1+2)_{n-1}$, $N_{a+2} < (a_1+3)_{n-1}$, ... Let N_{a+p} be the first of the numbers N_a, N_{a+1}, ..., such that $N_{a+p} < (a_1+p)_{n-1}$.

Put $N_{a+q} = (a_1+q)_{n-1} + N'_{a+q}$, $q = 0$ to $p-1$, where N'_{a+q} is the number in scale $n-2$ with the same digits as N_{a+q} omitting the first a_1+q. Compare the system of pairs

$$\begin{array}{cccccc} N_a & N_{a+1} & N_{a+2} & \dots & N_b & 0 \\ 0 & Q(N_a) & Q(N_{a+1}) & \dots & Q(N_{b-1}) & Q(N_b) \end{array}$$

with the substituted system

$$\begin{array}{ccccccccc} (a_1)_{n-1} & (a_1+1)_{n-1} & \dots & (a_1+p-1)_{n-1} & N_{a+p} & N_{a+p+1} & \dots & N_b & 0 \\ 0 & (a_1+1)_{n-1} & \dots & (a_1+p-1)_{n-1} & (a_1+p)_{n-1} & Q(N_{a+p}) & \dots & Q(N_{b-1}) & Q(N_b) \end{array}$$

in which $N_{a+p} < (a_1+p)_{n-1}$ by hypothesis. Here we have diminished

the whole number N by $N'_a+N'_{a+1}+\ldots+N'_{a+p-1}$, or N', and the total derived number by the sum of the higher numbers in the pairs

$$\begin{matrix} N'_a & N'_{a+1} & \ldots & N'_{a+p-1} & 0 \\ 0 & Q_{n-2}(N'_2) & \ldots & Q_{n-2}(N'_{a+p-2}) & Q_{n-2}(N'_{a+p-1}), \end{matrix}$$

that is by a number $\geqslant Q(N')$, assuming the theorem for n variables. This comes by observing that

$$N_{a+q}=(a_1+q)_{n-1}+N'_{a+q}, \quad Q(N_{a+q-1})=(a_1+q)_{n-1}+Q_{n-2}(N'_{a+q-1}),$$

so that the higher of the pair N_{a+q}, $Q(N_{a+q-1})$ is greater than the higher of the substituted pair $(a_1+q)_{n-1}$, $(a_1+q)_{n-1}$ by the higher of the pair N'_{a+q}, $Q_{n-2}(N'_{a+q-1})$. Note that

$$N'=N_a+\ldots+N'_{a+p-1}<(a_1+1)_{n-2}+\ldots+(a_1+p)_{n-2}<(a_1+p)_{n-1}.$$

Put

$$N_{a+p}+\ldots+N_b=(a_1+p)_{n-1}+(a_1+p+1)_{n-1}+\ldots$$
$$+(a_1+t-1)_{n-1}+R_t, \quad 0\leqslant R_t<(a_1+t)_{n-1},$$

or it is possible to have only R_p on the right with $0\leqslant R_p<(a_1+p)_{n-1}$.

Then, since

$$N_{a+p}<(a_1+p)_{n-1}, \quad Q^{-1}(N_{a+p+1})\leqslant N_{a+p}, \quad \ldots, \quad Q^{-1}(N_b)\leqslant N_{b-1},$$

we have

$$Q(N_{a+p})+\ldots+Q(N_b)\geqslant(a_1+p+1)_{n-1}+\ldots+(a_1+t)_{n-1}+Q(R_t),$$

Lemma III, and the total derived number of the substituted system above

$$\{\geqslant(a_1)_{n-1}+\ldots+(a_1+p)_{n-1}+Q(N_{a+p})+\ldots+Q(N_{b-1})+Q(N_b)\}$$

is equal to or greater than that of the system

$$\begin{matrix} (a_1)_{n-1} & (a_1+1)_{n-1} & \ldots & (a_1+t-1)_{n-1} & R_t & 0 \\ 0 & (a_1+1)_{n-1} & \ldots & (a_1+t-1)_{n-1} & (a_1+t)_{n-1} & Q(R_t). \end{matrix}$$

To this we add at the end the number $N'<(a_1+t)_{n-1}$ which was taken away earlier; and according as $R_t+N'<(a_1+t)_{n-1}$ or $\geqslant(a_1+t)_{n-1}$ we put $R_t+N'=R'$ or $(a_1+t)_{n-1}+R'$, $0\leqslant R'<(a_1+t)_{n-1}$. In doing this

we increase the total derived number by $Q(N')$ at most (II, i), whereas we decreased it in the earlier step by $Q(N')$ at least. Finally if $a_1 > 1$ we add on numbers $(1)_{n-1}$, $(2)_{n-1}$, ..., $(a_1-1)_{n-1}$ at the beginning and subtract their sum from the end, again with no increase to the total derived number. We then have

$$N = (1)_{n-1}+(2)_{n-1}+\ldots+(b_1)_{n-1}+R,\ 0 \leqslant R < (b_1+1)_{n-1},$$

$$= (b_1)_n+(b_2, b_3, \ldots)_{n-1},\ \text{where}\ R = (b_2, b_3, \ldots)_{n-1},\ b_2 \leqslant b_1,$$

$$= (b_1, b_2, b_3, \ldots)_n;$$

and the final total derived number

$$= (1)_{n-1}+(2)_{n-1}+\ldots+(b_1+1)_{n-1}+Q(R)$$

$$= (b_1+1)_n+(b_2+1, b_3+1, \ldots)_{n-1} = Q_n(N).$$

Hence the original total derived number $\geqslant Q_n(N)$. Q.E.D.

The final partitioning of N into $(1)_{n-1}$, $(2)_{n-1}$, ..., $(b_1)_{n-1}$, R is such that $N_{p+1} = Q(N_p)$ except for the remainder R. There is a still neater partition such that $N_p = Q^{-1}(N_{p+1})$ throughout, having the same total derived number $Q_n(N)$, but not having a remainder. If $N = (b_1, b_2, \ldots, b_r)_n$, the last partial number is $N_{b_1} = (b_1, b_2, \ldots, b_r)_{n-1}$, and the others are $Q^{-1}(N_{b_1})$, ..., $Q^{b_1+1}(N_{b_1})$. If $r = n$, the last digit b_n in N_{b_1}, though irregular, becomes significant, because it does not disappear till we come to $Q^{-b_n}(N_{b_1})$. The higher of the pair N_{p+1}, $Q(N_p)$ is N_{p+1}; so the total derived number with this partition is

$$\Sigma N_n + Q(N_{b_1}) = (b_1, \ldots, b_r)_n + (b_1+1, \ldots, b_r+1)_{n-1}$$

$$= (b_1+1, \ldots, b_r+1)_n = Q_n(N).$$

III. *Special polynomial-ideals.*

In this section we shall give a few results concerning some special polynomial-ideals* and one or two particular examples.

Ideal of rank n. An H-ideal of rank n in n variables is a point-ideal

* Some references to No. 19 of the "Cambridge Tracts in Mathematics and Mathematical Physics" (*The algebraic theory of modular systems*, Cambridge Univ. Press, 1916) will be made under the reference letter (T).

whose point is the origin $(0, 0, \ldots, 0)$. A non-H-ideal of rank n is composed of a finite number of point-ideals, which may or may not have the origin as one of its points. The conditions for the H-ideal are simply those already found, viz. $D_{l+1} \geqslant Q(D_l)$ with the addition that $\chi(l) = 0$, or H_l vanishes when $l \geqslant \gamma$ (*i.e.* H_l vanishes when l is sufficiently great, the lowest value of l for which this happens being denoted by γ). The conditions for the non-H-ideal are $\Delta D_{l+1} \geqslant Q(\Delta D_l)$ with $\Delta\chi(l) = 0$, or $\chi(l)$ a constant.

To construct a non-H-ideal of rank n with any assigned D series satisfying the necessary conditions above, take the p.p.-ideal P whose D series is ΔD_0, ΔD_1, ΔD_2, ..., *i.e.* the ideal P whose members of any and every degree l are the first ΔD_l p.p.'s in $(x_1, x_2, \ldots, x_n)^l$, and in every member $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ of the basis of P change $x_i^{p_i} (i = 1, 2, \ldots, n)$ to $x_i(x_i - 1) \ldots (x_i - p_i + 1)$. This will give the basis of the required non-H-ideal. It will have precisely the same (finite) number of separate points $(q_1, q_2, \ldots, q_n)$ as there are p.p.'s $x_1^{q_1} x_2^{q_2} \ldots x_n^{q_n}$ which are not members of P. Again, if in P we change $x_1, x_2, \ldots, x_n$ to $x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n$ we obtain a non-H-point-ideal having the same assigned D series.

Ideal of the principal class. This term was used by Kronecker, though it seems to have gone out of use and no other term has replaced it. It is not what is called a principal ideal (or ideal of rank 1 with a basis (F) consisting of a single member) but an ideal of rank r with a basis $(F_1, F_2, \ldots, F_r)$ consisting of r members only. In an H-ideal $(F_1, F_2, \ldots, F_r)$ of rank r the generating function of the H series is

$$(1 - x^{l_1})(1 - x^{l_2}) \ldots (1 - x^{l_r})(1 - x)^{-n},$$

where $l_1, l_2, \ldots, l_r$ are the degrees of $F_1, F_2, \ldots, F_r$. This is proved in (T., § 58) and was known previously.

There is no corresponding formula for a non-H-ideal unless its equivalent H-ideal is also of the principal class. Thus the H-ideal equivalent to $(x_1^2, x_2 + x_1 x_3)$ is $(x_1^2, x_1 x_2, x_2^2, x_0 x_2 + x_1 x_3)$, which is not of the principal class.

Other H-ideals whose generating functions $\Sigma H_l x^l$ can be found (and written down like the formula above) are the ρ-th power $(F_1, F_2, \ldots, F_r)^\rho$ of an H-ideal of the principal class, and an H-ideal of rank $k - \rho + 1$ whose basis consists of all the determinants of a matrix with k columns and ρ rows such that the differences of the degrees of the elements in any row of the matrix are the same for every row. These ideals are perfect (T., §§ 50, 53, 89); and the generating function of a perfect ideal can always be said to be known, as will be explained immediately.

Unmixed ideals and perfect ideals. An *unmixed* ideal M of rank r is one which includes all polynomials F as members for which a polynomial $\phi(x_{r+1}, \ldots, x_n)$ in $x_{r+1}, \ldots, x_n$ exists such that

$$F\phi(x_{r+1}, \ldots, x_n) = 0(M),$$

the variables $x_1, x_2, \ldots, x_n$ having been subjected to a general homogeneous linear substitution beforehand. In other words an ideal M of rank r is unmixed if (and only if) $F\phi(x_{r+1}, \ldots, x_n) = 0(M)$ always requires $F = 0(M)$. If (after the preliminary substitution) we regard $x_{r+1}, \ldots, x_n$ as parameters and $x_1, x_2, \ldots, x_r$ only as variables, ϕ may be regarded as a constant (footnote p. 531).

Let R, R_p, R_{pq} denote polynomials in $x_{r+1}, \ldots, x_n$ only, and ω_1 $(= 1)$, $\omega_2, \omega_3, \ldots$, p.p.'s in $x_1, x_2, \ldots, x_r$ only. Then it is shown (T., §§ 77–80) that any given unmixed H-ideal M of rank r has a system of members

$$\text{(A)} \qquad R\omega_p + R_{p1}\omega_1 + R_{p2}\omega_2 + \ldots + R_{p\mu}\omega_\mu \quad (p = \mu+1, \mu+2, \ldots, \infty)$$

to which corresponds what is called an r-dimensional inverse system

$$\text{(B)} \qquad R\omega_i^{-1} - \sum_{p>\mu}^{\infty} R_{pi}\omega_p^{-1} \quad (i = 1, 2, \ldots, \mu)$$

where R and all the R_{pq} are unique, with H.C.F. equal to 1, the number μ, and the p.p.'s $\omega_1, \omega_2, \ldots, \omega_\mu$ are fixed, and ω_p is any p.p. other than $\omega_1, \omega_2, \ldots, \omega_\mu$. The relation $R_1\omega_1 + R_2\omega_2 + \ldots + R_\mu\omega_\mu = 0(M)$ requires $R_1 = R_2 = \ldots = R_\mu = 0$.

The members of (A) and (B) are to include not only those which are written but also any and every linear combination of them with multiples of type R or ϕ, deprived of any factor R_t that will divide out. When (A) and (B) are complete in this sense the number of members of (A) of degree l which are linearly independent as regards all the letters $x_1, x_2, \ldots, x_n$ is D_l; and the like number for (B) will be denoted by G_l. Thus G_l, unlike D_l, is not necessarily zero when l is negative, since the negative powers of $x_1, x_2, \ldots, x_r$ in a member of (B) may overpower the positive powers of $x_{r+1}, \ldots, x_n$. Each member of (B) is a homogeneous infinite power series in $x_1, x_2, \ldots, x_n$; for $R\omega_i^{-1}$, $R_{pi}\omega_p^{-1}$ are of the same degree, since $R\omega_p$, $R_{pi}\omega_i$ in (A) are of the same degree.

The H-ideal M is said to be *perfect* when (and only when) $R = 1$, a property which is not uncommon, the most important examples being those which have been mentioned above. Assuming M perfect, if we put $x_{r+1} = \ldots = x_n = 0$ in (B) we get the system inverse to the point-ideal

$M(x_{r+1} = \ldots = x_n = 0)$, (T., § 80), whose H series we may denote by $1, h_1, h_2, \ldots, h_{\gamma-1}$ ($h_l = 0$ when $l \geqslant \gamma$). There are h_p members actually written in (B) of degree $-p$, which will be brought to degree l by multiplying each of them by any and every p.p. in $x_{r+1}, \ldots, x_n$ of degree $l+p$. In this way we can get all the G_l members of (B) of degree l, and they are all linearly independent in $x_1, x_2, \ldots, x_n$. Hence G_l is the coefficient of x^l in

$$(1+h_1x^{-1}+h_2x^{-2}+\ldots+h_{\gamma-1}x^{-\gamma+1})(1-x)^{-n+r},$$

while H_l is the coefficient of x^l in

$$(1+h_1x+h_2x^2+\ldots+h_{\gamma-1}x^{\gamma-1})(1-x)^{-n+r}.$$

These are then the generating functions of the G and H series for a perfect H-ideal M of rank r, where $1, h_1, h_2, \ldots, h_{\gamma-1}$ is the H series of the point-ideal $M(x_{r+1} = \ldots = x_n = 0)$. In non-perfect ideals the G and H series are to a large extent independent; ideals with the same G series may have very different H series. In perfect ideals they are dependent; and the dependence can be expressed by the relation

$$H_l - \chi(l) = (-1)^{n-r} G_{-l-n+r},$$

as may be shown from the above.

In an H-ideal of the principal class there is a further simplification, viz. the series $1, h_1, h_2, \ldots, h_{\gamma-1}$ reads the same forwards and backwards, so that $G_l = H_{l-\gamma+1}$, where $\gamma-1 = (l_1-1)+(l_2-1)+\ldots+(l_r-1)$, $l_1, l_2, \ldots, l_r$ being the degrees of the members of the basis of the H-ideal. This, of course, may occur for other perfect H-ideals.

Since $1, h_1, \ldots, h_{\gamma-1}$ is the H series of an H-point-ideal the conditions that $D_0, D_1, D_2, \ldots$ may be the D series of some actual perfect H-ideal of rank r in n variables are $\Delta^{n-r} D_{l+1} \geqslant Q(\Delta^{n-r} D_l)$ with $\Delta^{n-r} H_l = 0$ when $l \geqslant \gamma$.

We give finally a very general and fundamental formula connecting *any two H-ideals which are mutually residual with respect to an H-ideal of the principal class.* Such ideals are necessarily unmixed and of the same rank; but they are perfectly general unmixed ideals.

If M is any unmixed H-ideal of rank r, and $\bar{M}$ is the ideal $(F_1, F_2, \ldots, F_r)$, also of rank r, where $F_1, F_2, \ldots, F_r$ are r members of M (of degrees $l_1, l_2, \ldots, l_r$), then M and its residual, M', with respect to $\bar{M}$ are mutually

residual with respect to $\bar{M}$; and the following relations hold between the terms of the G and H series of M, M', $\bar{M}$:

$$G_{l-\gamma+1}+H'_l = G'_{l-\gamma+1}+H_l = \bar{G}_{l-\gamma+1} = \bar{H}_l,$$

where $\gamma-1 = (l_1-1)+\ldots+(l_r-1)$. This follows from T., § 86, though it is not stated there. The relations determine the G and H series of M' if the G and H series of M are known; for $\bar{H}_l$ is known, viz. $\bar{H}_l$ is the coefficient of x^l in $(1-x^{l_1})\ldots(1-x^{l_r})(1-x)^{-n}$.

Notes and Examples.

It may have been observed that we have only found the conditions which govern the terms of the D series in the two cases of the general ideal and a perfect ideal and some special cases of the latter. We have not found them for the general unmixed ideal, primary ideal, the ideal with no multiple spread, and prime ideal. Each of these cases is more difficult to solve than the previous one, and I doubt whether the solution can be found for any of them, since there seems to be no law governing the discontinuities which occur. In the rather important case of an H-point-ideal whose inverse system is a principal system, the conditions are that the latter half of its finite H series 1, H_1, ..., $H_{\gamma-1}$ is the first half reversed (T, § 70), while the conditions for the first half are only those of the general H-ideal $D_{l+1} \geqslant Q(D_l)$.

The only method that I know of, which can be called in any sense general, for finding the value of D_l for an ideal whose basis $(F_1, F_2, \ldots, F_k)$ is given is the following:—Find its rank r, which is usually known, and, if not known, is easy to find. If $(F_1, F_2, \ldots, F_k)$ is a non-H-ideal, change it into its equivalent H-ideal. This may be difficult, but it is an essential preliminary for working purposes. Then modify the basis if necessary so that its first r members $(F_1, F_2, \ldots, F_r)$ shall not belong to any ideal of rank $< r$. The rest of the work will be sufficiently clear by taking an example (Ex. i).

Example (i). Find D_l for the H-ideal (F_1, F_2, F_3, F_4) of rank 2:—

$$F_1 = v_1w_2-v_2w_1, \qquad F_2 = u_2v_1w_1-u_1v_1w_2+v_2w_2,$$

$$F_3 = u_2v_1^2-u_1v_1v_2+v_2^2, \quad F_4 = u_2w_1^2-u_1w_1w_2+w_2^2,$$

u_1, v_1, w_1 being linear and u_2, v_2, w_2 quadratic homogeneous polynomials in n variables.

The D_l of (F_1, F_2, F_3, F_4) is the number of linearly independent polynomials of (F_1, F_2) of degree l, that is the D_l of (F_1, F_2), increased by the number of polynomials A_3F_3 of degree l of which no linear combination equals $0(F_1, F_2)$, increased by the number of polynomials A_4F_4 of degree l of which no linear combination equals $0(F_1, F_2, F_3)$. To find the first of these two increases we consider the equation

$$X_3F_3 = 0(F_1, F_2)$$

where X_3F_3 is to be of degree l, and X_3 of degree $l-4$. Now $X_3 = 0(w_1, w_2)$, since $(F_1, F_2) = 0(w_1, w_2)$ and $F_3 \neq 0(w_1, w_2)$. Also both w_1F_3 and $w_2F_3 = 0(F_1, F_2)$. Hence it is not only necessary but *sufficient* that $X_3 = 0(w_1, w_2)$. Hence the number of linearly independent polynomials X_3 of degree $l-4$ is the D_{l-4} of (w_1, w_2); and the number of the A_3 (or A_3F_3 above) is the H_{l-4} of (w_1, w_2), *i.e.* the coefficient of x^{l-4} in $(1-x)(1-x^2)(1-x)^{-n}$, or the coefficient of x^l in $x^4(1-x)(1-x^2)(1-x)^{-n}$. It so happens that the second increase, the number of polynomials A_4, is equally easily found in this example; for $X_4F_4 = 0(F_1, F_2, F_3)$ gives $X_4 = 0(v_1, v_2)$, since $(F_1, F_2, F_3) = 0(v_1, v_2)$; so that the number of the polynomials A_4 equals that of the A_3. We can express the general result symmetrically in the form:—the H_l of $(F_1, F_2, \ldots, F_k)$ of rank r, $F_1, F_2, \ldots, F_k$ being of degrees $l_1, l_2, \ldots, l_k$, is $H_l^{(r)} - H_{l-l_{r+1}}^{(r+1)} - \ldots - H_{l-l_k}^{(k)}$, where $H_l^{(r)}$ is the known H_l of $(F_1, F_2, \ldots, F_r)$, and $H_{l-l_p}^{(p)}$ is the H_{l-l_p} of the ideal (X_p) determined by $X_pF_p = 0(F_1, F_2, \ldots, F_{p-1})$. In the example (F_1, F_2, F_3, F_4), H_l is the coefficient of x^l in

$$\{(1-x^3)(1-x^4) - x^4(1-x)(1-x^2) - x^4(1-x)(1-x^2)\}(1-x)^{-n}.$$

The solution would have been more difficult if the second member F_2 had not been the one actually chosen, but one of the other two F_3, F_4.

Example (ii). Find the basis of the p.p.-ideal P corresponding to the H-ideal (F_1, F_2), where F_1, F_2 are of degrees 2, 4 in 4 variables.

First find the last principal member $x_2^{a_2}x_3^{a_3}x_4^{a_4}$ of P. The Hilbert function (p. 536) is

$$(l+1)_2 - (l-a_2, l-a_2-a_3+1)_2$$

$$= \text{the coefficient of } x^l \text{ in } (1-x^2)(1-x^4)(1-x)^{-4} = 8l-8,$$

i.e. $a_2 l - \frac{1}{2}a_2(a_2-3)+a_3 = 8l-8$, so that $a_2 = 8$, $a_3 = 12$. The value of a_4 (or a_n) is zero as it generally is. Thus we have to go to degree 20 before obtaining the last member of the basis of P, viz. $x_2^8 x_3^{12}$.

To write down the basis of P we have to provide that the p.p.'s of P of degree l shall be the first D_l p.p.'s of $(x_1, x_2, x_3, x_4)^l$, but we have only to write down the principal members, that is, those members which are not derivable from the D_{l-1} members of P of degree $l-1$. In this way we find that the basis is

$$(x_1^2,\ x_1 x_2^3,\ x_1 x_2^2 x_3^2,\ \ldots,\ x_2^8 x_3^{12}).$$

Thus the last *derived* member of degree 5 is $x_1 x_2^3 x_4$ (the last member of degree 4 multiplied by x_n or x_4); this gives 23 derived members of degree 5, but as there should be 24 members in all, there is one principal member $x_1 x_2^2 x_3^2$, the next after the last derived member $x_1 x_2^3 x_4$. The chief interest of this example is that it shows the extraordinary height to which the basis of P must ascend in general if P is to have the same D series as a given ideal.

Example (iii). As an example of the application of the formula of p. 551 consider the H-ideal determined by a generators of the same system on a quadric surface.

When l is sufficiently high each generator supplies $l+1$ conditions for a polynomial of degree l, independently of the conditions supplied by the other generators, so that $\chi(l) = a(l+1)$. But when l is low enough the conditions become those of containing the quadric surface; *i.e.* H_l is the coefficient of x^l in $(1-x^2)(1-x)^{-4}$, *i.e.* $(l+1)^2$ so long as this $\leqslant a(l+1)$, or so long as $l \leqslant a-1$; and is $a(l+1)$ when $l \geqslant a$. Hence the generating function $1+H_1 x+H_2 x^2+\ldots$ is

$$1+2^2 x+3^2 x^2+\ldots+a^2 x^{a-1}+a(a+1)x^a+a(a+2)x^{a+1}+\ldots$$

or

$$(1+2x+\ldots+2x^{a-1}-\overline{a-1}x^a)(1-x)^{-2}.$$

The degree of the least polynomial which contains the generators but not the quadric is a, and its residual section with the quadric consists of a generators of the other system. The residual section of a general polynomial F_{a+b} of degree $a+b$ $(b>0)$ through the a generators will be an irreducible curve C of order $a+2b$.

The inverse system (B) of p. 549 corresponding to the a generators is

$$\text{(B)} \qquad \sum_{p_1, p_2} (\alpha_i x_3+\beta_i x_4)^{p_1}(\gamma_i x_3+\delta_i x_4)^{p_2}(x_1^{p_1} x_2^{p_2})^{-1} \quad (i=1, 2, \ldots, a),$$

where $x_1 = \alpha_i x_3 + \beta_i x_4$, $x_2 = \gamma_i x_3 + \delta_i x_4$ are the equations of the i-th generator. These a members of (B) are all of degree zero, and quite independent, because the generators have no points of intersection. Hence $G_l = a(l+1)$ when $l \geqslant 0$, and $G_l = 0$ when $l < 0$, *i.e.* G_l is the coefficient of x^l in $a(1-x)^{-2}$. Applying the formula of p. 551, taking M, M' to be the ideals which correspond to the generators and the curve C, and $\bar{M} = (F_2, F_{a+b})$, where F_2 is the quadric, we have

$$H'_l = \bar{H}_l - G_{l-\gamma+1} = \bar{H}_l - G_{l-a-b}$$

$$= \text{coefficient of } x^l \text{ in } (1-x^2)(1-x^{a+b})(1-x)^{-4} - ax^{a+b}(1-x)^{-2}$$

$$= \text{coefficient of } x^l \text{ in } (1+2x+\ldots+2x^{a+b-1}-\overline{a-1}x^{a+b})(1-x)^{-2}.$$

Similarly for G'_l. Thus $H'_l = H_l$, and similarly $G'_l = G_l$, when $b = 0$; this is as it should be, since M, M' are exactly alike when $b = 0$.

[*Note added July*, 1927, *on the Hilbert function of a prime ideal.* There are many reasons for supposing that the Hilbert function $\chi(l)$ of a prime ideal *having no point singularities* (or the "postulation" of the corresponding irreducible non-singular "variety" for hypersurfaces of sufficiently high degree l) is expressible in terms of certain *genera* (so called by analogy with the genus of a curve or surface). I suggest that this formula is the following :

$$\chi(l) = (l+1)_m + p_0(l)_m - p_1(l)_{m-1} + p_2(l)_{m-2} - \ldots + (-1)^m p_m,$$

where m is the number of dimensions of the spread (or variety), and p_i is the *arithmetic* genus of the section of it, of i dimensions, made by $m-i$ general hyperplanes $u_1 = u_2 = \ldots = u_{m-i} = 0$. Thus p_m is the arithmetic genus of the variety itself, and is further defined as follows : Project the variety into $(m+1)$-space, preserving its order unchanged; it will then have a double spread, in general of $m-1$ dimensions, not free from singularities [unless the original variety is already in $(m+1)$-space, in which case it will be already projected and have no double spread.] Let $\psi(l)$ be the polynomial in l to which the D_l of the double spread becomes equal when l is large enough [*i.e.* ψ is the function complementary to the Hilbert function of the double spread for $(m+1)$-space] ; then $p_m = \psi(p_0 - m - 1)$, and $p_i = \Delta^{m-i}\psi(p_0 - i - 1)$, $p_0 = a-1$ (the order of the variety less 1). If the D_l of the double spread is actually

equal to $\psi(l)$ when $l \geqslant p_0 - m - 1$, the variety and all its sections are *regular;* otherwise the variety is *irregular.*

The evidence so far collected for the correctness of the formula is that it is true (i) for a principal ideal (F), the consideration of which first suggested the formula; (ii) for an H-ideal (F_1, F_2) of the principal class; (iii) for the intersection of three quadrics, and that of four quadrics; (iv) when $p_0 = 0$, as is evident, or $p_1 = 0$; (v) when $m = 1$ (Noether), or $m = 2$ (Severi). I was not aware that Severi had proved the last case until Professor H. F. Baker drew my attention to it. It is the only case among the above which includes any irregular varieties. The formula requires further testing for irregular varieties for three or more dimensions.]

CORRIGENDA.

Vol. 25, pp. 265–282. The following corrections should be made in the paper on "The maximum of a certain bilinear form", by G. H. Hardy, J. E. Littlewood, and G. Pólya:

For 3 March, 1925, *read* 12 March, 1925, *throughout.*

P. 279, l. 8: *for* $\alpha = 1$ *read* $\alpha = \lambda$.

P. 281, l. 7: *for* when *read* When.

Vol. 26, pp. 35–80. The following corrections should be made in the paper on "Some further contributions to the theory of null series and their connexion with null integrals", by C. Fox:

P. 55. *Add to the enunciation of Theorem I*:

$$\text{(v)} \quad a = \phi(x, 0), \quad m = 0,$$
$$a = 0, \qquad m > 0.$$

P. 71. *In equation* (3) *for*

$$\frac{\frac{1}{2}}{\Gamma(\nu+1)} + \sum_{n=1}^{\infty} (-1)^n \frac{J_\nu\{x\sqrt{(n^2+a^2)}\}}{\{\frac{1}{2}x\sqrt{(n^2+a^2)}\}^\nu}$$

read

$$\frac{\frac{1}{2}J_\nu(xa)}{a^\nu} + \sum_{n=1}^{\infty} (-1)^n \frac{J_\nu\{x\sqrt{(n^2+a^2)}\}}{(n^2+a^2)^{\frac{1}{2}\nu}}.$$

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